Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 1, 97-112.

Compositions of Quadratic Forms with Quadratic, Bilinear or Hermitian Forms

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1. Introduction

In this paper we construct compositions of quadratic spaces with quadratic, symplectic or hermitian spaces by using properties of the Clifford-Algebra. Over fields of characteristic not equal to 2, we know by the Radon-Hurwitz formula that there exists a composition $(M, q) \times (E, p) \to (E, p)$ for some quadratic spaces M of dimension m and E of dimension l if and only if $m \leq \rho(m)$ where ρ is the Radon-Hurwitz function (see e.g. [3], p.133ff). We want to generalize this result by considering arbitrary rings (we do not suppose $\frac{1}{2} \in R$), but also by replacing the quadratic space (E, p) by a symplectic space or by a hermitian space over a quadratic or quaternion algebra.

Throughout this paper we fix a commutative ring R. In order to simplify notations, we assume that R is connected, so the rank of a projective module is well-defined (in the general case we decompose R into connected components). All bilinear and hermitian forms are supposed to be nonsingular.

A quadratic space of rank m is a pair (M, q), where M is a projective R-module of rank m and q is a regular quadratic form on M if m is even, and a semiregular quadratic form (in the sense of Kneser ([4]; see also [1], p.207ff) if m is odd. Let $C(M, q)$ (resp. $C_0(M, q)$) denote the Clifford algebra (resp. the even Clifford algebra) of the quadratic space (M, q). We call the involution on $C(M, q)$ which extends the identity on M the *canonical involution* and the involution which extends the map $x \mapsto -x$ on M the *standard involution.*

If b is a bilinear form on a projective module E and $g \in \mathrm{End}_R(E)$, there is a homomorphism $\overline{g}$ such that $b(g(x), y) = b(x, \overline{g}(y))$ holds for all x, y in E. If b is symmetric or skew-symmetric, the map $g \mapsto \overline{g}$ is an involution, which we call the adjoint of b. Conversely, let σ be an involution on $\mathrm{End}_R(E)$. By the Skolem-Noether Theorem, σ is the adjoint of a bilinear form b_σ on E with values in an invertible module. We say that σ is of orthogonal (resp. symplectic) type if b_σ is symmetric (resp. skew-symmetric). An orthogonal (resp. symplectic) involution σ is even if there is a bilinear form on S^m such that $b_\sigma(x, y) = c(x, y) + c(y, x)$

(resp. $b_\sigma(x,y) = c(x,y) - c(y,x)$). If τ is an involution on an arbitrary Azumaya algebra A, and if S is a faithfully flat R-algebra such that $A \otimes S \simeq M_m(R)$, then we call τ orthogonal (resp. symplectic, resp. even) if $\sigma \otimes 1_S$ is orthogonal (resp. symplectic, resp. even).

2. Hyperbolic Spaces

Let P be a finitely generated projective module of rank m, and let $P^* = \mathrm{Hom}_R(P, R)$. We call $(M, q) = (P \oplus P^*, ((x, f) \mapsto f(x))$ a hyperbolic space. The Clifford-Algebra $C(M, q)$ of a hyperbolic space is given by the following

Theorem 2.1. (a) *There is an isomorphism*

$$\alpha : C(q) := C(P \oplus P^*, q) \to \mathrm{End}_R(\bigwedge P),$$

where $\bigwedge P$ denotes the exterior algebra of P.
(b) *Let τ be the canonical involution of $C(M, q)$. Then we have*

$$\alpha\tau\alpha^{-1}(g) = u^{-1}(g^* \otimes 1_I)u,$$

where $I = \bigwedge^m(P)$ is invertible and $u : \bigwedge P \to \mathrm{Hom}((\bigwedge P), I)$ is the adjoint of the bilinear form b given by

$$b(x,y) = \begin{cases} \sigma(x)y & (r+s=m) \\ 0 & (r+s \neq m) \end{cases},$$

where $x \in \bigwedge^r P$, $y \in \bigwedge^s P$ and σ is the involution of $\bigwedge P$ which extends the identity of P.
(c) *Let τ' be the standard involution of $C(q)$. Then we have*

$$\alpha\tau\alpha^{-1}(g) = u'^{-1}(g^* \otimes 1)u',$$

where $u' : \bigwedge P \to \mathrm{Hom}((\bigwedge P), I)$, is the adjoint of the bilinear form given by

$$b'(x,y) = \begin{cases} \sigma'(x)y & (r+s=m) \\ 0 & (r+s \neq m) \end{cases},$$

where $x \in \bigwedge^r P$, $y \in \bigwedge^s P$ and σ' denotes the involution of $\bigwedge P$ which extends the map $P \to P$, $x \mapsto -x$.

(d)
$$b(x,y) = \begin{cases} b(y,x) & (m \equiv 0,1 \pmod 4) \\ -b(y,x) & (m \equiv 2,3 \pmod 4) \end{cases}.$$

(e)
$$b'(x,y) = \begin{cases} b'(y,x) & (m \equiv 0,3 \pmod 4) \\ -b'(y,x) & (m \equiv 1,2 \pmod 4) \end{cases}.$$

Proof. See [5]. □

We will show now that the form b or b' in the above theorem is not only symmetric (resp. skew-symmetric), but that b is the polar of a quadratic form on $\bigwedge P$ (resp. is an alternating form) even if 2 is not a unit in R.

Lemma 2.2. *Let $\bigwedge P$ be generated by $e_1, \dots, e_l$ where $e_i = x_{i_1} \wedge \cdots \wedge x_{i_{r_i}}$. Let b be the bilinear form b or b' in the above theorem and $\mu_{ij} := b(e_i, e_j)$. For $x = \lambda_1 e_1 + \cdots + \lambda_l e_l$ we consider $b(x,x)$ as a polynomial in $\mathbf{Z}[\lambda_1, \dots, \lambda_l, \mu_{12}, \dots]$.*

(a) *If b is symmetric, there exists $g \in \mathbf{Z}[\lambda_1, \dots, \lambda_l, \mu_{12}, \dots]$ such that $b(x,x) = 2g(\lambda_1, \dots, \lambda_l, \mu_{12}, \dots)$.*

(b) *If b is skew-symmetric, then we have $b(x,x) = 0$ $\forall x \in \bigwedge P$.*

Proof. We have $b(e_i, e_i) = 0$. Thus $b(x,x) = \sum_{i,j} \lambda_i \lambda_j b(e_i, e_j) = \sum_i \lambda_i^2 b(e_i, e_i) + \sum_{i \neq j} \lambda_i \lambda_j b(e_i, e_j) = 2 \sum_{i<j} \lambda_i \lambda_j \mu_{ij}$. Now (a) follows if we set

$$g(\lambda_1, \dots, \mu_{12}, \dots) := \sum_{i<j} \lambda_i \lambda_j \mu_{ij},$$

and (b) follows immediately from $b(e_i, e_j) = -b(e_j, e_i)$. □

Theorem 2.3. *Let (M,q) be a quadratic space of rank $2m$ such that there is an isomorphism $\alpha : C(M,q) \to \mathrm{End}_R(E)$. Let τ denote the canonical or standard involution of $C(M,q)$. Then $\alpha\tau\alpha^{-1} = \tau_c$ where τ_c is the adjoint of a bilinear form on E which is either alternating or the polar of a quadratic form p. Moreover, in the second case, if a linear map $g : E \to E$ satisfies $c(x,y) = c(g(x), g(y)) \cdot a$ for some $a \in R$, we have $p(x) = p(g(x)) \cdot a$.*

Proof. For (M,q) hyperbolic, the assertions are in 2.1 and 2.2. In the general case, there exists a splitting

$$\beta : (M,q) \otimes S \simeq H(P)$$

(S a commutative, faithfully flat R-algebra) and an isometry

$$(E \otimes S, c \otimes S) \to (\textstyle\bigwedge P, b).$$

where b denotes the form b or b' of 2.1. The form b is even. It follows by descent that c is even, which proves the first statement. The second follows from the same reasons and the fact that polynomial rings over $\mathbf{Z}$ are integral domains. □

A variation of this proof shows that the theorem remains correct if there is an isomorphism $\alpha : C(M,q) \otimes Q \to \mathrm{End}_R(E)$ where Q is a quaternion algebra (i.e. an Azumaya-algebra of rank 4) and the involution is $\tau \otimes \sigma$ where σ is the standard involution of Q).

We consider now quadratic spaces of odd rank. We begin with a special case:

Lemma 2.4. *Let P a projective module of rank m. Let q' be the quadratic form $q'(v,f,\mu) = f(v) + a\mu^2$ on $M = P \oplus P^* \oplus R$ ($a \in R^\bullet$). Let τ be the standard involution auf $C_0(M,q)$. Then the following holds:*

(a) *There is an isomorphism $\alpha' : C_0(q) \to \mathrm{End}_R(\bigwedge P)$,*

(b) *$\alpha'\tau\alpha'^{-1}(g) = u'^{-1}(x^* \otimes 1)u'$, where u' is the homomorphism defined in 2.1 (c).*

Proof. The lemma follows immediately from 2.1 and the fact that $C(U,p)$ and $C_0(U \oplus R, p \perp \langle a \rangle)$ are isomorphic as algebras with involutions for every quadratic space (U,p) if $C(U,p)$ carries the standard involution. □

In the same way as 2.3 we can prove:

Theorem 2.5. *Let (M, q) be a quadratic space of rank $2m+1$ such that there is an isomorphism $\alpha : C_0(M, q) \to \mathrm{End}_R(E)$. Let τ denote the standard involution of $C_0(M, q)$. Then $\alpha\tau\alpha^{-1} = \tau_c$, where τ_c is the adjoint of a bilinear form on E which is either alternating or the polar of a quadratic form p. Moreover, in the second case, if a linear map $g : E \to E$ satisfies $c(x, y) = c(g(x), g(y)) \cdot a$ for some $a \in R$, we have $p(x) = p(g(x)) \cdot a$.*

3. Compositions with quadratic forms

Let (M, q) and (E, p) be quadratic spaces. A *composition* of (M, q) with (E, p) is a linear map $\phi : M \times E \to E$ such that the following holds:

$$p(\phi(y)) = p(y)q(x)\ \forall x \in M, y \in E.$$

We shall write $\phi : (M, q) \times (E, p) \to (E, p)$ for such a composition. Of course, more general definitions are possible, e.g. compositions $(M, q) \times (E, p) \to (E', p')$. We allow (E, p) to take values in some invertible module I (i.e. p is a map $E \to I$ with the usual properties of a quadratic form), but we always suppose (M, q) to be R-valued. By the type of a composition we mean the triple (rank(M), rank(E), rank(E)). We often shall write ϕ_x for the map $y \mapsto \phi(x, y)$.

Let m be a positive integer. We will denote by $\pi_+(m)$ the following function: let $m = 8a + b$ where a, b are integers with $0 \leq b < 8$. Then we have

b	0	1	2	3	4	5	6	7
$\pi_+(m)$	2^{4a-1}	2^{4a}	2^{4a+1}	2^{4a+2}	2^{4a+2}	2^{4a+3}	2^{4a+3}	2^{4a+3}

If R is a field of characteristic not equal to 2, it is known that $\pi_+(m)$ is the smallest integer l such that there exists a composition of type (m, l, l) (see [3], p.133ff). We will show in the following that over an arbitrary ring this result is still true except for obvious counterexamples.

We will consider the algebra $C(M, q)$ resp. $C_0(M, q)$ as an element of the Brauer group of R. We will write $[A] = 1$ if the Azumaya algebra A is Brauer-equivalent to R, i.e. if $A \simeq \mathrm{End}_R(E)$ for some projective R-module E, and $[A] = Q$ will mean that A is equivalent to a quaternion algebra Q. Moreover, $Z(B)$ will denote the center of the algebra B.

Theorem 3.1. *Let (M, q) be a quadratic space of rank m. We suppose that (M, q) is pointed, i.e. that there exists an element $z \in M$ with $q(z) = 1$, and that the following conditions are satisfied:*

(a) *If $m \equiv 0 \pmod 8$ then $[C(M, q)] = 1$ in $\mathrm{Br}(R)$ and $Z(C_0(M, q)) \simeq R \times R$.*
(b) *If $m \equiv \pm 1 \pmod 8$ then $[C_0(M, q)] = 1$ in $\mathrm{Br}(R)$.*
(c) *If $m \equiv \pm 2 \pmod 8$ then $[C_0(M, q)] = 1$ in $\mathrm{Br}(Z)$ where $Z = Z(C_0(M, q))$.*
(d) *If $m \equiv \pm 3 \pmod 8$ then $[C(M, q)] = Q$ in $\mathrm{Br}(R)$ for a quaternion algebra Q.*
(e) *If $m \equiv 4 \pmod 8$ then $[C(M, q)] = Q$ in $\mathrm{Br}(R)$ for a quaternion algebra Q, and $Z(C_0(M, q)) \simeq R \times R$.*

Then (M, q) admits a composition of type $(m, \pi_+(m), \pi_+(m))$.

Proof. (a) Let $\alpha : C(M,q) \to \mathrm{End}_R(E)$ be an isomorphism, and let e be a nontrivial idempotent of $Z(C_0(M,q))$. We define

$$E_1 = (\alpha(e))(E),\ E_2 = (\alpha(1-e))(E).$$

It is easy to show that $E = E_1 \oplus E_2$, $\alpha(C_0(M,q)) = \mathrm{End}_R(E_1) \times \mathrm{End}_R(E_2)$ and $\mathrm{rank}(E_1) = \mathrm{rank}(E_2) = 2^{\frac{m}{2}-1} = \pi_+(m)$. Since $m \equiv 0 \pmod 8$, the standard involution of $C(M,q)$ is of orthogonal type. Thus we have

$$\alpha\tau\alpha^{-1}(g) = u^{-1}(g^* \otimes 1)u \quad \forall g \in \mathrm{End}_R(E)$$

for an isomorphism $u : E \to \mathrm{Hom}_R(E,I)$, where I is an invertible R-module, and we have $u = u^* \otimes 1_I$. Thus u is the adjoint of a symmetric bilinear form on E which is, by 2.3, polar of a quadratic form p. We can show that the decomposition $E = E_1 \to E_2$ is orthogonal with respect to p if we use the fact that the standard involution of $C_0(M,q)$ induces the identity on $Z(C_0(M,q))$. Now set

$$\phi(x,y) = \alpha(zx(y))\ (x \in M, y \in E_1).$$

We show that ϕ is a composition:

$$\begin{aligned}
(u(\phi(x,y)))(\phi(x,y')) &= (u\alpha(zx)y)(\alpha(zx)y') \\
&= (\alpha(zx)^* \otimes 1_I)(u\alpha(zx)y)(y') \\
&= (u\alpha(\tau\alpha^{-1}(\alpha(zx)))\alpha(zx)y)(y') \\
&= (u\alpha(\tau(x)\tau(z)zx)y)(y') \\
&= (u\alpha(xzzx)y)(y') \\
&= (u(y))(y')q(z)q(x)
\end{aligned}$$

By 2.3, ϕ is a composition $(M,q) \times (E_1,p) \to (E_1,p)$. The type of the composition is $(m, \pi_+(m), \pi_+(m))$.

(b) Let $\alpha : C_0(M,q) \to \mathrm{End}_R(E)$ be an isomorphism, and let $u : E \to \mathrm{Hom}(E,I)$ be as in (a). Let p denote the polar of the adjoint bilinear form of u. Then we can prove as in (a) that $\phi : M \times E \to E$, $\phi(x,y) = \alpha(zx(y))$ is a composition of the wanted type.

(c) Let $\beta : C_0(M,q) \to \mathrm{End}_Z(E)$ be an isomorphism. The homomorphism $\gamma : M \to \mathrm{End}_R(E \oplus E), \gamma(x)(y,y') = (\beta(xz)(y'), \beta(zx)(y))$ can (by the universal property of the Clifford algebra) be extended to a homomorphism of algebras

$$\gamma : C(M,q) \to \mathrm{End}_R(E \oplus E).$$

Now we define

$$Q = \{g \in \mathrm{End}_R(E \oplus E');\ ag = ga \ \ \forall a \in \gamma(C(M,q))\}.$$

By well-known results on Azumaya algebras, Q is a quaternion algebra and there exists a homomorphism

$$\alpha : C(M,q) \otimes Q \to \mathrm{End}_R(E \oplus E').$$

It is defined by $\alpha(c \otimes g) = \gamma(c)g$. The checker-board grading of $\mathrm{End}_R(E \oplus E)$ induces, by an easy calculation, via α a grading on $C(M,q)$ which coincides with the usual one, and a nontrivial grading on Q. Therefore we have an isomorphism

$$\alpha : C_0(M,q) \otimes Q_0 \oplus C_1(M,q) \otimes Q_1 \to \mathrm{End}_R(E) \oplus \mathrm{End}_R(E').$$

Let τ be the standard involution of $C(M,q)$ and σ the standard involution of Q. Since both involutions are even and of symplectic type, $\alpha(\tau \otimes \sigma)\alpha^{-1}$ is an even orthogonal involution on $E \oplus E$. Thus the corresponding bilinear form is the polar of a quadratic form p with values in some invertible module I (as in (a)). We will show now that the decomposition $E \oplus E$ is orthogonal with respect to p. Let e be the inverse image under α of the first projection of $E \oplus E$. Then we have $e \in Z(C_0(M,q) \otimes Q_0 \oplus C_1(M,q) \otimes Q_1) \subset C_0(M,q) \otimes Q_0$. Using the fact that the restriction of τ on $Z(C_0(M,q))$ is the standard involution of $Z(C_0(M,q))$, we get $(\tau \otimes \sigma)(e) = e$. Now let be y_i in the ith component of $(E \oplus E)$. Then we have

$$\begin{aligned} u(y_1)(y_2) &= u((\alpha(e))(t))(\alpha(1-e))(t')) \ (t, t' \in E \oplus E) \\ &= (\alpha(1-e))^* u((\alpha(e))(t))(t') \\ &= u(\alpha(\tau \otimes \sigma)(1-e))(\alpha(e))(t)(t') \\ &= u\alpha((1-e)e)(t)(t') = 0. \end{aligned}$$

Thus p is a nonsingular quadratic form on E. As in (a), we have a composition $(M,q) \times (E,p) \to (E,p)$ defined by $\phi(x,y) = \beta(zx)(y)$.

(d) Let $\alpha : C_0(M,q) \otimes Q \to \mathrm{End}_R(E)$ be an isomorphism. Let τ be the standard involution of $C_0(M,q)$ and σ the standard involution of Q. Then $\tau \otimes \sigma$ is an even orthogonal involution. Now we can continue as in (a).

(e) is proved by the same arguments as in (a) and (d). □

We now want to prove the converse of the above theorem. We use the following construction: If $\phi : (M,q) \times (E,p) \to (E,p)$ is a composition, the map

$$\alpha : M \to \mathrm{End}_R(E \oplus E), \ \ \alpha(x)(y,y') = (u^{-1}\phi_x^* u(y'), \phi_x(y)),$$

where $\phi_x(y) = \phi(x,y)$ and u is the adjoint of the polar of p, can be extended by the universal property of Clifford algebras to an algebra homomorphism

$$\alpha : C(M,q) \to \mathrm{End}_R(E \oplus E)$$

whose restriction yields a homomorphism

$$\alpha : C_0(M,q) \to \mathrm{End}_R(E).$$

This homomorphism preserves the involution if $C(M,q)$ carries the canonical (resp. standard) involution and $\mathrm{End}_R(E \oplus E)$ the involution induced by $\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}$ (resp. $\begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$.

Theorem 3.2. *For a quadratic space (M, q) of rank m, the conditions of Theorem 3.1 (without the existence of z with $q(z) = 1$) are necessary for the existence of a composition of type $(m, \pi_+(m), \pi_+(m))$ except if 2 is a zero divisor in R and $m \equiv 4 \pmod 8$.*

Proof. (a) The map α defined above is a homomorphism of Azumaya algebras of the same rank. Therefore it is an isomorphism. We have $\alpha(C_0(M, q)) = \mathrm{End}_R(E) \times \mathrm{End}_R(E)$. The restriction on the center yields $Z(C_0(M, q)) \simeq R \times R$.

(b) The map $\alpha : C_0(M, q) \to \mathrm{End}_R(E)$ defined above is a homomorphism of Azumaya algebras of the same rank and therefore an isomorphism.

(c) The map $\alpha : C_0(M, q) \to \mathrm{End}_R(E)$ is a homomorphism of Azumaya algebras. We have a Z-module structure on E via α and therefore a Z-algebra homomorphism $\alpha : C_0(M, q) \to \mathrm{End}_Z(E)$. Since both algebras have the same rank and $C_0(M, q)$ is Azumaya over Z, it suffices to show that the Z-module E is faithful. Now suppose that R is local, and let $\{1, z\}$ be a basis of Z which satisfies $z^2 = z + r,\ 1 + 4r \in R^\bullet$ and $\overline{z} = 1 - z$ ($z \mapsto \overline{z}$ denotes the standard involution of Z) (see [1]). Since the standard involution of $C_0(M, q)$ induces the standard involution on Z, we have $\beta(z) = 1 - z$ and therefore $\beta(\overline{sz + t}) = \beta(-sz + s + t) = 0$. This implies $\beta(s + 2t) = 0$ and $s + 2t = 0$.

Thus we get $\beta(sz + t) = \beta(t(-2z + 1)) = 0$ and $\beta(t(-2z + 1)^2) = \beta(t)\beta(4r + 1) = 0$. This equality implies $t = 0,\ sz + t = 0$. If R is arbitrary, the statement is true since it's true in every localization.

(d) The map $\alpha : C_0(M, q) \to \mathrm{End}_R(E)$ defined above is a homomorphism of Azumaya algebras. We define

$$Q = \{g \in \mathrm{End}_R(E);\ gc = cg \ \ \forall c \in \beta(C_0(M, q))\}.$$

Then Q is a quaternion algebra and there is a homomorphism

$$\beta : C_0(M, q) \otimes Q \to \mathrm{End}_R(E).$$

The Azumaya algebras $C_0(M, q) \otimes Q$ and $\mathrm{End}_R(E)$ are of the same rank and therefore β is an isomorphism.

(e) The existence of an isomorphism $\alpha : C(M, q) \otimes Q \to \mathrm{End}_R(E \oplus E))$ is proved as in (d). The checker-board grading on $\mathrm{End}_R(E \oplus E)$ induces via α a grading of $C(M, q) \otimes Q$, and we can show that it induces also a grading of Q. It is not difficult to show that the involution ρ on $\mathrm{End}_R(E \oplus E)$ induces the standard involution τ on $C(M, q)$, which is of symplectic type, and an involution σ on Q which is symplectic since ρ is orthogonal and $\rho = \alpha(\tau \otimes \sigma)\alpha^{-1}$. Thus σ is the standard involution of Q, and it induces the standard involution on $Z(Q_0)$. We have

$$Z(C_0(M, q) \otimes Q_0 \oplus C_1(M, q) \otimes Q_1) \simeq R \times R.$$

Using the fact that τ induces the identity on $Z = Z(C_0(M, q))$ we get

$$\begin{aligned} Z(C_0(M, q) \otimes Q_0 \oplus C_1(M, q) \otimes Q_1) &\subset \{z \in Z \otimes Z(Q_0);\ (\tau \otimes \sigma)(z) = z\} \\ &= \{z \in Z \otimes Z(Q_0);\ (1 \otimes \sigma)(z) = z\} \\ &= Z \end{aligned}$$

(The last equality holds because 2 is not a zero divisor). Since $\operatorname{rank}(Z(C_0(M,q))) = 2$ we conclude $Z(C_0(M,q)) \simeq R \times R$. □

Note that in the case $m \equiv 4 \pmod 8$ the condition that 2 is not a zero divisor cannot be dropped. However the statement of Theorem 3.2 holds for $m = 4$, see [1]. Now we show that the compositions constructed in 3.1 are the best possible:

Theorem 3.3. *Let m be a positive integer. The smallest number l such that there exists a composition with quadratic spaces of type (m, l, l) is exactly $\pi_+(m)$ except in the cases $m \equiv 3, 4, 5 \pmod 8$ and $2 = 0$ in R. In these cases we have $l = \frac{\pi_+(m)}{2}$.*

Proof. If $m \equiv 0$ (resp. 1,7, resp. 2,6) (mod 8), the theorem follows by considering the injective homomorphism of Azumaya algebras (see 3.2) $C(M,q) \to \operatorname{End}_R(E \oplus E)$ (resp. $C_0(M,q) \to \operatorname{End}_R(E)$, resp. $C_0(M,q) \to \operatorname{End}_Z(E)$) and comparing the ranks. If $m \equiv 4$ (resp. 3,5) and $2 \neq 0$ this homomorphism cannot be surjective because the standard involution of $C(M,q)$ (resp. $C_0(M,q)$) and the involution on $\operatorname{End}_R(E \oplus E)$ (resp. $\operatorname{End}_R(E)$) are not of the same type. For the exceptional cases, see Theorem 4.1. □

4. Compositions of quadratic forms with alternating forms

Let (M,q) be a quadratic space and (E,b) a symplectic space. A *composition* of (M,q) with (E,b) is a linear map $\phi : M \times E \to E$ such that the following holds:

$$b(\phi(x,y), \phi(x,y')) = b(y,y')q(x) \quad \forall x \in M, y, y' \in E.$$

We will write $\phi : (M,q) \times (E,b) \to (E,b)$. The bilinear form b may have values in some invertible module I.

Let m be a positive integer. We will denote by $\pi_-(m)$ the following function: let $m = 8a + b$ where a, b are integers with $0 \leq b < 8$. Then we have

b	0	1	2	3	4	5	6	7
$\pi_-(m)$	2^{4a}	2^{4a+1}	2^{4a+1}	2^{4a+1}	2^{4a+1}	2^{4a+2}	2^{4a+3}	2^{4a+4}

If R is a field of characteristic not 2, it is known that $\pi_-(m)$ is the smallest integer l such that there exists a composition with an alternating form of type (m,l,l). This result is still true over an arbitrary ring with the same exceptions as in 3.3. The following results are analogous to those for composition with quadratic forms and have nearly the same proofs as 3.1, 3.2 and 3.3.

Theorem 4.1. *Let (M,q) be a quadratic space of rank m. We suppose that (M,q) ist pointed, i.e. that there exists an element $z \in M$ with $q(z) = 1$, and that the following conditions are satisfied:*

- (a) *If $m \equiv 4 \pmod 8$ then $[C(M,q)] = 1$ in $\operatorname{Br}(R)$ and $Z(C_0(M,q)) \simeq R \times R$.*
- (b) *If $m \equiv \pm 3 \pmod 8$ then $[C_0(M,q)] = 1$ in $\operatorname{Br}(R)$.*
- (c) *If $m \equiv \pm 2 \pmod 8$ then $[C_0(M,q)] = 1$ in $\operatorname{Br}(Z)$ where $Z = Z(C_0(M,q))$.*
- (d) *If $m \equiv \pm 1 \pmod 8$ then $[C(M,q)] = Q$ in $\operatorname{Br}(R)$ for a quaternion algebra Q.*

(e) *If* $m \equiv 0 \pmod 8$ *then* $[C(M,q)] = Q$ *in* $\mathrm{Br}(R)$ *for a quaternion algebra* Q*, and* $Z(C_0(M,q)) \simeq R \times R$.

Then (M,q) *admits a composition of type* $(m, \pi_-(m), \pi_-(m))$ *with an alternating form.*

Theorem 4.2. *For a quadratic space* (M,q) *of rank* m*, the conditions of Theorem 4.1 (without the existence of* z *with* $q(z) = 1$*) are necessary for the existence of a composition of type* $(m, \pi_-(m), \pi_-(m))$ *except if 2 is a zero divisor in* R *and* $m \equiv 0 \pmod 8$.

Theorem 4.3. *Let* m *be a positive integer. The smallest number* l *such that there exists a composition of a quadratic space with a symplectic space of type* (m,l,l) *is exactly* $\pi_-(m)$ *except in the case* $m \equiv 7, 0, 1 \pmod 8$ *and* $2 = 0$ *in* R*. In that case we have* $l = \frac{\pi_-(m)}{2}$.

5. Composition with hermitian forms

We consider hermitian spaces (E,b) over a quadratic R-algebra S. Let (M,q) be a quadratic space. Compositions $(M,q) \times (E,b) \to (E,b)$ are defined as in the preceding section. We suppose that a composition is S-linear in the second variable, i.e. the equality $\phi(x, sy) = s\phi(x,y)$ holds for $x \in M, y \in E$ and $s \in S$. The hermitian form may have values in an invertible S-module.

For a positive integer m, let $\pi_h(m)$ be defined as follows: if m is even we set $\pi_h(m) = 2^{(m-2)/2}$, and if m is odd we set $\pi_h(m) = 2^{(m-1)/2}$. We want to show that $\pi_h(m)$ is the smallest S-rank of a hermitian space (E,b) such that there exist a composition $(M,q)\times(E,b) \to (E,b)$ for some quadratic space (M,q) of rank m.

Theorem 5.1. *Let* (M,q) *be a quadratic space of rank* m*. We suppose that* (M,q) *is pointed. Then the following holds:*

(a) *If* $m \equiv 0 \pmod 4$*,* $[C(M,q)] = Q$ *in* $\mathrm{Br}(R)$*, where* Q *is a quaternion algebra with a nontrivial grading* $Q = Q_0 \oplus Q_1$*, and* $Z(C_0(M,q)) \simeq R \times R$*, then* (M,q) *admits a composition with a hermitian space of rank* $\pi_h(m)$ *over* Q_0.
(b) *If* $m \equiv \pm 1 \pmod 4$ *and* $[C(M,q)] = Q$ *in* $\mathrm{Br}(R)$*, where* Q *is a quaternion algebra with a nontrivial grading* $Q = Q_0 \oplus Q_1$*, then* (M,q) *admits a composition with a hermitian space of rank* $\pi_h(m)$ *over* Q_0.
(c) *If* $m \equiv \pm 2 \pmod 8$ $[C_0(M,q)] = Z$ *in* $\mathrm{Br}(Z)$*, where* $Z = Z(C_0(M,q)$*, then* (M,q) *admits a composition with a hermitian space of rank* $\pi_h(m)$ *over* Z.

Proof. (a) As in Theorem 3.1 (e), we have an isomorphism

$$\alpha : C_0(M,q) \otimes Q \to \mathrm{End}_R(E_1) \times \mathrm{End}_R(E_2).$$

Now let

$$S = Z(C_0(M,q) \otimes Q_0) \oplus (C_1(M,q) \otimes Q_1)).$$

Let ρ be the standard involution of $Z(C_0(M,q))$ and σ that of Q. We have $S = \{s \in Z(C_0(M,q) \otimes Q_0);\ (\rho \otimes \sigma)(s) = s\}$. The map α induces a S-module-structure on $E_1 \oplus E_2$. For $c \in (C_0(M,q) \otimes Q_0) \oplus (C_1(M,q) \otimes Q_1)$ we get immediately

$$\alpha(c) \in \mathrm{End}_{S'}(E).$$

By the theory of quadratic algebras with involution (see e.g. [1], p.129), and using $Z(C_0(M,q)) \simeq R \times R$, we conclude $S \simeq Q_0$ and therefore

$$C_0(M,q) \otimes Q_0 \simeq \mathrm{End}_{Q_0}(E_1) \times \mathrm{End}_{Q_0}(E_2).$$

The standard involution τ on $C_0(M,q)$ is of the first kind, and σ induces the standard involution of Q_0. Thus $\tau \otimes \sigma$ is of the second kind. Therefore $\tau \otimes \sigma$ is the adjoint of a nonsingular hermitian form on $E_1 \oplus E_2$ or, as in Theorem 3.1 (a), on E_1. Now we define the composition $\phi : M \times E_1 \to E_1$ by $\phi(x,y) = (\alpha(zx \otimes 1))(y)$. As in the preceding sections we can prove that ϕ is a composition of the wanted type.

(b) Let $\alpha : C_0(M,q) \otimes Q \to \mathrm{End}_R(E)$ be an isomorphism. Via α, E becomes a Q_0-module of rank $\pi_h(m)$. The standard involution of $C_0(M,q)$ and the standard involution of Q_0 induce an involution of the second kind on $\mathrm{End}_R(E)$ and therefore a nonsingular hermitian form b on E. As in the preceding sections we can show that $\phi : M \times E \to E,\ \phi(x,y) = (\alpha(zx) \otimes 1)(y)$ is a composition of (M,q) with (E,b).

(c) Let $\alpha : C_0(M,q) \to \mathrm{End}_Z(E)$ be an isomorphism. The standard involution of $C_0(M,q)$ is of the second kind and therefore induces a hermitian form on E. Now we continue as in (b). □

Theorem 5.2. *For a quadratic space (M,q) of rank m, the sufficient conditions (a), (b) and (c) of Theorem 5.1 are necessary for the existence of a composition $M \times E \to E$ with a hermitian space over a quadratic algebra S, where rank$(M) = m$ and rank$_S(E) = \pi_h(m)$.*

Proof. (a) As in the bilinear case, the composition induces an isomorphism $\alpha : C(M,q) \otimes Q \to \mathrm{End}_R(E \otimes E)$ for a quaternion algebra Q. Since the composition is S-linear, multiplication with $s \in S$ of E defines an element of $C(M,q) \otimes Q$ which lies in $R \otimes Q$ because of the S-linearity of α. Thus Q contains a copy of S which is denoted by Q_0. Now α induces an isomorphism

$$\alpha : C_0(M,q) \otimes Q_0 \to \mathrm{End}_S(E) \times \mathrm{End}_S(E)$$

Let ρ be the involution on $\mathrm{End}_R(E \oplus E)$ induced by the hermitian form on E. Then $\alpha^{-1}\rho\alpha = \tau \otimes \sigma$, where τ is the standard involution of $C_0(M,q)$. Since ρ is of the second and τ of the first kind, σ is the standard involution of Q_0. Now we set $\pi : E \oplus E \to E \oplus E,\ \pi(y,y') = (y,0)$ and $e = \alpha^{-1}$. We have $\rho(\pi) = \pi$ and therefore $(1 \otimes \sigma)(e) = e$. Thus $e \in Z(C_0(M,q))$, and e is idempotent because π is. Therefore we have $Z(C(M,q)) \simeq R \times R$.

(b) The composition induces an injective homomorphism $\alpha : C_0(M,q) \to \mathrm{End}_R(E)$. Now, if $Q = \{g \in \mathrm{End}_R(E);\ gc = cg\ \forall c \in \alpha(C_0(M,q))\}$, then Q is a quaternion algebra and the first statement is proved. As above one shows that Q contains a copy of S.

(c) The composition induces an injective homomorphism

$$\alpha : C_0(M,q) \to \mathrm{End}_R(E)$$

whose image is $\mathrm{End}_Z(E)$. It is easy to prove that $S \simeq Z$. □

By similar arguments as in the bilinear case, we can prove:

Theorem 5.3. *Let m be a positive integer. The smallest number l such that there exists a composition of a quadratic space of rank m with a hermitian space of S-rank l over a quadratic algebra S is exactly $\pi_h(m)$.*

6. Composition of quadratic forms with hermitian forms over quaternion algebras

In this section we consider compositions with hermitian spaces (E, b) over a quaternion algebra Q. We suppose that E is a left Q-module and the involution on Q is always the standard involution.

In the bilinear case, we used the fact that an involution on $\mathrm{End}_R(E)$ induces a bilinear form on E with values in an invertible module (this is a consequence of a generalized version of the Skolem-Noether theorem). In order to generalize our results for composition with hermitian forms over quaternion algebras (or, more generally, over Azumaya algebras with involution) we need the following

Lemma 6.1. *Let A be an Azumaya algebra with involution σ, and let E be a faithfully projective A-module. If τ is an involution on $\mathrm{End}_A(E)$. Then $\tau(g) = u^{-1}g^*u$ for a semilinear map $u : E \to \mathrm{Hom}_A(E, A \otimes I)$ where I is an invertible R-module. Moreover u is the adjoint of a hermitian oder skew-hermitian form.*

Proof. See [2]. □

Let (E, b) be a symplectic space. The form

$$B(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}) = \begin{pmatrix} b(y, x') & -b(x, x') \\ b(y, y') & -b(x, y') \end{pmatrix}$$

on $E \oplus E$ is a hermitian form over $M_2(R)$. Conversely we can prove by Morita-theory that if (F, B) is a hermitian space over $M_2(R)$ then $F \simeq E \oplus E$ and B can be constructed from an alternating form b as above. Thus there is a 1-1-correspondence between hermitian spaces over $M_2(R)$ and symplectic spaces over R or between compositions with hermitian spaces over $M_2(R)$ and compositions with symplectic spaces over R. If a composition of a quadratic space with a hermitian space over a quaternion algebra Q is given, we choose a faithfully flat commutative R-algebra S such that $Q \otimes S \simeq M_2(S)$. Therefore the properties of such a composition can be taken over from the theory of compositions with alternating forms:

Theorem 6.2. *Let Q be a quaternion algebra. Let (M, q) a pointed quadratic space of rank m which statisfies the following conditions:*

(a) *If $m \equiv 4 \pmod 8$ then $[C(M, q)] = Q$ in $\mathrm{Br}(R)$ and $Z(C_0(M, q)) \simeq R \times R$.*
(b) *If $m \equiv \pm 3 \pmod 8$ then $[C_0(M, q)] = Q$ in $\mathrm{Br}(R)$.*
(c) *If $m \equiv \pm 2 \pmod 8$ then $[C_0(M, q)] = Q \otimes Z$ in $\mathrm{Br}(Z)$ if we denote by Z the algebra $Z(C_0(M, q))$.*
(d) *If $m \equiv \pm 1 \pmod 8$ then $[C(M, q)] = Q \otimes Q'$ in $\mathrm{Br}(R)$ for a quaternion algebra Q'.*
(e) *If $m \equiv 0 \pmod 8$ then $[C(M, q)] = Q \otimes Q'$ in $\mathrm{Br}(R)$ for a quaternion algebra Q, and $Z(C_0(M, q)) \simeq R \times R$.*

Then (M, q) admits a composition with a hermitian space (E, B) over Q, and the rank of E as R-module is $2\pi_-(m)$.

Theorem 6.3. *For a quadratic space (M, q) of rank m, the conditions of Theorem 6.2 (without the existence of z with $q(z) = 1$) are necessary for the existence of a composition of (M, q) with a hermitian space of R-rank $2\pi_-(m)$ over a quaternion algebra except if 2 is a zero divisor in R and $m \equiv 0 \pmod 8$.*

Theorem 6.4. *Let m be a positive integer. The smallest number l such that there exists a composition of a quadratic space of rank m with a hermitian space of rank l (as an R-module) over a quaternion algebra Q is exactly $2\pi_-(m)$ excepted in the cases $m \equiv 7, 0, 1 \pmod 8$ and $2 = 0$ in R. In these cases we have $l = \pi_-(m)$.*

Note that if we take a symmetric bilinear form instead of an alternating one, the construction above yields compositions with skew-hermitian forms. We get analogous results if we use the theorems of Section 3.

These results can be easily generalized for compositions of quadratic spaces with hermitian or skew-hermitian spaces over an Azumaya algebra with involution.

If B is an even hermitian form over a quaternion algebra, then $p(x) = B(x, x)$ is a quadratic form over R. Let $(M, q) \times (E, B) \to (E, B)$ be a composition with a hermitian form over a quaternion algebra. Then we get a composition of quadratic spaces $(M, q) \times (E, p) \to (E, p)$. We want to compare this composition with those constructed in Section 3, which will be denoted $(M, q) \times (E_0, p_0) \to (E_0, p_0)$. Thus we have $\mathrm{rank}(E_0) = \pi_+(m)$ where $m = \mathrm{rank}(M)$. As usual we suppose that (E, p) and (E_0, p_0) have values in an invertible R-module.

Theorem 6.5. *Let $\phi : (M, q) \times (E, B) \to (E, B)$ be a composition of a quadratic space (M, q) of rank m, with a hermitian space (E, B) over a fixed quaternion algebra Q such that the rank of E is minimal. Then*

(a) *If $m \equiv 0$, (resp. 1,7) $\pmod 8$, and if $[C(M, q)] = 1$ (resp. $[C_0(M, q)] = 1$) in $\mathrm{Br}(R)$, we have $(E, p) \simeq (E_0, p_0) \otimes (Q, r)$, where r is a symmetric bilinear form on Q with values in an invertible R-module.*

(b) *If $m \equiv 3, 4, 5 \pmod 8$, then (E, p) is similar to (E_0, p_0).*

Proof. (a) We only give the proof for $m \equiv \pm 1 \pmod 8$. The other case is similar. We have isomorphisms

$$\alpha : C_0(M, q) \to^{\simeq} \mathrm{End}_{\mathrm{End}_R Q} E$$

and, by Theorem 3.2,

$$\beta : C_0(M, q) \to \mathrm{End}_R(E).$$

By Morita-theory we conclude that $E = E_0 \otimes Q$. Let u be the adjoint of B and $\overline{u} = u + u^* : E \to \mathrm{Hom}_R(E, Q \otimes I)$. We have (as in the bilinear case): $u^{-1}(\beta(c \otimes d))^* u = (\tau \otimes \sigma')(c \otimes d)$, and therefore $u^{*-1}(\beta(c \otimes d))^* u^* = (\tau \otimes \sigma')(c \otimes d)$. For $d = 1$ we conclude from these formulas

$$(\alpha(c))^* \overline{u} = \overline{u} \tau(c).$$

As in the proof of Theorem 3.1, this formula can be used to construct a composition $(M, q) \times (E, \overline{B}) \to (E, \overline{B})$ with a bilinear form $\overline{B}$ such that $\overline{u}$ is adjoint of $\overline{B}$. We obviously have

$\overline{B}(x,y) = B(x,y) + B(y,x)$, and $\overline{B}$ is the polar of p. We will show below (see Theorem 7.2) that

$$(E, \overline{B}) = (E_0, b_0) \otimes (Q, r)$$

where b_0 is the polar of p_0 and r is a symmetric bilinear form on Q.

(b) We only give the proof for $m \equiv \pm 3 \pmod 8$. The other case is similar. We have an isomorphism

$$\alpha : C_0(M,q) \to \simeq \mathrm{End}_{\mathrm{End}_R Q} E.$$

If u is the adjoint of B and $\overline{u} = u + u^* : E \to \mathrm{Hom}_R(E, Q \otimes I)$, we prove as in (a) that

$$(\alpha(c \otimes 1))^* \overline{u} = \overline{u}(\tau \otimes \sigma)(c \otimes 1).$$

By an easy calculation, using the fact that the trace tr on Q satisfies $\mathrm{tr}(ed) = \mathrm{tr}(de)$, the formula holds also for $c \otimes d$ $(d \in Q)$. We now conclude as in (a). □

Note that there is no analogous statement in (a) if $[C(M,q)] \neq 1$ in $\mathrm{Br}(R)$, and there is no similar result if $m \equiv 2 \pmod 4$. It can be proved that the symmetric bilinear space (Q,r) in (a) is a tensor product of two symplectic spaces of rank 2.

7. Tensor products of compositions

We fix a quadratic space of rank m which admits a composition with a bilinear space of minimal rank (symmetric or alternating). In this section we compare such a "minimal composition" with a composition of the same quadratic space with an arbitrary bilinear space. In the following, let

$$\phi_0 : (M,q) \times (E_0, b_0) \to (E_0, b_0)$$

denote this minimal composition. Thus b_0 is symmetric and $\mathrm{rank}(E_0) = \pi_+(m)$ if $m \equiv 0, 1, 7 \pmod 8$, and b_0 is skew-symmetric and $\mathrm{rank}(E_0) = \pi_-(m)$ if $m \equiv 3, 4, 5 \pmod 8$. In the case $m = 4k + 2$ the type of b_0 is not determined, but we should exclude this case because there is no analogous results to the following theorem.

Lemma 7.1. *Let $\phi : (M,q) \times (E,b) \to (E,b)$ be a composition. Let m be the rank of M and suppose $m \neq 4k + 2$. Let (E_0, b_0) be as above. Then we have $E \simeq E_0 \otimes F$ for an R-module F.*

Proof. The composition ϕ induces a homomorphism

$$C_0(M,q) \to \mathrm{End}_R(E)$$

Thus E is a $C_0(M,q)$-module. Since (E_0, b_0) admits a composition and the rang of E_0 is minimal, we have by Theorem 3.2 or 4.2

$$C_0(M,q) \simeq \begin{cases} \mathrm{End}_R(E_0) & m = 2k+1 \\ \mathrm{End}_R(E_0) \oplus \mathrm{End}_R(E_0') & m = 4k \end{cases}$$

Therefore E is an $\mathrm{End}_R(E_0)$-module. The statement follows by Morita-theory. □

In the following, for invertible modules I and J, let $B_{I,J}$ denote the bilinear form $I \times J \to I \otimes J$, $(i,j) \mapsto i \otimes j$.

Theorem 7.2. *Let (M,q) a quadratic space of rank m, which satisfies one of the following conditions:*

(1) $\operatorname{rank}(M) = 2k+1$ *and* $\alpha : C_0(M,q) \to \operatorname{End}_R(E_0)$ *is an isomorphism.*
(2) $\operatorname{rank}(M) = 4k$ *and* $\alpha : C(M,q) \to \operatorname{End}_R(E_0)$ *is an isomorphism, and* $Z((C_0(M,q)) \simeq R \times R$.

Let $\phi_0 : (M,q) \times (E_0, b_0) \to (E_0, b_0)$ be a composition where the rank of E_0 is minimal (see above).

(a) *Suppose that $E = E_0 \otimes F$ and $\phi = \phi_0 \otimes 1_F : M \times E \to E$. Let b' be an arbitrary nonsingular bilinear form on E with values in an invertible module I'. Then the following holds: the map ϕ is a composition $(M,q) \times (E,b') \to (E,b')$ if and only if $b' = b_0 \otimes c$ where c is a bilinear form on F (with values in an invertible module.)*
(b) *Let (E,b') be a bilinear space. Then we have the following: There exists a composition $\phi' : (M,q) \times (E,b') \to (E,b')$ if and only if b' is isometric with $b = b_0 \otimes c$ where c is a bilinear form on some R-module F.*

Proof. (a): The "if"-part is obvious. For the proof of the "only if", let $\tilde{c}$ be a nonsingular bilinear form on F with values in an arbitrary invertible module. We put $b = b_0 \otimes \tilde{c}$. The form b has values in an invertible module, say I. Let $u : E \to \operatorname{Hom}(E,I)$ (resp. $u' : E \to \operatorname{Hom}(E,I')$) be the adjoint of b (resp. b'.) Since ϕ is a composition, we have $u(\phi_x)(y)((\phi_x)(y')) = (\phi_x \otimes 1_I)^*(u(\phi_x)(y))(y') = q(x)u(y)(y')$ for all $x \in M$ and $y, y' \in E$, therefore

$$(\phi_x \otimes 1_I)^* u \phi_x = q(x) \cdot u$$

and in the same way

$$(\phi_x \otimes 1_{I'})^* u' \phi_x = q(x) \cdot u'.$$

We define the invertible R-module L by $L = I^* \otimes I'$. We get

$$1_L \otimes ((\phi_x \otimes 1_I)^* u \phi_x u^{-1}) = (\phi_x \otimes 1_{I'})^* u' \phi_x u'^{-1}$$

and after some calculations

$$(\phi_x \otimes 1_L)((u^{-1} \otimes 1_L)u') = ((u^{-1} \otimes 1_L)u')\phi_x \qquad \forall x \in M.$$

Now the set $\{\phi_x; x \in M\}$ generates the algebra $\alpha(C_0(M,q))$. Thus $(u^{-1} \otimes 1_L)u' \in \operatorname{End}_R(E)$ commutes with $\alpha(C_0(M,q))$. In the case $m = 2k+1$ we have $\alpha(C_0(M,q)) = \operatorname{End}_R(E_0)$, therefore $(u^{-1} \otimes 1_L)u'$ commutes with $\operatorname{End}_R(E_0)$. Because of $\operatorname{Hom}(E, E \otimes L) \simeq \operatorname{End}_R(E) \otimes L \simeq \operatorname{End}_R(E_0) \otimes \operatorname{End}_R(F) \otimes L \simeq \operatorname{End}_R(E_0) \otimes \operatorname{Hom}(F, F \otimes L)$ we conclude

$$(u^{-1} \otimes 1_L)u' \in \operatorname{Hom}(F, F \otimes L).$$

In other words, there is a $f \in \operatorname{Hom}(F, F \otimes L)$ such that $u' = uf$. Thus, for $y = y_0 \otimes z,\ y' = y'_0 \otimes z' \in E = E_0 \otimes F$ we have

$$b'(y_0 \otimes z, y'_0 \otimes z') = (b \otimes B_{L,R})(y_0 \otimes f(z), y'_0 \otimes z') = b_0(y_0, y'_0) \otimes (\tilde{c} \otimes B_{L,R})(f(z), z').$$

Therefore, for c we can take

$$c(z,z') = (\tilde{c} \otimes B_{L,R})(f(z),z) \qquad (z,z' \in F).$$

The case $m = 4k$ is similar (we use $\alpha(C_0(M,q)) = \mathrm{End}_R(E_0) \oplus \mathrm{End}_R(E_0'))$.

(b): The "if"-part is clear. Let us prove "only if" in the case $m = 2k+1$. Let

$$\beta' : C_0(M,q) \to \mathrm{End}_R(E)$$

be the homomorphism of algebras with involution induced by ϕ' (see Section 3). We set $D = \{g \in \mathrm{End}_R(E),\ gc = cg\ \forall g \in \beta'(C_0(M,q))\}$. We get an isomorphism

$$\gamma : \beta'(C_0(M,q)) \otimes D \simeq \mathrm{End}_R(E),$$

and, using the theory of Azumaya algebras, we can prove

$$D \simeq \mathrm{End}_R(F')$$

where $E_0 \otimes F' \otimes U \simeq E$ for an invertible module U. We define $F = F' \otimes U$. Then there is an isomorphism

$$\delta : D \to \mathrm{End}_R(F)$$

and an isomorphism

$$\epsilon : \alpha(C_0(M,q)) \otimes \mathrm{End}_R(F) \to \mathrm{End}_R(E)$$

which is induced by $E_0 \otimes F \simeq E$. Now let $\tilde{c}$ be a nonsingular bilinear form on F with values in an invertible module. We define $b = b_0 \otimes \tilde{c}$. The form b has values in an invertible module, say I. Let $u : E \to \mathrm{Hom}(E,I)$ (resp. $u' : E \to \mathrm{Hom}(E,I')$) be the adjont of b (resp. b'). Set

$$\beta : C_0(M,q) \to \mathrm{End}_R(E) \simeq \mathrm{End}_R(E_0) \otimes \mathrm{End}_R(F),\ \beta(a) = \epsilon(\alpha(a) \otimes 1)$$

for $a \in C_0(M,q)$. We consider $C_0(M,q)$ to be equipped with the standard involution τ and $\mathrm{End}_R(E)$ with the involution induced by u. Then β is a homomorphism of algebras with involution, therefore

$$\beta(\tau(a)) = u^{-1}((\beta(a))^* \otimes 1_I)u,\ \beta'(\tau(a)) = u'^{-1}((\beta'(a))^* \otimes 1_I')u'.$$

Now we define

$$\psi : \mathrm{End}_R(E) \to \mathrm{End}_R(E),\ \psi = \gamma(\beta'\alpha^{-1} \otimes \delta^{-1})\epsilon^{-1}.$$

Then, for $a \in C_0(M,q)$, we have

$$\psi\beta(a) = \gamma(\beta'\alpha^{-1} \otimes \delta^{-1})\epsilon^{-1}\beta(a) = \gamma(\beta'\alpha^{-1} \otimes \delta^{-1})(\alpha(a) \otimes 1_F) = \gamma(\beta'(a) \otimes 1_F) = \beta'(a).$$

Therefore, by a generalized version of the Skolem-Noether theorem, there exists an invertible module J and an isomorphism $h : E \otimes J \to E$ such that

$$\psi(g) = h(g \otimes 1_J)h^{-1}.$$

Because $\beta' = \psi\beta = h(\beta \otimes 1_J)h^{-1}, \quad \forall x \in M$, we have

$$h(u^{-1}(\beta(a)^* \otimes 1_I)u \otimes 1_J)h^{-1} = u'^{-1}(h^{*-1}(\beta(a)^* \otimes 1_J^*)h^* \otimes 1_J)u',$$

and by an easy calculation

$$u^{-1}(\beta(a)^* \otimes 1_I)u = (h^{-1}u'^{-1}(h^{*-1} \otimes 1_{I'})) \otimes 1_{J^*}(\beta(a)^* \otimes 1_{J^* \otimes I'})((h^* \otimes 1_{I'})u'h \otimes 1_J.$$

Set

$$u'' = (h^* \otimes 1_{I'})u'h \otimes 1_{J^*} : E \to \mathrm{Hom}_R(E, I' \otimes J^* \otimes J^*).$$

We have

$$u^{-1}(\beta(a)^* \otimes 1_I)u = u''^{-1}(\beta(a)^* \otimes 1_{J^* \otimes J^* \otimes I'})u''$$

and conclude, using that u and u'' are symmetric oder skew-symmetric,

$$(\beta(a)(u^{-1}(u'' \otimes 1_L))) = (u^{-1}(u'' \otimes 1_L))(\beta(a) \otimes 1_L),$$

where L is the invertible module $L = J \otimes J \otimes I'^* \otimes I$. Therefore the homomorphism $u^{-1}(u'' \otimes 1_L) \in \mathrm{Hom}(E \otimes L, E)$ commutes with $\beta(C_0(M,q)) = \mathrm{End}_R(E_0)$. From $E \simeq E_0 \otimes F$ we get that $f = u^{-1}(u'' \otimes 1_L)$ lies in $\mathrm{Hom}(F \otimes L, F)$. Now u'' is the adjoint of a bilinear form b''. For $y = y_0 \otimes z,\ y' = y_0' \otimes z' \in E = E_0 \otimes F$ we have

$$b''(y_0 \otimes z, y_0' \otimes z') = (b \otimes B_{L,R})(y_0 \otimes f(z), y_0' \otimes z') = b_0(y_0, y_0') \otimes (\tilde{c} \otimes B_{L,R})(f(z), z').$$

We now can take for c the bilinear form

$$c(z, z') = (\tilde{c} \otimes B_{L,R}(f(z), z) \qquad (z, z' \in F)$$

since $b'' = b_0 \otimes c$ and b' isometric with b''. This concludes the proof. The case $m = 4k$ is similar (we replace $\mathrm{End}_R(E)$ by $\mathrm{End}_R(E_0) \oplus \mathrm{End}_R(E_0)$ in the first part of the proof). $\square$

Corollary 7.3. *Let (M,q) be a quadratic space of rank $m \neq 4k+2$. Bilinear spaces (E_0, b_0) of minimal rank which admit compositions $\phi_0 : (M,q) \times (E_0, b_0) \to (E_0, b_0)$ are determined up to isometry by (M,q).*

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Received December 1, 1997