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Generalized Hopf Constructions and Eigenmaps between Spheres

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Abstract. In this paper, we give two kinds of generalizations of Hopf constructions, which can be used to produce many quadratic eigenmaps between spheres (Theorem 2.3) including some well-known examples (e.g. Veronese map) and some new ones. We also obtain some non-existence results (Theorems 3.2, 3.3) for eigenmaps into low dimensional spheres. Finally, we give an existence result (Theorem 3.6) for quadratic eigenmaps $S^{n-2} \times S^1 \to S^{n-1}$.

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1. Introduction

A map between spheres $\varphi: S^m \to S^n$ is called an eigenmap with eigenvalue λ if it is a harmonic map satisfying $|d\varphi|^2 \equiv \lambda$. It is well-known that if $\phi: R^{m+1} \to R^{n+1}$ is a map whose components are given by harmonic polynomials of homogeneous degree k and ϕ restricts to a map of spheres $\varphi: S^m \to S^n$ the φ is an eigenmap with eigenvalue $\lambda_k = k(k+m-1)$. In fact, the description of all eigenfunctions of the Laplacian on S^m implies that any eigenmap $\varphi: S^m \to S^n$ is of this form [4]. We say φ is a λ_k-eigenmap induced by ϕ. A map $\varphi: S^m \to S^n$ is said to be *full* if the image of φ is not contained in any proper great sphere of S^n. Clearly, if φ is induced by $\phi: R^{m+1} \to R^{n+1}$, the φ is full if and only if the $n+1$ component functions of ϕ are linearly independent.

The classification of eigenmaps between spheres is an interesting problem in the study of harmonic maps, see R. T. Smith [16], Eells and Lemaire [8,9,10], and the recent works [12], [19]. The problem is still far from being solved except for the following few cases:

(i) For $\varphi: S^2 \to S^n$, we have Calabi's rigidity theorem [5] which asserts that any full λ_k-eigenmap from S^2 must be the standard minimal immersion $f_{\lambda_k}: S^2 \to S^{2k}$.

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(ii) For $\varphi: S^3 \to S^n$, Toth [17] obtains a complete classification of full λ_2-eigenmaps.
(iii) For $\varphi: S^m \to S^2$, Gauchman and Toth [12] give a rigidity theorem saying that any full λ_2-eigenmap is isometric to the classical Hopf fibration $S^3 \to S^2$.

Concerning the construction of λ_2-eigenmaps, the following methods are known:

a) Gradients of cubic eiconals (see e.g. [11, VIII (1.23)]): A k-homogeneous polynomial $F: R^{m+1} \to R$ is an eiconal if its gradient satisfies $|\nabla F(x)|^2 = |x|^{2k-2}$ for all $x \in R^{m+1}$. Clearly $\varphi = \nabla F$ is a $(k-1)$-homogeneous polynomial map $S^m \to S^m$. Furthermore, if F is also harmonic, then φ is an eigenmap. All cubic harmonic eiconals were classified by Cartan [6], and their gradients produce λ_2-eigenmaps $S^m \to S^m$ for $m = 4, 7, 13, 25$.

b) The classical Hopf constructions: Let $\mu: R^p \times R^p \to R^n$ be an orthogonal multiplication (see [11] and [18] for existence of such orthogonal multiplications), i.e. an R-bilinear map satisfying $|\mu(x,y)|^2 = |x|^2|y|^2$, $\forall (x,y) \in R^p \times R^p$. Then the classical Hopf construction on μ is the map defined by $H_\mu: R^p \times R^p \to R^{n+1}$, $H_\mu(x,y) = \big(|x|^2 - |y|^2, 2\mu(x,y)\big)$; this induces an eigenmap $S^{2p-1} \to S^n$. The standard multiplications of the real algebras of real, complex, quaternionic and Cayley numbers are examples of orthogonal multiplications; their Hopf constructions induce, respectively, the well-known Hopf fibrations $S^3 \to S^2, S^7 \to S^4, S^{15} \to S^8$, which are eigenmaps with eigenvalues $\lambda_2 = 8, 16, 32$. The concept of orthogonal multiplications has been generalized, by Baird and the author [3], to multilinear norm-preserving mappings; these can be used to construct some interesting harmonic maps and morphisms. On the other hand, Ding [7] uses the so-called α-Hopf constructions to give many harmonic (not necessarily eigen-) maps between spheres.

c) In [12], a construction of raising the source dimension is given so that for any λ_2-eigenmap $\varphi: S^m \to S^n$ one can construct another $\overline{\varphi}: S^{m+1} \to S^{m+n+2}$; [12] also provides the so-called *harmonic product* construction.

d) Let $\varphi_1: S^{m-1} \to S^{p-1}$ and $\varphi_2: S^{n-1} \to S^{q-1}$ be two λ_2-eigenmaps, and let $\mu: R^m \times R^n \to R^r$ be an orthogonal multiplication. Then it is easily checked that $f(x,y) := \big(\varphi_1(x), \varphi_2(y), \sqrt{2}\mu(x,y)\big)$ induces a λ_2-eigenmap $S^{m+n-1} \to S^{p+q+r-1}$. By using the existence of orthogonal multiplication of the form $\mu: R^2 \times R^n \to R^r$, Ueno [19] is able to construct some new examples of λ_2-eigenmaps between spheres.

2. Generalized Hopf constructions

Let $\mu: R^p \times R^p \to R^n$ be an orthogonal multiplication, and $k \in N$. We introduce the following two kinds of generalized Hopf constructions:

Definition 2.1. i) *The kp-Hopf construction on μ is a map $F_1: R^{kp} \cong \underbrace{R^p \times \cdots \times R^p}_{k} \to R^{\frac{1}{2}k(k-1)(n+1)}$, $(k \geq 2)$, defined by*

$$F_1(x_1, \ldots, x_k) = \frac{1}{\sqrt{k-1}} \Big(|x_i|^2 - |x_j|^2, \sqrt{2k}\mu(x_i, x_j)\Big)_{1 \leq i < j \leq k}. \tag{1}$$

ii) *The 2kp-Hopf construction on μ is a map $F_2: R^{2kp} \cong \underbrace{R^p \times \cdots \times R^p}_{2k} \to R^{k(2k-1)n+k}$, $(k \geq 1)$, defined by*

$$F_2(x_1, \ldots, x_{2k}) = \Big(|x_{2i-1}|^2 - |x_{2i}|^2, 2\mu(x_{2i-1}, x_{2i}), \sqrt{2}\mu(x_t, x_s)\Big)_{\substack{1 \leq i \leq k,\, 1 \leq t < s \leq 2k: \\ (t,s) \neq (2i-1, 2i)}}. \tag{2}$$

Remark 2.2. Clearly, the kp-Hopf construction for $k=2$ and the $2kp$-Hopf construction for $k=1$ are both identical to the classical Hopf construction. When $p=n=2$, $\mu: C\times C\to C$ with $\mu(z,w)=z\overline{w}$, and $k=m+1$ (m as in [12]), then F_1 is the generalized Hopf map mentioned in [12], whilst F_2 is different from the generalized complex Veronese map given in [12].

Theorem 2.3. *Let $\mu: R^p\times R^p\to R^n$ be an orthogonal multiplication. Then the generalized Hopf constructions F_1 and F_2 induce, respectively, the following full λ_2-eigenmaps*

$$f_1: S^{kp-1}\to S^{\frac{k}{2}(k-1)n+k-2}, \tag{3}$$

$$f_2: S^{2kp-1}\to S^{k(2k-1)n+k-1}. \tag{4}$$

Proof. A direct computation shows that $|F_1(x_1,\ldots,x_k)|^2 = \left(|x_1|^2+\cdots+|x_k|^2\right)^2$, $|F_2(x_1,\ldots,x_{2k})|^2=\left(|x_1|^2+\cdots+|x_{2k}|^2\right)^2$. Thus, F_1 and F_2 indeed induce λ_2-eigenmaps between spheres. We need only to check the fullness and the dimensions of the range spheres. Note that the components of orthogonal multiplication $\mu(x_i,x_j)$ are all bilinear functions in x_t and x_s, and it is not difficult to see that those components of $F_1(x_1,\ldots,x_k)$ given by $\left(\sqrt{2k}\mu(x_i,x_j)\right)_{1\le i<j\le k}$ form a linearly independent set consisting of $\frac{1}{2}k(k-1)n$ elements. On the other hand, the set of components, given by $\left(|x_i|^2-|x_j|^2\right)_{1\le i<j\le k}$ contains only $k-1$ linearly independent elements. It is easily checked that no linear combination of these $k-1$ elements can be linearly represented by the above-mentioned linearly independent set. It follows that the maximum linearly independent set of all components of F_1 contains $\frac{1}{2}k(k-1)n+k-1$ elements. Therefore F_1 induces a λ_2-eigenmap f_1. The proof of the statement on F_2 is similar and is omitted. $\square$

Remark 2.4. The classical Hopf constructions can only produce full λ_2-eigenmaps from odd-dimensional spheres whilst the generalized Hopf constructions can produce full λ_2-eigenmaps from both even- and odd-dimensional spheres which include some well-known examples as well as some new ones as the following examples show.

Example 2.5. Let $\mu: R\times R\to R$, $\mu(x,y)=xy$ be the multiplication of real numbers. Then the kp-Hopf construction for $k=3$ is given by

$$F_1(x,y,z)=\frac{1}{\sqrt{2}}\left(x^2-y^2,\, x^2-z^2,\, y^2-z^2, \sqrt{6}xy, \sqrt{6}xz, \sqrt{6}yz\right),$$

which induces a λ_2-eigenmap $f_1: S^2\to S^5$. However, f_1 is not full as one can check that its image is contained in the proper great sphere of S^5 cut out by the hyperplane orthogonal to the vector $\nu=(1,-1,1,0,0,0)\in R^6$. In fact, one can further check that after the orthogonal transformation of coordinates

$$(\varepsilon_1,\ldots,\varepsilon_6)=(e_1,\ldots,e_6)\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & & & \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} & & 0 & \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & & & \\ & & & 1 & 0 & 0 \\ & 0 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{pmatrix},$$

F_1 can be written as $F_1(x,y,z) = \big(0, \frac{1}{2}(x^2+z^2-2y^2), \frac{1}{2}\sqrt{3}(x^2-z^2), \sqrt{3}xy, \sqrt{3}xz, \sqrt{3}yz\big) \in S^4$. This is the well-known Veronese map.

Example 2.6. For $k=2$, the $2kp$-Hopf construction on the real multiplication is given by $F_2\colon R^4 \to R^8$, with $F_2(x_1,x_2,x_3,x_4) = (x_1^2-x_2^2, x_3^2-x_4^2, 2x_1x_2, 2x_3x_4, \sqrt{2}x_1x_3, \sqrt{2}x_1x_4, \sqrt{2}x_2x_3, \sqrt{2}x_2x_4)$, which induces a full λ_2-eigenmap $S^3 \to S^7$.

Example 2.7. For $k=4$, the kp-Hopf construction on the real multiplication is defined by $F_1\colon R^4 \to R^{12}$ with

$$\begin{aligned} F_1(x_1,\dots,x_4) = \tfrac{1}{\sqrt{3}}\big(&x_1^2-x_2^2,\ x_1^2-x_3^2,\ x_1^2-x_4^2,\ x_2^2-x_3^2,\ x_2^2-x_4^2,\ x_3^2-x_4^2,\\ &\sqrt{8}x_1x_2, \sqrt{8}x_1x_3, \sqrt{8}x_1x_4, \sqrt{8}x_2x_3, \sqrt{8}x_2x_4, \sqrt{8}x_3x_4\big) \end{aligned}$$

which induces a λ_2-eigenmap $S^3 \to S^{11}$. Similarly to Example 2.4, one can check that the induced map is not full, and after an orthogonal change of coordinates, F_1 can be written as

$$\begin{aligned} F_1(x_1,\dots,x_4) = \Big(&\tfrac{1}{3}\left(3x_2^2-x_1^2-x_3^2-x_4^2\right), \tfrac{\sqrt{2}}{3}\left(2x_3^2-x_1^2-x_4^2\right), \tfrac{\sqrt{2}}{3}\left(x_1^2-x_4^2\right),\\ &\tfrac{\sqrt{8}}{\sqrt{3}}x_1x_2, \tfrac{\sqrt{8}}{\sqrt{3}}x_1x_3, \tfrac{\sqrt{8}}{\sqrt{3}}x_1x_4, \tfrac{\sqrt{8}}{\sqrt{3}}x_2x_3, \tfrac{\sqrt{8}}{\sqrt{3}}x_2x_4, \tfrac{\sqrt{8}}{\sqrt{3}}x_3x_4, 0,0,0\Big) \in S^8. \end{aligned}$$

Thus F_1, in fact, induces a full λ_2-eigenmap $S^3 \to S^8$, which is isometric to the standard minimal immersion as in the classification of Toth [17].

Corollary 2.8. *For natural numbers m and $k \geq 2$, and the pairs $(p,n) =$ (1,1), (2,2), (3,4), (3,7), (3,8), (3,9), (4,4), (5,8), (6,8), (7,8), (8,8), (9,16), (10,16), (11,20), (11,26), (12,20), (12,26), (13,24), (13,28), (14,24), (14,32), (15,24), (15,32), (16,24), (16,32), (17,32), etc., there exist full λ_2-eigenmaps*

$$\begin{aligned} f_1&\colon S^{kp-1} \to S^{\frac{k}{2}(k-1)n+k-2}, & f_2&\colon S^{2mp-1} \to S^{m(2m-1)n+m-1},\\ f_3&\colon S^{kp} \to S^{kp+\frac{1}{2}k(k-1)n+k-1}, & f_4&\colon S^{2mp} \to S^{m[2p+(2m-1)n+1]}, \end{aligned}$$

and λ_3-eigenmaps

$$\begin{aligned} &S^{kp-1} \to S^{\frac{1}{2}k(k-1)(nk+2)p-1}, & &S^{kp} \to S^{k(kp+1)[p+1+\frac{1}{2}(k-1)n]-1},\\ &S^{2mp-1} \to S^{2m^2p[(2m-1)n+1]-1}, & &S^{2mp} \to S^{(2mp+1)\{2mp+1+m[(2m-1)n+1]\}-1}. \end{aligned}$$

Proof. f_1 and f_2 are obtained by using Theorem 2.3 and the fact that for the above pairs of (p,n) values, there exist (see [1], [14]) full orthogonal multiplications of the form $R^p \times R^p \to R^n$, whilst f_3 and f_4 are given by raising the source dimensions (see [12]) of f_1 and f_2. Finally the degree raising method [12] applies to give λ_3-eigenmaps. $\square$

Remark 2.9. 1) Let $p = n = 2$. The generalized Hopf constructions provide full λ_2-eigenmaps $S^7 \to S^{14}$, $S^7 \to S^{13}$, which cannot be obtained by the methods of [12] or [19].

2) In [17], Toth shows that quadratic harmonic maps $S^3 \to S^n$ exist if and only if $2 \le n \le 8$ and $n \ne 3$, and that for such n, there is just one class of quadratic harmonic maps up to suitable equivalence. By appropriate choice of p, n and k, the generalized Hopf constructions produce representatives for four of the six classes, namely $n = 2, 4, 7, 8$.

By using Corollary 2 in [15] we immediately obtain the following

Corollary 2.10. *For any pairs of eigenmaps of the same degree constructed by Corollary* 2.8 *there exists an equivariant harmonic join, e.g.* $f_1 * f_2 \colon S^{3kp-1} \to S^{\frac{1}{2}k(5k-3)n+2(k-1)}$.

3. Some existence and non-existence results

In this section, the following lemma will be used for several times.

Lemma 3.1. *If* $m \ge 2n$, *then every polynomial map* $S^m \to S^n$ *is a constant map.*

Proof. See Eells and Ratto [11, Corollary VIII: 3.16]. □

Theorem 3.2. *For* $m \ge 8$, $n \le 7$, *there exists no full eigenmap* $S^m \to S^n$.

Proof. First note that a constant map is never full and that the composition of two full maps is again full. Thus, it follows from Lemma 3.1 that there exists no full polynomial map $S^m \to S^n$ $(m \ge 8,\ n = 1, 2, 3, 4)$. The proof of the theorem is completed by the following

Claim. *There exists no full polynomial map* $S^m \to S^n$ $(m \ge 8,\ n = 7, 6, 5)$.

Proof of Claim. By using the multiplication of quaternions and its restrictions we know that there exist full orthogonal multiplications $R^4 \times R^4 \to R^4$, $R^3 \times R^4 \to R^4$, $R^3 \times R^3 \to R^4$, on which the Hopf constructions induce full quadratic (not necessarily harmonic) maps: $S^7 \to S^4$, $S^6 \to S^4$, $S^5 \to S^4$ respectively. If there were a full polynomial map $S^m \to S^n$ $(m \ge 8,\ n = 7, 6, 5)$, then, by composing it with one of the above three full quadratic maps, there would be a full (and hence non-constant) polynomial map $S^m \to S^4$ $(m \ge 8)$, which is impossible by Lemma 3.1. This contradiction gives the proof of the claim. □

In contrast to the rigidity theorem [12] for full λ_2-eigenmaps $S^m \to S^2$, we have

Theorem 3.3. *There exists no full* λ_2*-eigenmap* $S^m \to S^3$ *for any* $m \in N$.

Proof. It follows from Lemma 3.1 that the possible domain dimensions of a full λ_2-eigenmap $\varphi \colon S^m \to S^3$ are $m = 5, 4, 3, 2, 1$. Note that the Hopf fibration $S^3 \to S^2$ is full. Thus, by using the composition and Lemma 3.1 again we can exclude the cases for $m = 5, 4$. On the other hand, it is known [17] that there exists no λ_2-eigenmap $S^3 \to S^3$. The case for

$m = 1$ can be excluded by the fact (see e.g. [12]) that the maximum range dimension for a full λ_2-eigenmap $S^m \to S^n$ is $\frac{m(m+3)}{2} - 1$. Now we are left with the only possible case $m = 2$, which, however, can be excluded by applying Calabi's rigidity theorem [5].

Remark 3.4. In their attempt to list all full λ_2-eigenmaps $S^5 \to S^n$, the authors of [12] and [19] raise the existence question for full λ_2-eigenmaps $S^5 \to S^3$. Theorem 3.3 gives a negative answer to that question.

For higher degree eigenmaps, Eells raised the following questions during the first International Conference on "Harmonic morphisms and the related topics" held in July 1997 in Brest (France):

Problem 1. Is there any λ_k-eigenmap $S^m \to S^n$ for $k \geq 3$ and $m \geq n$?

Problem 2. If $S^m \to S^n$ is a λ_k-eigenmap for $k \geq 3$, and $m \geq n$, then it is a locally trivial fibration. Is it a Hopf fibration?

Now we give an example which gives a positive answer to Problem 1 and a negative answer to Problem 2.

Example 3.5. Let $\phi: S^4 \to S^4$ be the gradient map of the first cubic harmonic eiconal $F: R^5 \to R$ with

$$F(x) = \tfrac{1}{3}x_1^3 - x_1x_2^2 + \tfrac{1}{2}x_1\left(x_3^2 + x_4^2 - 2x_5^2\right) + \tfrac{\sqrt{3}}{2}x_2\left(x_3^2 - x_4^2\right) + \sqrt{3}x_3x_4x_5,$$

and $\varphi: S^7 \to S^4$ be the Hopf fibration. It is known that ϕ is a λ_2-eigenmap and φ is a harmonic morphism (for definition of this interesting special harmonic maps see e.g. [10, (2.30)]) which is also a λ_2-eigenmap. By using the composition law [10, (2.31)] we see that $\phi \circ \varphi: S^7 \to S^4$ is an eigenmap of degree 4, which is obviously not a Hopf fibration.

Theorem 3.6. *For $n = 2s$, $(s \geq 2)$, there exist full quadratic eigenmaps $S^{n-2} \times S^1 \to S^{n-1}$.*

Proof. It is well-known (see e.g. [11]) that any orthogonal multiplication $R^p \times R^q \to R^n$ restricts to a bi-eigenmap [11, VIII (2.11)] $S^{p-1} \times S^{q-1} \to S^{n-1}$. On the other hand, it follows from [19] that there exists an orthogonal multiplication $R^p \times R^2 \to R^n$ for any even n and any p. Now let $n = 2s$ and $p = n - 1$ we obtain the required results. $\square$

Example 3.7. It can be checked that $\mu: R^7 \times R^2 \to R^8$ with

$$\begin{aligned}\mu(x_3, \ldots, x_9; x_1, x_2) = (&-x_1x_5 + x_2x_7,\ x_1x_6 - x_2x_8,\ x_1x_3 + x_2x_9,\ -x_1x_4, \\ &-x_1x_9 - x_2x_3,\ -x_2x_4,\ x_1x_7 + x_2x_5,\ -x_1x_8 - x_2x_6)\end{aligned}$$

is a full orthogonal multiplication. Thus, it restricts to a full quadratic eigenmap $S^6 \times S^1 \to S^7$.

Remark 3.8. Other related constructions of harmonic maps $S^{n-2} \times S^1 \to S^{n-1}$ have been found by Hsiang and Yu [13] and by Baird and Eells [2].

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