Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 2, 459-474.

Flow-symmetric Riemannian Manifolds

Eric Boeckx* **Peter Bueken*** **Lieven Vanhecke**

Department of Mathematics, Katholieke Universiteit Leuven
Celestijnenlaan 200B, 3001 Leuven, Belgium
e-mail: Eric.Boeckx@wis.kuleuven.ac.be
Peter.Bueken@wis.kuleuven.ac.be
Lieven.Vanhecke@wis.kuleuven.ac.be

Abstract. We introduce the notion of (locally) flow-symmetric Riemannian manifolds. These are Riemannian manifolds admitting a non-vanishing vector field ξ such that all local reflections with respect to integral curves of this vector field are (local) isometries. Locally φ-symmetric Sasakian manifolds, strongly φ-symmetric contact metric manifolds and (locally) Killing-transversally symmetric spaces are examples of such spaces. We give necessary and sufficient conditions for an analytic Riemannian manifold to be locally flow-symmetric, and construct a number of additional examples. Finally, we present a complete local classification of two-dimensional locally flow-symmetric spaces.

MSC 1991: 53C25
Keywords: locally flow-symmetric spaces, KTS-spaces, φ-symmetric Sasakian spaces, strongly φ-symmetric contact metric manifolds

1. Introduction

Local reflections with respect to curves in a Riemannian manifold (M, g) are natural generalizations of reflections with respect to straight lines in Euclidean geometry. It is well-known that the properties of these special local diffeomorphisms are very closely related to the geometry (in particular, the curvature) of the manifold (M, g) and to the geometric properties of the curves. (We refer to [20] for a survey of known results concerning local reflections in Riemannian geometry and for further references to related material.) As an example it is well-known that, if the local reflections with respect to a curve σ in a Riemannian manifold (M, g) are local isometries, the curve σ must be a geodesic. Moreover, it turns out that certain properties of these local reflections are even characteristic for special

* Postdoctoral Researcher of the Fund for Scientific Research – Flanders (FWO – Vlaanderen)

classes of Riemannian manifolds. Examples of such characteristic properties were obtained by T.J. Willmore and the third author in [21]. They prove that a (connected) Riemannian manifold has constant curvature if and only if all local reflections with respect to geodesics are local isometries. Moreover, the space is locally symmetric if and only if the local reflections with respect to all geodesics are volume-preserving (up to sign). The precise relation between the geometry of a Riemannian manifold and the properties of its local reflections has been investigated extensively, leading to a number of nice characterizations for certain special classes of Riemannian manifolds.

A particular case where this approach has led to interesting results is that of contact geometry. An *almost contact metric manifold* $(M, g, \xi, \eta, \varphi)$ is an odd-dimensional Riemannian manifold (M, g) equipped with a vector field ξ, a one-form η and a (1,1)-tensor field φ satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \eta, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{1}$$

for all vector fields X and Y tangent to M. This manifold is said to be *contact metric* if

$$d\eta(X, Y) = \phi(X, Y) = g(X, \varphi Y) \tag{2}$$

for all vector fields X and Y tangent to M. Furthermore, an almost contact metric manifold is *K-contact metric* if it satisfies the condition

$$\nabla_X \xi = -\varphi X \tag{3}$$

for all vector fields X and it is called *Sasakian* if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X \tag{4}$$

for all vector fields X and Y. (We refer to [1] for more detailed information on almost contact metric manifolds.)

On a Sasakian manifold $(M, g, \xi, \eta, \varphi)$ the characteristic vector field ξ is a Killing vector field of constant length and hence, its integral curves form a family of geodesics, called *ξ-geodesics*. In [18], the local reflections with respect to these ξ-geodesics were used by T. Takahashi to characterize the *locally φ-symmetric Sasakian manifolds*. These are Sasakian manifolds whose Riemann curvature tensor R satisfies the condition $\nabla_X R_{YZVW} = 0$ for all vector fields X, Y, Z, V, W orthogonal to ξ. In particular, it was shown in [18] that a Sasakian manifold is locally φ-symmetric if and only if the local reflections with respect to ξ-geodesics are automorphisms of the Sasakian structure (that is, they preserve the structure tensors g, η and φ). This result was extended by D. Blair and the third author in [2] where it was shown that a necessary and sufficient condition for a Sasakian manifold to be locally φ-symmetric is that the local reflections with respect to all ξ-geodesics are local isometries. In [6], it was later shown that this condition even characterizes locally φ-symmetric Sasakian spaces in the larger class of K-contact metric manifolds. Finally, in [13], [14], F. Narita studies Riemannian submersions whose local reflections with respect to the fibers are local isometries and he shows that, under certain conditions, such a manifold admits a locally φ-symmetric Sasakian structure. We also remark that in the complete and simply connected case, the locally φ-symmetric spaces were completely classified by O. Kowalski and J. Jiménez in [11].

The characteristic property of locally φ-symmetric Sasakian manifolds mentioned above recently led two of the present authors [4] to introduce *strongly φ-symmetric contact metric spaces*. These are defined as contact metric manifolds for which the local reflections with respect to the integral curves of the characteristic vector field ξ are local isometries. (It should be remarked here that, on a contact metric manifold, the vector field ξ is not necessarily a Killing vector field, but the integral curves of ξ are again geodesics.) In [4], it was shown that the unit tangent bundle of a space of constant curvature c, equipped with its natural contact metric structure, is strongly φ-symmetric. However, this contact metric structure is Sasakian only if $c = 1$, and hence this construction provides a large number of non-Sasakian examples. More recently, considering contact metric manifolds whose characteristic vector field ξ belongs to the (k, μ)-nullity distribution, the first author [3] constructed a more general class of examples of (non-Sasakian) strongly φ-symmetric contact metric manifolds. (We also refer to [5], [7] and [16] for related information.)

Also motivated by the situation in the Sasakian case (but now omitting the contact metric structure), J.C. González-Dávila, M.C. González-Dávila and the third author [10] introduced the *locally Killing-transversally symmetric manifolds* (or *locally KTS-spaces*). These are Riemannian manifolds admitting a Killing vector field ξ of unit length such that the local reflections with respect to the integral curves of ξ are local isometries. These manifolds are special examples of transversally symmetric Riemannian manifolds, introduced and studied by Ph. Tondeur and the third author in [19].

The first aim of the present paper is to introduce the notion of *locally flow-symmetric* Riemannian manifolds as Riemannian manifolds admitting a non-vanishing vector field ξ such that the local reflections with respect to the integral curves of this vector field are local isometries. After collecting some preliminary results we present, in Section 2, a system of necessary and sufficient conditions (on the Riemann curvature tensor and its covariant derivatives) for an analytic Riemannian manifold (M, g) to be locally flow-symmetric. As a consequence, we also find that the manifold can only be locally flow-symmetric with respect to the eigenvector fields of the Ricci operator.

It is easily seen from the discussion above that all locally φ-symmetric Sasakian manifolds, strongly φ-symmetric contact metric manifolds and locally KTS-spaces are examples of locally flow-symmetric Riemannian manifolds. It is also well-known that all these examples are locally homogeneous Riemannian manifolds (and hence, they are also analytic). In Section 3, we construct a number of new examples of locally flow-symmetric manifolds which are not strongly φ-symmetric contact metric spaces nor (locally) KTS-spaces. In particular, it turns out that certain warped product manifolds and doubly warped product manifolds, as well as some Lie groups (equipped with left-invariant Riemannian metrics), admit a unit length vector field ξ such that the local reflections with respect to its integral curves are local isometries. Furthermore, we find that, among these examples, there are locally flow-symmetric Riemannian manifolds which, contrary to all previously known examples, are not locally homogeneous and not analytic.

Finally, in Section 4, we present a complete classification of all two-dimensional locally flow-symmetric spaces. In particular, we show that, in dimension two, all locally flow-symmetric Riemannian manifolds can (locally) be represented as warped product manifolds.

2. Locally flow-symmetric manifolds

In what follows, we denote by (M,g) an n-dimensional, smooth, connected Riemannian manifold and by ∇ its associated Levi Civita connection. Furthermore, we define the Riemann curvature tensor R as the (1,3)-tensor field given by

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for all vector fields $X, Y \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields tangent to M, and we write

$$R_{XYZW} = g(R_{XY}Z, W)$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

Now, let $\sigma : [a,b] \to M$ be a smooth, embedded curve in M and denote by $\sigma^\perp$ the normal bundle of σ and by $\exp_\sigma$ the exponential map of this normal bundle, that is,

$$\exp_\sigma(\sigma(t), v) = \exp_{\sigma(t)}(v)$$

for all $t \in [a,b]$ and all (sufficiently small) $v \in \sigma(t)^\perp$, where $\sigma(t)^\perp$ denotes the fiber of $\sigma^\perp$ over $\sigma(t)$. The involutive local diffeomorphism ψ_σ, given (in a sufficiently small neighborhood of the curve σ) by

$$\psi_\sigma : p = \exp_\sigma(\sigma(t), v) \mapsto \psi_\sigma(p) = \exp_\sigma(\sigma(t), -v),$$

is called the *local reflection* with respect to σ.

In [8], B.Y. Chen and the third author proved the following

Proposition 1. *Let (M,g) be a Riemannian manifold and σ a curve in M. If the local reflection ψ_σ is an isometry, then σ is a geodesic and, for all $k \in \mathbf{N}$ and all vector fields X,Y and Z orthogonal to $\dot\sigma$, we have*

$$\begin{aligned}(\nabla^{2k}_{X\ldots X}R)_{XYX\dot\sigma} &= 0,\\ (\nabla^{2k+1}_{X\ldots X}R)_{XYXZ} &= 0,\\ (\nabla^{2k+1}_{X\ldots X}R)_{X\dot\sigma X\dot\sigma} &= 0\end{aligned}$$

where $\nabla^i R$ denotes the i-th covariant derivative of the Riemann curvature tensor. Moreover, if the Riemannian manifold (M,g) is real analytic, the converse also holds.

As mentioned before, a Riemannian manifold (M,g) is said to be *locally flow-symmetric* if it admits a non-vanishing vector field ξ such that the local reflections with respect to the integral curves of ξ are (local) isometries. Rescaling this vector field ξ if necessary and applying Proposition 1, we obtain the following

Theorem 2. *A necessary condition for a Riemannian manifold (M, g) to be locally flow-symmetric is the existence of a unit vector field ξ whose integral curves are geodesics (that is, $\nabla_\xi \xi = 0$) and such that*

$$\begin{aligned}(\nabla^{2k}_{X\ldots X} R)_{XYX\xi} &= 0,\\ (\nabla^{2k+1}_{X\ldots X} R)_{XYXZ} &= 0,\\ (\nabla^{2k+1}_{X\ldots X} R)_{X\xi X\xi} &= 0\end{aligned} \tag{5}$$

for all $k \in \mathbf{N}$ and all vector fields X, Y, Z orthogonal to ξ. If the Riemannian manifold is analytic, this necessary condition is also sufficient.

A straightforward argument yields the following

Corollary 3. *Let (M, g) be a Riemannian manifold which is locally flow-symmetric with respect to the unit vector field ξ. Then ξ is an eigenvector field of the Ricci operator, that is, $\rho(X, \xi) = \lambda g(X, \xi)$ for some function λ on M. Moreover, $X(\lambda) = 0$ for all vector fields X orthogonal to ξ.*

3. Examples of locally flow-symmetric spaces

We have already mentioned that locally φ-symmetric Sasakian manifolds and, more generally, strongly φ-symmetric contact metric manifolds and locally Killing-transversally symmetric spaces are examples of locally flow-symmetric manifolds. The aim of the present section is to construct other examples.

3.1. The semi-direct product $\mathbf{R} \ltimes_\alpha \mathbf{R}^n$

Given a linear map $\alpha : \mathbf{R}^n \to \mathbf{R}^n$ and a basis $\{e_0, e_1, \ldots, e_n\}$ for the vector space $\mathbf{R}^{n+1} = \mathbf{R} \times \mathbf{R}^n$, we construct a bracket operation on $\mathbf{R}^{n+1}$ by putting

$$[e_0, e_i] = \alpha e_i, \quad [e_i, e_j] = 0 \tag{6}$$

for all $i, j = 1, \ldots, n$. This bracket makes the space $\mathbf{R}^{n+1}$ into a Lie algebra $\mathbf{R} +_\alpha \mathbf{R}^n$, called the *semi-direct sum* of $\mathbf{R}$ and $\mathbf{R}^n$. The unique connected, simply connected Lie group $\mathbf{R} \ltimes_\alpha \mathbf{R}^n$ associated to the Lie algebra $\mathbf{R} +_\alpha \mathbf{R}^n$ is called the *semi-direct product* of the Lie groups $\mathbf{R}$ and $\mathbf{R}^n$ and it admits a unique left-invariant Riemannian metric g for which the vectors e_i form an orthonormal basis of left-invariant vector fields. Denoting by $\{\omega^0, \omega^1, \ldots, \omega^n\}$ the orthonormal coframe field dual to the frame field $\{e_0, \ldots, e_n\}$, a straightforward computation involving the Koszul formula shows that the Levi Civita connection associated to the Riemannian metric g is given, for all $i, j = 1, .., n$, by

$$\begin{aligned}
\nabla_{e_0}\omega^0 &= 0,\\
\nabla_{e_0}\omega^i &= \frac{1}{2}\sum_{j=1}^{n}(c_i^j - c_j^i)\,\omega^j,\\
\nabla_{e_i}\omega^0 &= -\frac{1}{2}\sum_{j=1}^{n}(c_i^j + c_j^i)\,\omega^j,\\
\nabla_{e_i}\omega^j &= \frac{1}{2}(c_i^j + c_j^i)\,\omega^0
\end{aligned} \tag{7}$$

where we have denoted $\alpha e_i = \sum_{j=1}^{n} c_i^j e_j$. The Riemann curvature tensor R takes the form

$$R = \sum_{i,k=1}^{n} A_{ik}\,\omega^0 \wedge \omega^i \otimes \omega^0 \wedge \omega^k + \sum_{i,j,k,l=1}^{n} B_{ijkl}\,\omega^i \wedge \omega^j \otimes \omega^k \wedge \omega^l \tag{8}$$

for some constants A_{ij} and B_{ijkl}.

The Lie group $\mathbf{R} \ltimes_\alpha \mathbf{R}^n$, equipped with the left-invariant metric g, is a homogeneous Riemannian manifold and as a consequence it is analytic. Moreover, it follows immediately from (7) that $\nabla_{e_0} e_0 = 0$, implying that integral curves of the vector field e_0 are geodesics. Finally, each term in (8) contains an even number of ω^0 components, while (7) implies that, for $i = 1, .., n$,

$$\begin{aligned}
\nabla\omega^0 &= -\frac{1}{2}\sum_{i,j=1}^{n}(c_i^j + c_j^i)\,\omega^i \otimes \omega^j,\\
\nabla\omega^i &= \frac{1}{2}\sum_{j=1}^{n}(c_i^j - c_j^i)\,\omega^0 \otimes \omega^j + (c_j^i + c_i^j)\,\omega^j \otimes \omega^0.
\end{aligned}$$

As a consequence, each term in $\nabla^{2k}R$ contains an even number of ω^0, while each term in $\nabla^{2k+1}R$ contains an odd number of ω^0. This implies that the infinite list of curvature conditions given in Theorem 2 is satisfied if we substitute $\xi = e_0$, and applying Theorem 2, we conclude that the manifold $\mathbf{R} \ltimes_\alpha \mathbf{R}^n$ admits a locally flow-symmetric structure given by the vector field e_0.

It follows from (7) that the vector field e_0 is a Killing vector field if and only if

$$c_i^j + c_j^i = 0,$$

that is, the linear mapping α is skew-symmetric. Hence, if α is not skew-symmetric, the examples constructed here are neither (locally) φ-symmetric Sasakian manifolds nor (locally) KTS-spaces. To show that the manifold cannot be a strongly φ-symmetric contact metric manifold with the integral curves of e_0 as characteristic curves, we remark that (2) implies that, for a contact metric structure (ξ, η, φ, g), we have

$$g([X, \varphi X], \xi) = 2$$

for any X orthogonal to ξ, while (6) implies that, for any two vectors X and Y orthogonal to e_0, $[X, Y] = 0$.

3.2. Warped products $\mathbf{R} \times_f M$

Consider a (strictly positive) function $f : \mathbf{R} \to \mathbf{R}$ and denote by $N = \mathbf{R} \times_f M$ the warped product of the one-dimensional manifold $\mathbf{R}$ (equipped with its standard Riemannian metric dt^2) and the n-dimensional Riemannian manifold (M, g), that is, the manifold $\mathbf{R} \times M$ equipped with the Riemannian metric

$$h = \pi_1^\star(dt^2) + (f \circ \pi_1)^2 \pi_2^\star(g)$$

where π_1 (resp. π_2) is the projection on $\mathbf{R}$ (resp. M). In what follows, we will determine necessary and sufficient conditions on the manifold (M, g) for this warped product manifold $\mathbf{R} \times_f M$ to be locally flow-symmetric with respect to the non-vanishing vector field $\xi = \frac{\partial}{\partial t}$ tangent to the fibers $\mathbf{R} \times \{m\}$, $m \in M$.

To do this, we consider an arbitrary point $p = (t_0, m) \in N$ and we construct a normal coordinate system $x = (x^1, \ldots, x^n)$ on M, centered at m. Let $\gamma : \mathbf{R} \to N : s \mapsto \gamma(s) = (t(s), x(s))$ be a unit speed geodesic in N which is *horizontal*, that is, orthogonal to ξ, at the point p. Then the local reflection with respect to the integral curve σ of ξ through p sends the point $\gamma(s)$ to the point $\gamma(-s)$. We now compute a coordinate expression for this local reflection.

If we denote by $\dot{}$ the derivative with respect to s and by $'$ the derivative with respect to t, the geodesic equation for γ can be split into its components tangent to $\mathbf{R}$ and M (see, for example, [15]), yielding

$$\left(\ddot{t} - \frac{f'}{f}\left(1 - \dot{t}^2\right)\right)\frac{\partial}{\partial t} + \left(\nabla^M_{\dot{x}}\dot{x} + 2\dot{t}\,\frac{f'}{f}\dot{x}\right) = 0. \tag{9}$$

It follows from (9) that the function $t(s) = \pi_1(\gamma(s))$ satisfies the differential equation

$$\ddot{t} = \frac{f'(t)}{f(t)}\left(1 - \dot{t}^2\right),$$

together with the initial conditions $t(0) = t_0$ and $\dot{t}(0) = 0$. This differential equation, and all its solutions, are independent of the coordinates $(x^1(s), \ldots, x^n(s))$. Hence, if $\gamma_1(s) = (t(s), x(s))$ and $\gamma_2(s) = (\bar{t}(s), \bar{x}(s))$ are two unit speed geodesics in N which are horizontal at the point $p = \gamma_1(0) = \gamma_2(0)$, then $t(s) = \bar{t}(s)$ for all values of s. Choosing $\dot{\gamma}_2(0) = -\dot{\gamma}_1(0)$, this implies that, for any geodesic γ which is horizontal at the point $p = \gamma(0)$, we have $t(s) = t(-s)$, that is, t is an *even* function of s and the local reflection with respect to the integral curve of ξ through the point $p = (t_0, m)$ maps every submanifold $\{t\} \times M \subset N$ into itself.

It also follows from (9) (see also [15]) that the projection $\tilde{\gamma} = \pi_2 \circ \gamma$ of the geodesic γ on the manifold M is a *pre-geodesic*, that is, a (unit speed) reparametrization of this curve is a geodesic in (M, g). Hence, the points $\tilde{\gamma}(s)$, $\tilde{\gamma}(0)$ and $\tilde{\gamma}(-s)$ lie on the same geodesic. Moreover, the velocity vector field of the curve $\tilde{\gamma}$ is given by $\dot{\tilde{\gamma}} = \pi_{2\star}\dot{\gamma}$ which has length

$$v = \sqrt{g(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}})} = \sqrt{(\pi_2^\star g)(\dot{\gamma}, \dot{\gamma})} = \frac{\sqrt{h(\dot{\gamma}, \dot{\gamma}) - (\pi_1^\star(dt^2))(\dot{\gamma}, \dot{\gamma})}}{f(t)} = \frac{\sqrt{1 - \dot{t}^2}}{f(t)}.$$

This is an even function of the parameter s, implying that the distance from $\tilde{\gamma}(s)$ to $\tilde{\gamma}(0)$ equals the distance from $\tilde{\gamma}(-s)$ to $\tilde{\gamma}(0)$. As $x = (x^1, \ldots, x^n)$ is a normal coordinate system centered at $\tilde{\gamma}(0)$ it now follows that, for all $s \in \mathbf{R}$,

$$x(\tilde{\gamma}(-s)) = -x(\tilde{\gamma}(s)).$$

We conclude that, with respect to the coordinate systems (t, x) constructed as above, the local reflections with respect to the integral curves of the vector field $\xi = \frac{\partial}{\partial t}$ are given by

$$\psi_\sigma : (t, x) \mapsto (t, -x).$$

We are now ready to prove

Theorem 4. *The warped product manifold $\mathbf{R} \times_f M$ is locally flow-symmetric if and only if (M, g) is a locally symmetric Riemannian manifold.*

Proof. Let $m \in M$ be a point in M and denote by σ the integral curve of the vector field ξ passing through m and by (t, x) the coordinate system constructed as above. With respect to such a coordinate system, the warped product metric h is given by

$$h = dt^2 + f^2(t) \sum_{i,j=1}^{n} g_{ij}(x) dx^i \otimes dx^j,$$

and it follows from the above coordinate expression that the local reflection with respect to σ is a local isometry if and only if

$$g(x) = g(-x)$$

for all $x \in M$, that is, the local reflection (in M) with respect to the point $m \in M$ is a local isometry. By applying this result at all points $m \in M$, the required result now follows immediately. $\square$

Remark 1. It is important to remark here that (choosing suitable functions f) the warped product spaces constructed here provide the first examples of locally flow-symmetric Riemannian manifolds which are not locally homogeneous.

Remark 2. We also point out that, for each $t \in \mathbf{R}$, the fiber $\{t\} \times M$ is a symmetric submanifold in the sense of Ferus and Strübing (see [9], [17]), and that the manifold $N = \mathbf{R} \times_f M$ is therefore foliated by such symmetric submanifolds.

The coordinate expression for the local reflections derived above clearly holds for any warped product $\mathbf{R} \times_f M$. As a consequence, an argument similar to the one used in Theorem 4 yields the following

Theorem 5. *Let $N = \mathbf{R} \times_f M$ be the warped product of the one-dimensional manifold $\mathbf{R}$ with a Riemannian manifold (M, g). Then the local reflections with respect to the integral curves of ξ are volume-preserving (up to sign) if and only if the manifold (M, g) is a D'Atri space, that is, the local reflections with respect to points in M are volume-preserving (up to sign).*

Remark 3. It is important to note that the proof of Theorem 4 does not require the analyticity of the manifold (M, g) or the warping function f to prove the local flow-symmetry of the warped product manifold $N = \mathbf{R} \times_f M$. As a consequence, taking a warping function f which is not analytic, the construction above yields the first examples of locally flow-symmetric spaces which are not necessarily analytic.

We remark that we may give an alternative proof for Theorem 4 which makes use of Theorem 2, and which therefore requires the analyticity of the warping function f. In the rest of this subsection, we sketch this alternative proof, which, in the next subsection, will be generalized to the case of doubly warped products.

Let $N = \mathbf{R} \times_f M$ be the warped product of $\mathbf{R}$ with the Riemannian manifold (M, g), let $\{E_1, \ldots, E_n\}$ be an orthonormal frame field along (M, g) and denote by $E_0 = \frac{\partial}{\partial t}$ the unit vector field tangent to $\mathbf{R}$. (In the rest of this subsection we assume that all roman indices $i, j, k, \ldots$ range from 1 to n, while greek indices $\alpha, \beta, \ldots$ range from 0 to n. To simplify the notations, we shall also denote $\nabla_{E_\alpha} E_\beta$ by $\nabla_\alpha \beta$ and write $(\nabla^k_{E_{\alpha_1} \ldots E_{\alpha_k}} R)_{E_\alpha E_\beta E_\gamma E_\delta}$ as $(\nabla^k_{\alpha_1 \ldots \alpha_k} R)_{\alpha\beta\gamma\delta}$.)

With respect to the frame field $\{E_0, E_1, \ldots, E_n\}$ on N, the Levi Civita connection ∇ of (N, h) is then given by

$$\begin{aligned} \nabla_{E_0} E_0 &= 0, \\ \nabla_{E_0} E_i &= \nabla_{E_i} E_0 = \frac{f'}{f} E_i, \\ \nabla_{E_i} E_j &= \nabla^M_{E_i} E_j - f f' g_{ij} E_0 \end{aligned} \tag{10}$$

while the Riemann curvature tensor R is completely determined by the components

$$\begin{aligned} R_{0i0j} &= -f'' f g_{ij}, \\ R_{0ijk} &= 0, \\ R_{ijkl} &= f^2 R^M_{ijkl} - f^2 f'^2 E^M_{ijkl} \end{aligned} \tag{11}$$

where $E^M_{ijkl} = g_{ik} g_{jl} - g_{il} g_{jk}$.

A straightforward computation using (10) and (11) now shows that

$$(\nabla_i R)_{jklm} = f^2 (\nabla^M_i R^M)_{jklm}.$$

If (N,h) is locally flow-symmetric, the local reflections with respect to the integral curves of ξ are local isometries, implying that

$$(\nabla_i R)_{jklm} = (\nabla_i^M R^M)_{jklm} = 0,$$

and hence, (M,g) is locally symmetric.

To prove the converse, let us assume that (M,g) is locally symmetric. Then M is analytic and hence, if the warping function f is analytic, the warped product $N = \mathbf{R} \times_f M$ will be analytic too. To prove that N is locally flow-symmetric, it suffices to prove that (N,h) satisfies the infinite list of curvature conditions given in Theorem 2. To this aim, we prove the following

Lemma 6. *With the notations introduced above we have, for all $k \in \mathbf{N}$ and at all points $p \in M$,*

$$(\nabla^k_{\alpha_1 \dots \alpha_k} R)_{\alpha\beta\gamma\delta}(p) = 0 \tag{12}$$

whenever an odd number of the indices $\alpha_1, \dots, \alpha_k, \alpha, \beta, \gamma, \delta$ are different from zero. If an even number of these indices differ from zero, $(\nabla^k_{\alpha_1 \dots \alpha_k} R)_{\alpha\beta\gamma\delta}$ can be written as a sum of products of the form $F(t)\, T^M$, where F is a function of t and T^M is some component of a tensor field on (M,g) which is parallel with respect to the connection ∇^M.

Proof. It follows immediately from (11) and the local symmetry of the manifold (M,g) that the result holds for $k = 0$.

We now proceed to prove the lemma by induction on the number k. By definition, at the point $p \in M$, $\nabla^{k+1} R$ takes the form

$$\begin{aligned}(\nabla^{k+1}_{\mu\alpha_1 \dots \alpha_k} R)_{\alpha\beta\gamma\delta}(p) =& (\nabla_\mu (\nabla^k_{\alpha_1 \dots \alpha_k} R)_{\alpha\beta\gamma\delta})(p) \\ & - (\nabla^k_{\nabla_\mu \alpha_1 \dots \alpha_k} R)_{\alpha\beta\gamma\delta}(p) - \dots \\ & - (\nabla^k_{\alpha_1 \dots \alpha_k} R)_{\alpha\beta\gamma\nabla_\mu\delta}(p).\end{aligned} \tag{13}$$

First, suppose that an odd number of the indices $\mu, \alpha_1, \dots, \alpha_k, \alpha, \beta, \gamma, \delta$ differ from zero. If $\mu = 0$, it is easily seen from (10) that each term in (13) contains an odd number of indices different from zero and the induction hypothesis then implies that the expression vanishes. If $\mu \neq 0$, a similar argument (but now specifying the frame field $\{E_1, \dots, E_n\}$ in such a way that $\nabla^M_{E_i} E_j(p) = 0$) shows that all terms except $(\nabla_\mu (\nabla^k_{\alpha_1 \dots \alpha_k} R)_{\alpha\beta\gamma\delta})(p)$ contain an odd number of indices different from zero, and hence (by induction) all these terms vanish. Again by the induction hypothesis, we know that $(\nabla^k_{\alpha_1 \dots \alpha_k} R)_{\alpha\beta\gamma\delta}$ is a sum of products of the form $F(t)\, T^M$, and by our choice of frame field and the fact that T^M is a component of some parallel tensor field, we see that

$$(\nabla_\mu (F(t)\, T^M))(p) = 0.$$

We conclude that the first part of the lemma holds.

If an even number of indices differ from zero, a similar induction argument shows that the second part of the lemma also holds. □

It is now easily seen from Lemma 6 that the warped product $N = \mathbf{R} \times_f M$ of the real line with a locally symmetric manifold (M, g) satisfies the infinite list of conditions given in Theorem 2, yielding the required result.

3.3. Doubly warped products $\mathbf{R} \times_f M_1 \times_g M_2$

Given two strictly positive functions $f, g : \mathbf{R} \to \mathbf{R}_0^+$ and two arbitrary Riemannian manifolds (M_1, g_1) and (M_2, g_2), we define a Riemannian metric h on the product manifold $\mathbf{R} \times M_1 \times M_2$ by putting

$$h = \pi_0^\star dt^2 + (f \circ \pi_0)^2 \pi_1^\star g_1 + (g \circ \pi_0)^2 \pi_2^\star g_2.$$

(For the sake of simplifying the notation, we omit, in the rest of this section, the projections π_i on the different components of the product.)

To construct an orthonormal frame field for this manifold, we start by choosing a unit vector field T tangent to $\mathbf{R}$ and an orthonormal frame field $\{E_1, \ldots, E_{n_1}\}$ (resp. $\{F_1, \ldots, F_{n_2}\}$) on M_1 (resp. M_2) and we denote the orthonormal coframe fields associated to these frame fields by τ, $\{\omega^1, \ldots, \omega^{n_1}\}$ and $\{\sigma^1, \ldots, \sigma^{n_2}\}$. Then an orthonormal coframe field on the doubly warped product manifold $\mathbf{R} \times_f M_1 \times_g M_2$ is given by

$$\{\tau, f\omega^1, \ldots, f\omega^{n_1}, g\sigma^1, \ldots, g\sigma^{n_2}\}.$$

A straightforward computation then yields that, for all $i = 1, \ldots n_1$ and all $j = 1, \ldots, n_2$, we have

$$\begin{aligned}
\nabla\tau &= ff'g_1 + gg'g_2,\\
\nabla\omega^i &= -\frac{f'}{f}(\tau \otimes \omega^i + \omega^i \otimes \tau) + \nabla^{M_1}\omega^i,\\
\nabla\sigma^j &= -\frac{g'}{g}(\tau \otimes \sigma^j + \sigma^j \otimes \tau) + \nabla^{M_2}\omega^j,
\end{aligned}$$

while the Riemann curvature tensor R takes the form

$$\begin{aligned}
R =& A\sum_i \tau \wedge \omega^i \otimes \tau \wedge \omega^i + B\sum_j \tau \wedge \sigma^j \otimes \tau \wedge \sigma^j\\
&+ CR^{M_1} + D\sum_{i,j} \omega^i \wedge \omega^j \otimes \omega^i \wedge \omega^j\\
&+ ER^{M_2} + F\sum_{i,j} \sigma^i \wedge \sigma^j \otimes \sigma^i \wedge \sigma^j\\
&+ G\sum_{i,j} \omega^i \wedge \sigma^j \otimes \omega^i \wedge \sigma^j
\end{aligned}$$

where $A, B, \ldots, G$ are functions of t.

By an induction argument similar to the one used for warped products in Subsection 3.2, it can now be shown that the manifold $\mathbf{R} \times_f M_1 \times_g M_2$ satisfies all the curvature conditions in Theorem 2 if and only if the manifolds (M_1, g_1) and (M_2, g_2) are locally symmetric. If (M_1, g_1) and (M_2, g_2) are locally symmetric and the functions f and g are analytic, the doubly warped product manifold is analytic and Theorem 2 implies the following

Theorem 7. *A necessary condition for the doubly warped product manifold* $\mathbf{R}\times_f M_1 \times_g M_2$ *to be locally flow-symmetric is that the Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2) *are locally symmetric Riemannian manifolds. If the functions* f *and* g *are analytic, this necessary condition is also sufficient.*

3.4. Three-dimensional Lie groups

Let (G, g) be a three-dimensional Lie group equipped with a left-invariant Riemannian metric g. First, if the Lie group G is unimodular, it follows from Milnor's classification [12] that we can always choose an orthonormal frame field $\{e_1, e_2, e_3\}$ consisting of left-invariant vector fields such that, with respect to these vector fields, the Lie bracket is given by

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2.$$

If we denote by $\{\omega^1, \omega^2, \omega^3\}$ the orthonormal coframe field dual to the frame field $\{E_1, E_2, E_3\}$, a straightforward computation shows that

$$\begin{aligned} \nabla\omega^1 &= -\mu_2\omega^2 \otimes \omega^3 + \mu_3\omega^3 \otimes \omega^2, \\ \nabla\omega^2 &= \mu_1\omega^1 \otimes \omega^3 - \mu_3\omega^3 \otimes \omega^3, \\ \nabla\omega^3 &= -\mu_1\omega^1 \otimes \omega^2 + \mu_2\omega^2 \otimes \omega^1 \end{aligned} \tag{14}$$

where $\mu_i = \frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_i)$, showing that the three left-invariant vector fields e_1, e_2 and e_3 are geodesic vector fields. Furthermore, it follows from (14) that the Riemann curvature tensor R is given by

$$\begin{aligned} R = -\,4\,\{&(\mu_1\mu_2 - \lambda_3\mu_3)\omega^1 \wedge \omega^2 \otimes \omega^1 \wedge \omega^2 \\ &+ (\mu_2\mu_3 - \lambda_1\mu_1)\omega^2 \wedge \omega^3 \otimes \omega^2 \wedge \omega^3 \\ &+(\mu_3\mu_1 - \lambda_2\mu_2)\omega^3 \wedge \omega^1 \otimes \omega^3 \wedge \omega^1\}, \end{aligned} \tag{15}$$

and it is easily seen from (15) that the left-invariant vector fields e_1, e_2 and e_3 are eigenvector fields of the Ricci operator, making them suitable candidates for locally flow-symmetric structures by Corollary 3.

We remark that the Lie group G, equipped with the left-invariant Riemannian metric g, is (locally) homogeneous and hence analytic. Moreover, it follows from (14) that every term in $\nabla\omega^i$ contains an even number of ω^i, while $\nabla\omega^j$ contains an odd number of ω^i if $i \neq j$, and we see from (15) that every term contained in the Riemann tensor consists of an even number of ω^i for all $i = 1, 2, 3$. Using an argument similar to the one used in Subsection 3.1, we find that each term in $\nabla^{2k}R$ contains an even number of ω^i for all $i = 1, 2, 3$, while each term in $\nabla^{2k+1}R$ contains an odd number. Hence, the conditions of

Theorem 2 are satisfied if we replace ξ by any of the vector fields e_1, e_2, e_3, and we conclude that the Lie group (G, g) is locally flow-symmetric with respect to the three left-invariant vector fields e_1, e_2, e_3.

In the case where the Lie group G is non-unimodular, again following [12], we can choose an orthonormal frame field consisting of left-invariant vector fields e_1, e_2, e_3 such that the Lie bracket is given by

$$[e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3, \quad [e_2, e_3] = 0$$

where $\alpha+\delta \neq 0$, which is a special case of the semi-direct products treated in Subsection 3.1.

Remark 4. It was shown by D. Perrone [16] that some of the three-dimensional Lie groups described here admit a homogeneous contact metric structure. In [5], the present authors have shown that, in the unimodular case, this contact metric structure is strongly φ-symmetric while, in the non-unimodular case, it is strongly φ-symmetric [5] only in the special case where the Milnor isomorphism invariant D (see [12] for the definition) is equal to zero.

4. Two-dimensional locally flow-symmetric spaces

The aim of this last section is to present a complete classification of all two-dimensional locally flow-symmetric manifolds (M, g). To do this, assume that (M, g) is a two-dimensional Riemannian manifold which is locally flow-symmetric with respect to the (unit) vector field ξ. Then, at least locally, we can choose a coordinate system (x, y) on M such that $\xi = \frac{\partial}{\partial x}$. With respect to such a coordinate system, the metric g takes the form

$$g = \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$$

where

$$\omega^1 = dx + A(x, y)dy, \quad \omega^2 = B(x, y)dy \tag{16}$$

form an orthonormal coframe field dual to the orthonormal frame field

$$E_1 = \xi = \frac{\partial}{\partial x}, \quad E_2 = \frac{1}{B(x, y)}(\frac{\partial}{\partial y} - A(x, y)\frac{\partial}{\partial x}).$$

The Levi Civita connection associated to the Riemannian metric g is given by

$$\begin{aligned}
\nabla_{E_1} E_1 &= \frac{1}{B}\frac{\partial A}{\partial x} E_2, \\
\nabla_{E_1} E_2 &= -\frac{1}{B}\frac{\partial A}{\partial x} E_1, \\
\nabla_{E_2} E_1 &= \frac{1}{B}\frac{\partial B}{\partial x} E_2, \\
\nabla_{E_2} E_2 &= -\frac{1}{B}\frac{\partial B}{\partial x} E_1.
\end{aligned}$$

Expressing the fact that the unit vector field ξ is geodesic yields that $\frac{\partial A}{\partial x} = 0$ and, if necessary, choosing new coordinates of the form

$$x' = x + \int A(y)dy, \quad y' = y,$$

we can always assume that $A = 0$. A straightforward computation then shows that

$$\begin{aligned} R_{E_1E_2E_1E_2} &= -\frac{1}{B}\frac{\partial^2 B}{\partial x^2}, \\ (\nabla_{E_2}R)_{E_1E_2E_1E_2} &= -\frac{1}{B}\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial^2 B}{\partial x^2}\right), \\ (\nabla^3_{E_2E_2E_2}R)_{E_1E_2E_1E_2} &= -\frac{1}{B}\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{1}{B}\frac{\partial^2 B}{\partial x^2}\right). \end{aligned} \tag{17}$$

If the local reflections with respect to the integral curves of ξ are local isometries, Proposition 1 and (17) imply

$$\frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial^2 B}{\partial x^2}\right) = \frac{\partial}{\partial y}\left(\frac{1}{B}\frac{\partial B}{\partial x}\right)\frac{\partial}{\partial x}\left(\frac{1}{B}\frac{\partial^2 B}{\partial x^2}\right) = 0. \tag{18}$$

Integrating the first equation in (18), we obtain

$$\frac{1}{B}\frac{\partial^2 B}{\partial x^2} = \alpha(x)$$

and hence,

$$B(x, y) = a_1(y)b_1(x) + a_2(y)b_2(x) \tag{19}$$

where b_1 and b_2 are two independent solutions of the ordinary second order differential equation $z''(x) = \alpha(x)z(x)$. Inserting (19) in the second part of (18), we get

$$W(a_1(y), a_2(y))W(b_1(x), b_2(x))\alpha'(x) = 0$$

where $W(f(x), g(x)) = f'(x)g(x) - g'(x)f(x)$ denotes the Wronskian of the functions f and g. As b_1 and b_2 are independent solutions of the differential equation $z'' = \alpha z$, $W(b_1(x), b_2(x)) \neq 0$ everywhere and hence

$$W(a_1(y), a_2(y))\alpha'(x) = 0$$

everywhere, which immediately implies that either $W(a_1, a_2) = 0$ everywhere or $\alpha'(x) = 0$ everywhere. First, if $\alpha'(x) = 0$, it follows from (17) that the manifold has constant sectional curvature α and it was shown in [21] that in this case the local reflections with respect to all geodesics are local isometries. If $W(a_1, a_2) = 0$ everywhere, then $a_2(y)$ is a constant multiple of $a_1(y)$, implying that

$$B(x, y) = a(y)b(x)$$

and, choosing a new coordinate system

$$x' = x, \quad y' = \int a(y)dy$$

if necessary, we can always make sure that $B = b(x)$. It then follows from (16) that the manifold is a warped product of the form $\mathbf{R} \times_f \mathbf{R}$. Conversely, as was shown in the previous section, every manifold of this type is locally flow-symmetric.

So, noting that every two-dimensional manifold of constant curvature is locally isometric to a warped product $\mathbf{R} \times_f \mathbf{R}$, we obtain the following

Theorem 8. *A two-dimensional Riemannian manifold is locally flow-symmetric if and only if it is locally isometric to a warped product manifold* $\mathbf{R} \times_f \mathbf{R}$ *for some function* $f : \mathbf{R} \to \mathbf{R}$.

References

[1] Blair, D.: *Contact manifolds in Riemannian geometry.* Lecture Notes in Math. **509**, Springer-Verlag, Berlin-Heidelberg-New York 1976.

[2] Blair, D.; Vanhecke, L.: *Symmetries and φ-symmetric spaces.* Tôhoku Math. J. **39** (1987), 373–383.

[3] Boeckx, E.: *A class of locally φ-symmetric contact metric spaces.* Preprint 1997.

[4] Boeckx, E.; Vanhecke, L.: *Characteristic reflections on unit tangent sphere bundles.* Houston J. Math. **23** (1997), 427–448.

[5] Boeckx, E.; Bueken, P.; Vanhecke, L.: *φ-symmetric contact metric spaces.* Preprint 1997.

[6] Bueken, P.; Vanhecke, L.: *Reflections in K-contact geometry.* Math. Rep. Toyama Univ. **12** (1989), 41–49.

[7] Calvaruso, G.; Perrone, D.; Vanhecke, L.: *Homogeneity on three-dimensional contact Riemannian manifolds.* Preprint 1997.

[8] Chen, B. Y.; Vanhecke, L.: *Isometric, holomorphic and symplectic reflections.* Geom. Dedicata **29** (1989), 259–277.

[9] Ferus, D.: *Symmetric submanifolds of Euclidean space.* Math. Ann. **247** (1980), 81–93.

[10] González-Dávila, J. C.; González-Dávila, M. C.; Vanhecke, L.: *Reflections and isometric flows.* Kyungpook Math. J. **35** (1995), 113–144.

[11] Jiménez, J.; Kowalski, O.: *The classification of φ-symmetric Sasakian manifolds.* Monatsh. Math. **115** (1992), 83–98.

[12] Milnor, J.: *Curvature of left-invariant metrics on Lie groups.* Adv. in Math. **21** (1976), 293–329.

[13] Narita, F.: *Riemannian submersions and isometric reflections with respect to submanifolds.* Math. J. Toyama Univ. **15** (1992), 83–94.

[14] Narita, F.: *Riemannian submersion with isometric reflections with respect to the fibers.* Kodai Math. J. **16** (1993), 416–427.

[15] O'Neill, B.: *Semi-Riemannian geometry with applications to relativity.* Academic Press, New York 1983.

[16] Perrone, D.: *Homogeneous contact Riemannian three-manifolds.* Illinois J. Math., to appear.

[17] Strübing, W.: *Symmetric submanifolds of Riemannian manifolds.* Math. Ann. **245** (1979), 37–44.

[18] Takahashi, T.: *Sasakian φ-symmetric spaces.* Tôhoku Math. J. **29** (1977), 91–113.

[19] Tondeur, Ph.; Vanhecke, L.: *Transversally symmetric Riemannian foliations.* Tôhoku Math. J. **42** (1990), 307–317.

[20] Vanhecke, L.: *Geometry in normal and tubular neighborhoods.* Rend. Sem. Fac. Sci. Univ. Cagliari, Supplemento al **58** (1988), 73–176.

[21] Vanhecke, L.; Willmore, T. J.: *Interaction of spheres and tubes.* Math. Ann. **163** (1983), 31–42.

Received May 5, 1998