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Projective Schemes with Degenerate General Hyperplane Section*

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Abstract. We study non-degenerate subschemes $Y \subseteq \mathbf{P}^n$ with degenerate general hyperplane section. Our results consist in some steps towards a classification of such schemes Y, in arbitrary characteristic when Y is a locally Cohen-Macaulay curve (with more details for $n = 4, 5$) and in characteristic zero when Y is an S_2 equidimensional scheme of dimension ≥ 2. Our main results are proved for connected schemes.

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Introduction

Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate pure one-dimensional locally Cohen-Macaulay scheme. Assume that the general hyperplane section of Y is degenerate. What can be said about Y?

For $n = 3$, in characteristic zero, Y must be either the union of two skew lines or a double line ([6], Th. 2.1), while in positive characteristic Y is either the union of two skew lines or a multiple structure on a line with arbitrary degree ([6], Th. 2.1 and Ex. 2.3).

For $n = 4$, in characteristic zero, such curves Y are classified in [2], 1.4 and 1.10; again, Y_{red} must contain a line and, if Y_{red} is connected, then Y must contain a multiple line.

The two quoted papers contain applications of this type of classification, for example to find upper bounds for the (arithmetic) genus or the Hartshorne-Rao module of Y (see in particular [2], Cor. 2.2).

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In the present paper we consider again this problem, with special regard to arbitrary characteristic and to higher-dimensional projective spaces, and we also deal with higher-dimensional projective schemes in characteristic zero.

In Section 1 we give several examples of non-degenerate curves whose general hyperplane section is degenerate, with the aim to convey to the reader a feeling about the problem. These examples also make us to believe that a classification of such curves should start with the connected ones and are a useful illustration of the results in Section 2.

We also introduce and discuss two basic definitions. The most important is the one of S_1-image of a linear projection, which allows us to use characteristic free inductive procedures by projecting a given curve $Y \subseteq \mathbf{P}^n$ into a lower-dimensional projective space from a linear space meeting Y. This notion must be taken with care because of the multiple points and of the lines, which can collapse to points.

The other useful definition says what we mean by minimal curve and is used mainly for some uniqueness statements.

In Section 2 we deal with curves in $\mathbf{P}^n$, and prove our main results for arbitrary n; moreover we give more precise results when $n = 4$ and $n = 5$. We summarize some of our results in the following Theorem A (where $\langle S \rangle$ denotes the linear span of a subscheme $S \subseteq \mathbf{P}^n$):

Theorem A. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate connected (equidimensional) curve with degenerate general hyperplane section. Put $m := dim\langle Y_{red} \rangle$. Then:*

(a) *if Y_{red} is irreducible, then $m \leq \frac{n-1}{2}$* (Th. 2.1);
(b) $m \leq n - 2$ (Prop. 2.3);
(c) *if D is an irreducible component of Y_{red} such that the corresponding component (with multiplicity) of Y is not contained in $\langle Y_{red} \rangle$, then $dim\langle D \rangle \leq n - m - 1$* (Prop. 2.4);
(d) *if $m = n - 2$, then every component as in* (c) *is a line;*
(e) *if $m = n - 2$, then the line as in* (d) *is unique if one of the following conditions is satisfied:*
  (e1) *Y is minimal* (Th. 2.6),
  (e2) $n = 4$ (Th. 2.9),
  (e3) *the characteristic of the ground field is zero* (Th. 2.7).

We want to note explicitly here that if $n \leq 4$ every non-degenerate curve Y with degenerate general hyperplane section contains a line, and contains a multiple line if it is connected. But if $n \geq 5$ this is no longer true because for any $n \geq 3$ there are curves Y for which the bound of Theorem A (a) is sharp (see Remark 2.2). Observe also that if Y is non-degenerate and connected, with degenerate general hyperplane section, then Y is necessarily non-reduced.

In Section 3 we turn our attention to the above problem for equidimensional S_2 schemes of dimension ≥ 2. Our results are obtained from the corresponding ones for curves by induction on the dimension. This is made possible by a consequence of the "Socle Lemma" of Huneke-Ulrich (Lemma 3.1) for which we need characteristic zero. Here we summarize some results:

Theorem B. *Assume that the characteristic of the ground field is zero. Let* $X \subseteq \mathbf{P}^n$ *be an equidimensional non-degenerate* S_2 *subscheme having degenerate general hyperplane section. Then:*

(a) $dim(X) < \frac{n}{2}$, *and* $dim(X) < \frac{n-1}{2}$ *if* $deg(X) \geq 3$ (Th. 3.3);
(b) *if* X *is connected, then* $dim\langle X_{red}\rangle \leq n - dim(X) - 1$ (Th. 3.5);
(c) *if* X *is connected and* $dim\langle X_{red}\rangle = n - dim(X) - 1$, *then there is a unique irreducible component* E *of* X_{red} *such that the corresponding component (with multiplicity) of* X *is not contained in* $\langle X_{red}\rangle$; *moreover* $deg(E) = 1$ *and* $deg(X') = deg(X) - 1$, *where* X' *is the largest equidimensional subscheme of* X *contained in* $X \cap \langle X_{red}\rangle$ (Th. 3.5);
(d) *if* X_{red} *is irreducible, then* $2dim(\langle X_{red}\rangle) \leq n - 1$ (Prop. 3.7).

We consider the above results as a first step towards a classification of the non-degenerate projective schemes with degenerate general hyperplane section. It is clear that such a classification encounters difficulties especially for schemes X with "small" $\langle X_{red}\rangle$, and particularly if they contain lines as components. The first non-complete classifications for curves are in $\mathbf{P}^4$ (positive characteristic) and $\mathbf{P}^5$ (even in characteristic zero). We feel however that such classifications should be pursued, as they will give, ultimately, a much better insight in the cohomological properties of projective schemes.

Standing conventions

We work over an algebraically closed field $\mathbf{K}$, and we will always state explicitly the restriction $char(\mathbf{K}) = 0$ whenever needed. By *curve* we will always mean a pure one-dimensional locally Cohen-Macaulay scheme. For every subscheme $S \subseteq \mathbf{P}^n$, $\langle S\rangle$ will denote the *linear span of* S, that is the least linear space containing S as a subscheme.

1. Definitions and examples

In this section we give several examples, and we introduce and discuss a few definitions, which we believe are rather significant for this type of problem. Furthermore, we fix some notation. We begin with an easy lemma.

Lemma 1.1. *Let* Y *be a non-degenerate curve, and assume that* $Y = A \cup B$, *where* $A, B \subseteq \mathbf{P}^n$ *are two curves with no common components. Then:*

(a) *if* $\langle A\rangle \cap \langle B\rangle = \emptyset$, *then the general hyperplane section of* Y *is degenerate;*
(b) *if* $\langle A\rangle \cap \langle B\rangle \neq \emptyset$ *and the general hyperplane section of* Y *is degenerate, then at least one among* A *and* B *has degenerate general hyperplane section relatively to its span.*

Proof. We have $\langle\langle A\rangle \cup \langle B\rangle\rangle = \mathbf{P}^n$. Hence, setting $dim(\emptyset) = -1$, we have $dim(\langle A\rangle) + dim(\langle B\rangle) = n + dim(\langle A\rangle \cap \langle B\rangle)$. For every general hyperplane H we have $\langle Y \cap H\rangle = \langle\langle A \cap H\rangle \cup \langle B \cap H\rangle\rangle$, $dim(\langle A\rangle \cap H) = dim(\langle A\rangle) - 1$ and $dim(\langle B\rangle \cap H) = dim(\langle B\rangle) - 1$. It follows:

$$dim(\langle Y \cap H\rangle) = dim(\langle A \cap H\rangle) + dim(\langle B \cap H\rangle) - dim(\langle A \cap H\rangle \cap \langle B \cap H\rangle).$$

Moreover $-1 \leq dim(\langle A \cap H\rangle \cap \langle B \cap H\rangle) \leq dim(\langle A\rangle \cap \langle B\rangle)$, and the conclusion follows with easy calculations. □

Remark 1.2. The converse of (b) of Lemma 1.1 is easily seen to be false.

Remark 1.3. By Lemma 1.1, the first important step in our search and classification of non-degenerate curves Y with degenerate general hyperplane section, is the case when Y_{red} is connected (decreasing if necessary the dimension of the ambient projective space). Indeed if Y is non-degenerate, but with degenerate general hyperplane section and if Y_{red} is connected, then for every one-dimensional subscheme Z of the linear space $\langle Y_{red}\rangle$ the general hyperplane section of $Y \cup Z$ is degenerate. Hence, if Y_{red} is not a line, every "minimal" example gives infinitely many examples, $Y \cup Z$, such that the one-dimensional part of $Y \cap Y_{red}$ is not an integral scheme. $\square$

Notation 1.4. If $Y \subseteq \mathbf{P}^n$ is a curve, for every irreducible component D of Y_{red}, D'' will denote the maximal subcurve of Y with $D''_{red} = D$.

If $Y \subseteq \mathbf{P}^n$ is a curve with Y_{red} irreducible, then the *multiplicity* of Y is the integer $deg(Y)/deg(Y_{red})$.

If $Y \subseteq \mathbf{P}^n$ is a curve with Y_{red} not irreducible and D is an irreducible component of Y_{red}, then the multiplicity of D (or of D'') in Y is the integer $deg(D'')/deg(D)$.

For every linear subspace M of $\mathbf{P}^n$ such that $dim(Y \cap M) > 0$, $deg(Y \cap M)$ will denote the degree of the maximal pure one-dimensional subscheme of the scheme $Y \cap M$.

Now we are ready to give some examples of non-degenerate curves with degenerate general hyperplane section.

Example 1.5. (Straightforward application of Lemma 1.1) Let $M,\ N \subseteq \mathbf{P}^n$ be two disjoint linear subspaces with $dim(M) \geq 1$, $dim(N) \geq 1$, and such that $dim(M) + dim(N) = n-1$. Let $Z \subseteq M$ and $W \subseteq N$ be two curves spanning M and N, respectively. Set $Y := Z \cup W$. Then Y is non-degenerate but its general hyperplane section is degenerate.

Example 1.6. Let $D \subseteq \mathbf{P}^n$ be a line. It is well-known that there are several double structures, Δ, on D such that Δ spans $\mathbf{P}^n$; if $n \geq 3$ the general hyperplane section of Δ is degenerate, since it is a zero-dimensional scheme of degree 2, hence it spans a line. Here is a classical construction of one such double structure. Fix a linear space $M \subseteq \mathbf{P}^n$ with $dim(M) = n-2$ and $D \cap M = \emptyset$. Fix a rational curve C spanning M (possibly singular) and take any birational correspondence t between C and D. For a general $Q \in D$, let $D(Q)$ be the line spanned by Q and $t(Q)$. Let S be the non-degenerate irreducible ruled surface which is the closure of the union of the lines $D(Q)$, with Q general in D. For general $Q \in D$ we have $Q \in S_{reg}$. Take as Δ the unique locally Cohen-Macaulay scheme which is a doubling of D inside S at every point of $D \cap S_{reg}$. Since C spans M, Δ spans $\mathbf{P}^n$.

Example 1.7. Let $n \geq 4$, and let Δ be any double line spanning $\mathbf{P}^n$. Let $Z \subseteq \mathbf{P}^n$ be a curve spanning a linear space W of dimension $\leq n-3$, and let $Y := D \cup Z$. Obviously Y is non-degenerate, but it is easy to see that if H is a general hyperplane, then $H \cap Y$ is contained in the union of $H \cap W$ and a line, and hence it is degenerate.

The construction described in Example 1.6 can be generalized and allows us to produce further examples.

Construction 1.8. Fix integers k, m, n with $k \geq 2$ and $1 \leq m \leq n-2$. Fix linear subspaces M, N of $\mathbf{P}^n$ with $dim(M) = m$, $dim(N) \geq 1$ and $\langle M \cup N \rangle = \mathbf{P}^n$. Fix an integral curve T with $\langle T \rangle = M$, an integral curve T' with $\langle T' \rangle = N$ and a rational dominant map f from T onto T'; we allow the case $deg(f) > 1$. Let $Z(T, T', f)$ be the closure of the union of the lines $\langle P, f(P) \rangle$ with P general in T, endowed with the reduced scheme structure induced by $\mathbf{P}^n$. Since T and T' are integral it is also clear that $Z(T, T', f)$ is irreducible, hence integral. If $Z(T, T', f)$ is smooth at a general point of T, we may take a unique k-order thickening, Y, of T inside $Z(T, T', f)$ (namely the closed subscheme of $Z(T, T', f)$ corresponding, locally, to the k-th symbolic power of the prime ideal of T). The surface $Z(T, T', f)$ is smooth at a general point of T if $M \cap N = \emptyset$ (i.e. if $dim(N) = n - m - 1$ and N is general). If $k \geq 2$ we have $\langle Y \rangle = \langle T \cup T' \rangle = \mathbf{P}^n$. By construction, at a general $P \in T_{reg}$ the Zariski tangent space of Y is spanned by $T_P T$ and $\langle P, f(P) \rangle$, i.e. generically Y has planar singularities.

Example 1.9. Let m, t be positive integers, with $m \leq t$ and $min\{m+t, 2t-1\} \leq n-2$. Let M be a linear subspace of $\mathbf{P}^n$ with $dim(M) = m$ and $T \subseteq M$ a reduced curve spanning M with $deg(T) = t$. By 1.8 there exists a non-degenerate curve $Y \subseteq \mathbf{P}^n$ such that $Y_{red} = T$, and every irreducible component of T has multiplicity 2 in Y. The general hyperplane section $Y \cap H$ consists of t distinct points of length 2, and hence it is contained in t double lines meeting M but not contained in M. Then $dim(\langle Y \cap H \rangle) \leq min\{m+t, 2t-1\} \leq n-2$ and $Y \cap H$ is degenerate.

Example 1.10. Fix integers k, n and n_i, $1 \leq i \leq k$, with $k \geq 2$, $n_i > 0$ for every i and $\sum_i n_i = n - k + 1$. Fix linear subspaces V_i, $1 \leq i \leq k$, with $dim(V_i) = n_i$ and $\langle \cup_i V_i \rangle = \mathbf{P}^n$. Hence these subspaces are complementary and disjoint. Fix curves $D_i \subseteq V_i$, $1 \leq i \leq k$, with $\langle D_i \rangle = V_i$. Set $Y := \cup_i D_i$. By Lemma 1.1, Y is non-degenerate but its general hyperplane section is degenerate. Now fix an integer t with $2 \leq t \leq k$ and assume $n_j = n_1$ and D_j projectively equivalent to D_1 for $1 \leq j \leq t$. Taking a flat family of such subspaces $\cup_{1 \leq j \leq t} V_j$ degenerating to a multiple structure of degree t on V_1, we may find such examples with a component of multiplicity t. If $t = k$, we obtain in this way connected examples with multiplicity t structures on any reduced D_1 and (in a suitable sense) a thickening of order t on any non-reduced D_1.

Example 1.11. Fix integers n, m, d with $2 \leq m \leq n-2$ and $d \geq n+1$. Let M be a linear subspace of $\mathbf{P}^n$ of dimension m. It is easy to construct non-degenerate curves $Y \subseteq \mathbf{P}^n$ with $deg(Y) = d$ and such that the one-dimensional part of the scheme $M \cap Y$ has degree $d - n + m + 1$. Obviously the general hyperplane section of any such Y is degenerate. If $n = 4$, $d \geq 4$ and $char(\mathbf{K}) = 0$, this is the only class of examples of non-degenerate curves with degenerate general hyperplane section (see [2], Th. 1.4).

Now we give some basic definitions which will be used in the sequel.

Definition 1.12. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve. We will say that Y is minimal if for every proper subcurve Y' of Y, with $Y_{red} \subseteq Y'$, we have $\langle Y_{red} \rangle = \langle Y' \rangle$.*

Remark 1.13. If Y is minimal, then Y is a generalized rope, i.e. a curve such that for a general point P of every irreducible component of Y_{red} and for a general hyperplane H

through P, the scheme $Y \cap H$ is contained in the first infinitesimal neighbourhood of P in H (see [1]).

In the literature there are several definitions of image for morphisms between schemes. In our very special situation we will use the following definition, which we will call S_1-image. It is used to give a suitable meaning to the linear projection of a curve Y from a linear space meeting Y.

Definition 1.14. *Fix integers n, s with $1 \leq s < n$. Let f be the linear projection from $\mathbf{P}^n := Proj(\mathbf{K}[x_0, \ldots, x_n])$ onto $\mathbf{P}^s := Proj(\mathbf{K}[x_0, \ldots, x_s])$ from the linear subspace $V := x_0 = \ldots = x_s = 0$ of $\mathbf{P}^n$. Let $\mathbf{I} \subseteq \mathbf{K}[x_0, \ldots, x_n]$ be the homogeneous ideal of a curve $Y \subseteq \mathbf{P}^n$. Set $\mathbf{I}' := \mathbf{I} \cap \mathbf{K}[x_0, \ldots, x_s]$, and let $\mathbf{J}$ be the intersection of the 1-dimensional primary components of $\mathbf{I}'$. The subscheme of $\mathbf{P}^s$ corresponding to $\mathbf{J}$ is called the S_1-image of Y and is denoted by $f(Y)$ (if there is no danger to confuse it with different notions).*

Remarks 1.15. We have the following easy consequences of the above definition:
(i) $f(Y)_{red} = f(Y_{red})$;
(ii) if Y_{red} is connected, then $f(Y)_{red}$ is connected;
(iii) $\mathbf{J} = \mathbf{I}'$ if V does not meet any irreducible component of Y which is a line;
(iv) if V does not meet any component of Y_{red} which is a line and Y is non-degenerate, then $f(Y)$ is non-degenerate.

2. The main results for curves

The important tools in this section are the just defined projections, that we use to study the case of curves $Y \subseteq \mathbf{P}^n$. In the first part we show the interplay between the dimension of $\langle Y_{red} \rangle$ and the dimension of the ambient projective space, and give a description of the extremal cases. In the second part we give more precise results for the cases $n = 4, 5$.

Theorem 2.1. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve with Y_{red} irreducible. Assume that the general hyperplane section of Y is degenerate and set $m := dim(\langle Y_{red} \rangle)$. Then $2m \leq n - 1$.*

Proof. We will use induction on n, the starting case $n = 2$ being trivial and the case $n = 3$ being [6], Theorem 2.1. Let $n \geq 4$ and assume $2m \geq n$. Take a general $P \in Y_{red}$ and a general $\mathbf{v} \in T_P Y$. Since Y is non-degenerate, $\mathbf{v}$ is not contained in $\langle Y_{red} \rangle$. Let $g : \mathbf{P}^n \cdots \to \mathbf{P}^{n-2}$ be the projection from the line $\langle \mathbf{v} \rangle$. From the assumption we have $m \geq 2$, whence Y_{red} is not a line. Thus $g(Y) \neq \emptyset$ and $g(Y)$ is non-degenerate by 1.15. Since the general hyperplane section of $g(Y)$ is degenerate and $dim(\langle g(Y)_{red} \rangle) = m - 1$, by induction we obtain $2(m - 1) = 2dim(\langle g(Y)_{red} \rangle) \leq n - 2 - 1$, whence $2m \leq n - 1$, a contradiction. $\square$

Remark 2.2. (i) If $char(\mathbf{K}) = 0$ and $n = 4$, Theorem 2.1 is a very particular case of [2], Theorem 1.4.

(ii) The bound given in Theorem 2.1 is sharp in a wide sense. Indeed for every $n \geq 3$ and for $m = \left[\frac{n-1}{2}\right]$ there is a non-degenerate curve $Y \subseteq \mathbf{P}^n$ with degenerate general hyperplane section, Y_{red} irreducible and $m = dim(\langle Y_{red}\rangle)$. For this it is sufficient to consider Example 1.9, with T an integral curve and $t = m$ if n is odd, or $t = m+1$ if n is even.

If Y_{red} is connected but not irreducible, we are able to prove only the following two weaker results:

Proposition 2.3. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve with Y_{red} connected. Assume that the general hyperplane section of Y is degenerate and set $m := dim(\langle Y_{red}\rangle)$. Then $m \leq n-2$.*

Proof. Since Y_{red} is connected, we have $\langle Y_{red}\rangle \neq \mathbf{P}^n$ ([2], Prop. 1.1). Assume $dim(\langle Y_{red}\rangle) = n-1$. Since for a general hyperplane H we have $dim(\langle Y \cap H\rangle) \leq n-2$ and $dim(\langle Y_{red} \cap H\rangle) = n-2$ ([2], Prop. 1.1), it follows $\langle Y \cap H\rangle = \langle Y_{red} \cap H\rangle$. Since $\langle Y\rangle \neq \langle Y_{red}\rangle$, there is an irreducible component D of Y_{red} with D'' not contained in $\langle Y_{red}\rangle$. Since $\langle Y \cap H\rangle = \langle Y_{red} \cap H\rangle$ for general H, we have $\langle D'' \cap H\rangle \subseteq \langle Y_{red}\rangle$. This implies that $deg(D'') = deg(D')$ where D' is the locally C-M curve associated to $D'' \cap \langle Y_{red}\rangle$. Since D'' is locally Cohen-Macaulay, this implies $D'' = D' \subseteq \langle Y_{red}\rangle$, a contradiction. □

The above proposition implies that there is at least one irreducible component D of Y_{red} such that D'' is not contained in the span of Y_{red}. The next propositions give upper bounds for the dimension of the span of such components.

Proposition 2.4. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve, with Y_{red} connected. Assume that the general hyperplane section of Y is degenerate and set $m := dim(\langle Y_{red}\rangle)$. Let D be an irreducible component of Y_{red} such that D'' is not contained in $\langle Y_{red}\rangle$. Then $dim(\langle D\rangle) \leq n-m-1$.*

Proof. We use induction on n. If $n = 2$ the statement is empty and if $n = 3$ it follows from Proposition 2.3. Now assume $n \geq 4$ and let us argue by contradiction, by assuming $dim(\langle D\rangle) \geq n-m$.

Fix a general $P \in D$ and a general $\mathbf{v} \in T_P D''$. Let $g : \mathbf{P}^n \cdots \to \mathbf{P}^{n-2}$ be the projection from the line $\langle \mathbf{v}\rangle$. By our choice of P and $\mathbf{v}$, the line $\langle \mathbf{v}\rangle$ does not meet any component of Y_{red} different from D and meets D transversally at P and at no other point. Moreover by Proposition 2.3, D is not a line, whence $g(D'') \neq \emptyset$ and $g(Y)$ is non-degenerate by 1.15. On the other hand $g(Y)$ has degenerate general hyperplane section and $g(Y)_{red}$ is connected. By the above we have also that $dim(\langle g(D)\rangle) = dim(\langle D\rangle) - 1$, $dim(\langle g(Y)_{red}\rangle) = m-1$ and $g(D'')$ is not contained in $\langle g(Y)_{red}\rangle$, a contradiction. □

The same inductive proof gives the following more general result:

Proposition 2.5. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve with Y_{red} connected. Assume that the general hyperplane section of Y is degenerate and set $m := dim(\langle Y_{red}\rangle)$. Assume the existence of an integer $t \geq 1$ and of irreducible components $D_1, \ldots, D_t$ of Y_{red} such that* *the scheme D_1'' is not contained in $\langle Y_{red}\rangle$ and for every integer i with $2 \le i \le t$ the scheme D_i'' is not contained in $\langle Y_{red} \cup D_1 \cup \ldots \cup D_{i-1}''\rangle$. Then $dim(\langle D_1 \cup \ldots \cup D_t''\rangle) \le n - m - 1$.*

The next theorems give a description for the extremal case of 2.3.

Theorem 2.6. *Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve with Y_{red} connected. Assume that the general hyperplane section of Y is degenerate and set $m := dim(\langle Y_{red}\rangle)$. Suppose $m = n - 2$. Then every irreducible component D of Y_{red} with D'' not contained in $\langle Y_{red}\rangle$ is a line. If Y is minimal, then there is a unique such component.*

Proof. The first assertion is the case $m = n - 2$ of Proposition 2.4. Take two such components, D and R, and let Y' be the largest curve contained in $Y \cap \langle Y_{red}\rangle$. If Y is minimal we have $Y = D'' \cup Y'$ and $Y = R'' \cup Y'$. Thus $R = D$. □

Theorem 2.7. *Assume $char(\mathbf{K}) = 0$. Let $Y \subseteq \mathbf{P}^n$ be a non-degenerate curve, with Y_{red} connected. Assume that the general hyperplane section of Y is degenerate and set $m := dim(\langle Y_{red}\rangle)$. Suppose $m = n - 2$. Then:*

(a) *there is a unique irreducible component, D, of Y_{red} with D'' not contained in $\langle Y_{red}\rangle$ and such component is a line.*

Let now k (obviously ≥ 2) be the multiplicity of D in D'' and in Y. Then:

(b) *$deg(D'' \cap \langle Y_{red}\rangle) = k - 1$, and $deg(Y \cap \langle Y_{red}\rangle) = deg(Y) - 1$.*

Proof. (a) The fact that every such component is a line follows from Theorem 2.6. Now we check the uniqueness of D. If $n = 3$ the statement is trivial, and if $n = 4$ it follows from [2], 1.10. Assume $n \ge 5$ and the existence of two such components, D and R. Since $n \ge 5$, then Y_{red} contains an irreducible component Z different from D and R. Call f the projection into $\mathbf{P}^{n-1}$ from a general point P of Z. Then the general hyperplane section of $f(Y)$ is degenerate. Moreover $f(D'')$ and $f(R'')$ are contained in $f(Y)$, but not in $\langle f(Y)_{red}\rangle$, and $f(D'') \ne f(R'')$. We have also, by 1.15, that $f(Y)_{red}$ is connected and $dim\langle f(Y)_{red}\rangle = n - 3$. Hence if $f(Y)$ is non-degenerate the conclusion follows by the inductive hypothesis. We assume now that $f(Y)$ is degenerate, and we seek for a contradiction. Set $L := \langle f(Y)\rangle$. Since $n - 2 \ge dim(L) > dim(\langle f(Y)_{red}\rangle) = n - 3$, it follows that $dim(L) = n - 2$. Since $f(D'')$ is not contained in $\langle f(Y)_{red}\rangle$ we have $dim(\langle f(Y)_{red} \cup D''\rangle) = n - 2$, whence $\langle f(Y)_{red} \cup D''\rangle = L$. Let H be the hyperplane of $\mathbf{P}^n$ joining L and P. Then $Y_{red} \cup D'' \subseteq H$. This holds for every component D of Y_{red} such that $D'' \nsubseteq \langle Y_{red}\rangle$, and hence $Y \subseteq H$, a contradiction.

(b) We use induction on n. If $n = 3$, then Y is a multiple line and since $char(\mathbf{K}) = 0$, Y is indeed a double line by [6], 2.1. Assume now $n \ge 4$. Since $dim(\langle Y_{red}\rangle) = n - 2 \ge 2$, there is an irreducible component A of Y_{red} with $A \ne D$. Fix a general $P \in A$ and let $f : \mathbf{P}^n \setminus \{P\} \to \mathbf{P}^{n-1}$ be the projection from P. The S_1-image $f(Y)$ has degenerate general hyperplane section, and, as in the proof of (a), $f(Y_{red}) = f(Y)_{red}$ is connected and $dim(\langle f(Y)_{red}\rangle) = n - 3$. Since $A'' \subseteq \langle Y_{red}\rangle$, its existence does not affect the non-degeneracy of Y, and hence $f(Y)$ is non-degenerate. Since P is general in A, we have $P \notin D$ and hence $f(D)$ is a line and $f(D)''$ is not contained in $\langle f(Y_{red})\rangle$. By the inductive assumption $deg(f(D)'' \cap \langle f(Y_{red})\rangle) = k - 1$. Hence $deg(D'' \cap \langle Y_{red}\rangle) \ge k - 1$. Since Y is locally C-M and D'' is not contained in $\langle Y_{red}\rangle$, we have $deg(D \cap \langle Y_{red}\rangle) \leq k-1$ and we conclude. □

Now we prove a more precise result for curves in $\mathbf{P}^4$.

Remark 2.8. Let $Y \subseteq \mathbf{P}^4$ be a non-degenerate curve with degenerate general hyperplane section and assume Y_{red} connected; then we have $dim(\langle Y_{red}\rangle) \leq 2$, by 2.3.

If $dim(\langle Y_{red}\rangle) = 1$, then Y is a multiple line. Since by [6], Ex. 2.3, we have several wild examples of multiple lines in positive characteristic, but not a full classification of them (not even in $\mathbf{P}^3$), we will assume in the following that $dim(\langle Y_{red}\rangle) = 2$.

Theorem 2.9. *Let $Y \subseteq \mathbf{P}^4$ be a non-degenerate curve with degenerate general hyperplane section.*

(a) *Assume Y reduced: then Y is the disjoint union of a line and a plane curve.*
(b) *Assume Y_{red} not connected: then Y is the union either of two skew lines (one at least of multiplicity > 1), or of a curve spanning a plane Π and a reduced line, skew with Π.*
(c) *Assume Y_{red} connected and $deg(Y_{red}) > 1$ (i.e. Y not a multiple line): then there are a plane $\Pi \subseteq \mathbf{P}^4$ and a unique component $D \subseteq Y_{red}$ such that Π does not contain the scheme D''. Moreover D is a line.*

In any case Y contains a multiple line.

Proof. (b) is an easy consequence of Lemma 1.1, and (a) follows from (b) and [2], Prop. 1.1. To prove (c) set $\Pi := \langle Y_{red}\rangle$. By Proposition 2.3 and Theorem 2.6 we have that Π is a plane and Y_{red} contains a line D such that D'' is not contained in Π. Put $Y = D'' \cup Z$ (whence Z_{red} is the union of the components of Y_{red} different from D). To conclude our proof we have to prove that $Z \subseteq \Pi$. Fix a general $P \in D$ and let $\mathbf{v}$ be a general tangent vector to D'' at P. By assumption $\langle \mathbf{v}\rangle \cap \Pi = \{P\}$. Let $g : \mathbf{P}^4 \setminus \langle \mathbf{v}\rangle \to \mathbf{P}^2$ be the projection. Since Y_{red} is not a line we have $g(Y) \neq \emptyset$. The general hyperplane section of the S_1-image $g(Y) \subseteq \mathbf{P}^2$ is degenerate, i.e. $g(Y)$ is a line with multiplicity 1, and it follows easily that $Z \subseteq g^{-1}(g(Y)) = \Pi$. □

Next, we deal with curves in $\mathbf{P}^5$.

Proposition 2.10. *Let $Y \subseteq \mathbf{P}^5$ be a non-degenerate curve with degenerate general hyperplane section. Assume $dim(\langle Y_{red}\rangle) = 2$ and that Y is minimal. Then there is a unique component D of Y_{red} such that D'' is not contained in $\langle Y_{red}\rangle$, i.e. Y is the curve associated to $D'' \cup (Y \cap \langle Y_{red}\rangle)$. If D is not a line, then $\langle D''\rangle = \mathbf{P}^5$. If D is a line, then $dim(\langle D''\rangle) \geq 4$.*

Proof. Since Y is minimal there is a unique component D of Y_{red} with D'' not contained in $\langle Y_{red}\rangle$ (see Th. 2.6), whence $\langle D''\rangle \cup \langle Y_{red}\rangle = \langle D'' \cup \langle Y_{red}\rangle\rangle = \mathbf{P}^5$.

Then $5 = dim(\langle D''\rangle) + dim(\langle Y_{red}\rangle) - dim(\langle D''\rangle \cap \langle Y_{red}\rangle)$, whence $dim(\langle D''\rangle) = dim(\langle D''\rangle \cap \langle Y_{red}\rangle) + 3$. Since $D'' \cap \langle Y_{red}\rangle$ contains D, the last assertion follows. □

Notice that if $dim(\langle Y_{red}\rangle) = 2$, then Y_{red} is connected. If Y_{red} is connected, we have $dim(\langle Y_{red}\rangle) \leq 3$ by Proposition 2.3, and the case $dim(\langle Y_{red}\rangle) = 3$ is settled by Theorem 2.6. In summary, we have the following result:

Corollary 2.11. *Let $Y \subseteq \mathbf{P}^5$ be a non-degenerate curve with degenerate general hyperplane section and with Y_{red} connected. Assume that Y is minimal. Then there is a unique component D of Y_{red} such that D'' is not contained in $\langle Y_{red} \rangle$.*

Remark 2.12. By using Example 1.5 it is easy to construct minimal non-degenerate curves $Y \subseteq \mathbf{P}^5$ with degenerate general hyperplane section, but which are not of the type described in Corollary 2.11 (even in characteristic zero). However, at least we know that any such curve must contain a curve of that type.

We observe that a complete classification of the non-degenerate curves in $\mathbf{P}^5$ with degenerate general hyperplane section is not yet complete, not even in characteristic zero, the major problem being given by the multiple lines.

3. The higher-dimensional case

In this section we consider again the problem: if $X \subseteq \mathbf{P}^n$ is a non-degenerate scheme with degenerate general hyperplane section, what can be said about X?

We consider S_2 schemes of pure dimension ≥ 2. The assumption S_2 (instead of locally Cohen-Macaulay), is due to the fact that we can use a strong form of Bertini's Theorem proved in [4].

The main tool is [3], Lemma 5.9, of which we need the following particular case:

Lemma 3.1. *Assume $char(\mathbf{K}) = 0$. Let $X \subseteq \mathbf{P}^n$ be an equidimensional subscheme with $dim(X) \geq 2$. Take two general hyperplanes H and H'. Set $Y := X \cap H$ and $Z := Y \cap H'$. If Y is degenerate in H, then Z is degenerate in $\langle Y \rangle \cap H'$.*

Proof. The result follows from [3], Lemma 5.9, applied for $b = 1$. □

Lemma 3.2. *Let $X \subseteq \mathbf{P}^n$ be an equidimensional subscheme and H a general hyperplane. Set $Y := X \cap H$. Then:*

(a) *$X_{red} \cap H = Y_{red}$;*
(b) *if X is S_2, then Y is S_2 and equidimensional;*
(c) *if $dim(X) \geq 2$ and X is S_2 and connected, then Y is connected.*

Proof. (a) By [4], Cor. 2, the scheme $X_{red} \cap H$ is reduced. Since Y and $X_{red} \cap H$ coincide set-theoretically, we have the conclusion.

(b) Y is S_2 by [4], Cor. 2. It follows that Y has no embedded components and then Y is equidimensional because X is equidimensional and H is general.

(c) Let X_i, $1 \leq i \leq s$, be the irreducible components of X with their multiple structure. Set $Y_i := X_i \cap H$. By Bertini's Theorem Y_i, $1 \leq i \leq s$, are the irreducible components of the scheme Y. Suppose that Y is not connected. Then we may assume that $(Y_1 \cup \ldots \cup Y_r) \cap (Y_{r+1} \cup \ldots \cup Y_s) = \emptyset$ for some r with $1 \leq r < s$. Hence $dim((X_1 \cup \ldots \cup X_r) \cap (X_{r+1} \cup \ldots \cup X_s)) = 0$. Thus by Hartshorne's Connectedness Theorem [5] each point of $(X_1 \cup \ldots \cup X_r) \cap (X_{r+1} \cup \ldots \cup X_s)$ has depth ≤ 1, contradicting the fact that X is S_2 and with pure dimension ≥ 2. □

Theorem 3.3. *Assume char*$(\mathbf{K}) = 0$*. Let* $X \subseteq \mathbf{P}^n$ *be an equidimensional non-degenerate* S_2 *subscheme. Assume that either:*
(a) $dim(X) \geq \frac{n}{2}$*, or*
(b) $dim(X) \geq \frac{n-1}{2}$ *and* $deg(X) \geq 3$*.*
Then the general hyperplane section Y *of* X *is non-degenerate.*

Proof. (a) We use induction on n. The cases with $n = 1, 2, 3$ and the case with $n = 4$ and $dim(X) = 3$ are trivial. If $n = 4$ and $dim(\langle Y \rangle) = 2$, then Y is a hypersurface in $\langle Y \rangle$ and hence it is arithmetically Cohen-Macaulay. Therefore X is arithmetically Cohen-Macaulay and in particular it is cohomologically connected (i.e. $h^1(\mathbf{I}_X) = 0$, or, equivalently, $h^0(\mathcal{O}_X) = 1$), a contradiction. Assume $n \geq 5$ and that the statement is true for all integers n' with $4 \leq n' < n$. Suppose Y degenerate and set $h := dim(\langle Y \rangle) \leq n-2$. We have $dim(Y) = dim(X) - 1 \geq \frac{n}{2} - 1 \geq \frac{h}{2}$. Moreover by Lemma 3.2, Y is equidimensional and S_2. We may apply to $Y \subseteq \langle Y \rangle$ the inductive assumption for the pair of integers $(h, dim(Y))$. Hence if H' is a general hyperplane of $\mathbf{P}^n$ the scheme $Z := Y \cap H'$ is a non-degenerate subscheme of $\langle Y \rangle$, contradicting Lemma 3.1.

(b) We use induction on n. If $n = 3$ and $dim(X) = 1$ the statement is [6], Th. 2.1. If $n = 3$ and $dim(X) = 2$ the result is trivial. If $n = 4$ and $dim(X) \geq 2$ part (b) follows from part (a). As we did in proving part (a) assume $n \geq 5$ and that the statement holds for all integers n' with $4 \leq n' < n$. Suppose Y degenerate and set $h := dim(\langle Y \rangle) \leq n-2$. We have $dim(Y) = dim(X) - 1 \geq \frac{n-1}{2} - 1 \geq \frac{h-1}{2}$. Moreover, $deg(Y) = deg(X) \geq 3$. Hence we may proceed by induction as in the proof of part (a) and get the conclusion. □

Remark 3.4. If $deg(X) = 2$, the general hyperplane section of X may be degenerate: just take X to be the union of two disjoint planes in $\mathbf{P}^5$.

There are also several double structures X, on a plane, which span $\mathbf{P}^5$, but whose general hyperplane section is degenerate. It is sufficient to take the cone on a double line spanning $\mathbf{P}^4$. By Example 1.6 for every integer $g < 0$ we may find such double line with arithmetic genus g. Hence we may find such examples with X of arbitrarily negative sectional genus. However no such cone is locally Cohen-Macaulay at its vertex.

The following result generalizes 2.3 and 2.7.

Theorem 3.5. *Assume char*$(\mathbf{K}) = 0$*. Let* $X \subseteq \mathbf{P}^n$ *be an equidimensional non-degenerate* S_2 *connected subscheme with degenerate general hyperplane section. Then:*
(a) $dim(\langle X_{red} \rangle) \leq n - dim(X) - 1$*;*
(b) *if* $dim(\langle X_{red} \rangle) = n - dim(X) - 1$*, there is a unique irreducible component* E *of* X_{red} *such that* $E'' \nsubseteq \langle X_{red} \rangle$*; moreover* $deg(E) = 1$*,* $deg(E') = deg(E'') - 1$ *and* $deg(X') = deg(X) - 1$*, where* E' *(resp.* X'*) is the largest equidimensional subscheme of* $E'' \cap \langle X_{red} \rangle$ *(resp.* $X \cap \langle X_{red} \rangle$*).*

Proof. (a) Induction on $d := dim(X)$. If $d = 1$, the conclusion follows from Proposition 2.3. Assume $d \geq 2$ and the statement to be true for $d - 1$. Let $Y := X \cap H$, where H is a general hyperplane. By Lemma 3.2, Y is connected, S_2 and equidimensional, and $Y_{red} = X_{red} \cap H$. By assumption $dim(\langle Y \rangle) \leq n - 2$, and Y is non-degenerate in $\langle Y \rangle$, while its general hyperplane section is degenerate by Lemma 3.1. Then by the inductive hypothesis we get $dim(\langle Y_{red}\rangle) \leq n-3-dim(Y)$. Now $dim(\langle Y_{red}\rangle) = dim(\langle X_{red}\rangle)-1$ because X_{red} is connected (seen by [2], Proposition 1.1), and the conclusion follows.

(b) We use again induction on d, the case $d=1$ being in Theorem 2.7. As before we have $dim(\langle Y_{red}\rangle) = dim(\langle X_{red}\rangle)-1 = (n-1)-dim(Y)-1$, and by Lemmas 3.1 and 3.2 the induction assumption is satisfied. Moreover by Bertini's Theorem there is a natural bijection between the irreducible components of X and the irreducible components of Y, and the conclusion follows easily. $\square$

We conclude the paper with the following results:

Proposition 3.6. *Assume $char(\mathbf{K})=0$. Let $X \subseteq \mathbf{P}^n$ be an equidimensional non-degenerate S_2 subscheme with degenerate general hyperplane section. Assume $dim(X) \geq \frac{n}{2}-2$. Then X_{red} has an irreducible component of degree 1.*

Proof. If $n=4$ the result is a particular case of [2], 1.10. If $n>4$ we use induction on n, exactly as in the proof of Theorem 3.3. $\square$

Proposition 3.7. *Assume $char(\mathbf{K})=0$. Let $X \subseteq \mathbf{P}^n$ be an equidimensional non-degenerate S_2 subscheme with degenerate general hyperplane section. If X_{red} is irreducible, then $2dim(\langle X_{red}\rangle) \leq n-1$.*

Proof. Induction on $d := dim(X)$. If $d=1$ the result is in Theorem 2.1. The induction argument is similar to the previous ones, and is left to the reader. $\square$

References

[1] Ballico, E.: *On the linearly general position of a general hyperplane section of nonreduced curves.* J. Algebra **183** (1996), 82–89.

[2] Chiarli, N.; Greco, S.; Nagel, U.: *On the genus and Hartshorne-Rao module of projective curves.* Math. Z., to appear.

[3] Chiarli, N.; Greco, S.; Nagel, U.: *Regularity bounds for projective subschemes of codimension two.* Preprint 1998.

[4] Cumino, C.; Greco, S.; Manaresi, M.: *An axiomatic approach to the second theorem of Bertini.* J. Algebra **98** (1986), 171–182.

[5] Hartshorne, R.: *Complete intersections and connectedness.* Amer. J. Math. **84** (1962), 497–508.

[6] Hartshorne, R.: *The genus of space curves.* Ann. Univ. Ferrara VII (N.S.) **40** (1994), 207–223, 1996.

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