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A Noteworthy Property of the Tridens Curve in the Isotropic Plane

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Abstract. We consider the pencil $\mathbf{P}$ of the cubic parabolas through the vertices of a triangle Δ in the isotropic plane. The locus of the inflection points is a tridens curve $\mathbf{T}$ of third order [4] through the vertices and the side-midpoints of Δ. Every tridens curve has such inscribed triangles Δ. They are formed by the points of contact of the non-isotropic tangents of $\mathbf{T}$ which pass through a point of the asymptote of the tridens curve.

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1. Let $\mathbf{A}$ be the affine plane of a triangle $\Delta := \Delta(A_1A_2A_3)$ and $\mathbf{r}$ a fixed direction not parallel to any one of the sides of Δ. So Δ is an admissible triangle [5] of the isotropic plane $\mathbf{I}_2(\mathbf{r}, \mathbf{R}) \subset \mathbf{A}$.

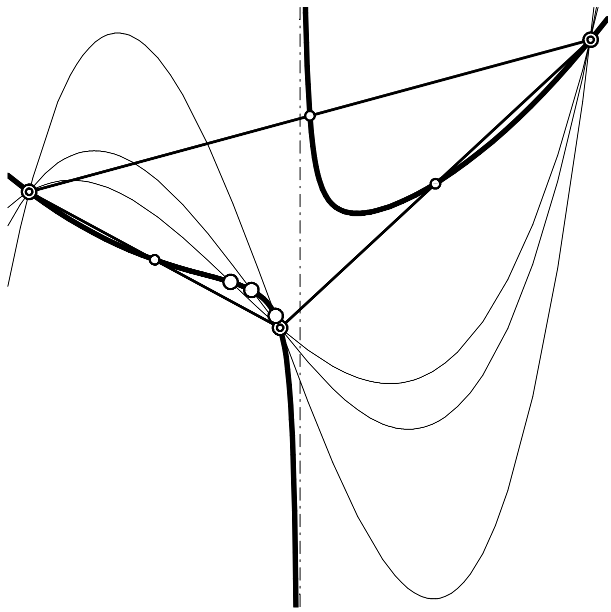
We define an affine ξ, η-coordinate system using two sides of Δ.

$$A_1 : (0,0),\ A_2 : (a,0),\ A_3 : (0,b);\ a, b \in \mathbf{R},\ ab \neq 0.$$

In the homogeneous coordinates (t, ξ, η) let $(0, \lambda, -1)$ be the infinite point assigned to the direction $\mathbf{r}$. So we have $\lambda(a - \lambda b) \neq 0$. We take the three parametric isotropic motion group B_3 as the fundamental group [5], and use the x, y-coordinates

$$x = \xi + \lambda\eta,\ \ y = \eta. \tag{1}$$

Now $t = 0$ represents the absolute line f and (0:0:1) the absolute point F in the coordinate system (1) of the isotropic plane. An algebraic curve of third order for which f is a tangent and F a node (a cusp) is called a *tridens curve* of third order (*cubic parabola*) [4, 7].



For the pencil **P** of the cubic parabolas $p(\mu) \in \mathbf{P}$ through the vertices of Δ we have

$$y = x(x-a)\{\lambda^{-1}(\lambda b - a)^{-1} + \mu(x - \lambda b)\}. \tag{2}$$

Hence

$$x_w(\mu) = \frac{1}{3}\{a + \lambda b - \mu^{-1}\lambda^{-1}(\lambda b - a)^{-1}\} \tag{3}$$

is the abscissa and

$$y_w(\mu) = \frac{1}{9}x_w(\mu)\{\lambda b - 2a - \frac{1}{\mu\lambda(\lambda b - a)}\}\{\frac{2}{\lambda(\lambda b - a)} + \mu(a - 2\lambda b)\} \tag{4}$$

the ordinate of the inflection point of $p(\mu)$.

Theorem 1. *Let Δ denote an admissible triangle of the isotropic plane $I_2(\mathbf{R})$ and* **P** *the pencil of the cubic parabolas through the vertices of Δ. Then the locus of the inflection points of the cubic parabolas of* **P** *is a tridens curve of third order $\mathbf{T}(\boldsymbol{\Delta})$. The vertices and the side midpoints of Δ are points of $\mathbf{T}(\Delta)$. The isotropic asymptote incides with the centroid of Δ. The radius of the asymptotic circle of $\mathbf{T}(\Delta)$ is 2/3 of the radius of the circumscribed circle of Δ.*

We outline the steps of the proof: For the analytic representation of the locus of the inflection points of **P** we get with (3) and (4)

$$y(x - x_s) = \frac{2}{3}\frac{1}{\lambda(\lambda b - a)}x(x-a)(x - \frac{a}{2}) \text{ with } x_s := \frac{1}{3}(a + \lambda b). \tag{5}$$

So this locus is a tridens curve of third order $\mathbf{T}(\Delta)$ (see [4]) passing through the vertices and side midpoints of Δ, x_s in (5) is the abscissa of the centroid S of Δ. Setting $\mu = 0$ in (2) we obtain the equation of the circumscribed circle of Δ. From this and the definition of the asymptotic circle of $\mathbf{T}(\Delta)$ (see [4, p. 39] and also [9]) we obtain the result on the radii. $\square$

Remark 1. Let us suppose that $\mathbf{T}(\Delta)$ is irreducible. This means $x_s(x_s - a)(x_s - a/2) \neq 0$. Furthermore, let $R(\Delta)$ denote the radius of the circumscribed circle of Δ. Then the equations of the tangents $t(A_i)$ of $\mathbf{T}(\Delta)$ in the vertices of Δ are

$$t(A_1) : y = R(\Delta)\frac{-a^2}{\lambda b + a}x, \quad t(A_2) : y = R(\Delta)\frac{-a^2}{\lambda b - 2a}(x - a), \tag{6}$$

$$t(A_3) : y = b - R(\Delta)\frac{a^2 - 3ab\lambda + 3b^2\lambda^2}{a - 2b\lambda}(x - \lambda b).$$

This means: The tangents of $\mathbf{T}(\Delta)$ in the vertices A_1, A_2, A_3 of Δ are copunctual. For this common point $K = (x_K, y_K)$ we have

$$x_K = x_s, \quad y_K = -\frac{1}{3}R(\Delta)a^2, \tag{7}$$

and K is therefore a point of the isotropic asymptote of $\mathbf{T}(\Delta)$. Using the Yaglom-Yzeren point $Y(x_Y, y_Y)$ i.e. the point of contact of the isotropic Feuerbach circle and the inscribed circle of Δ (see [11], [12], and also [10, p. 200]) we deduce that

$$x_Y = x_s, \quad y_Y = \frac{1}{9}R(\Delta)(a - 2b\lambda)^2.$$

Thus we find for the isotropic span σ

$$\sigma(K, Y) = 4\sigma(S, Y) \tag{8}$$

and from this a construction for the point K.

2. The above considerations inspire the investigation of those inscribed triangles $\Delta(P_1P_2P_3)$ of an irreducible tridens curve $\mathbf{T}$ of third order, for which the vertices P_i are the points of contact of the non-isotropic tangents of $\mathbf{T}$ from the point $P_s(x_s, y_s)$ on the isotropic asymptote of $\mathbf{T}$. We start with the following

Lemma. *Let $\mathbf{T}$ denote an irreducible tridens curve of third order of the isotropic plane with the equation*

$$y(x - x_s) = Qx(x - A)(x - B), \text{ where } Qx_s(A - x_s)(B - x_s) \neq 0, \tag{9}$$

$P_0(x_0, y_0) \neq P_1(x_1, y_1)$ two arbitrary points of $\mathbf{T}$, and $\mathbf{K}$ the isotropic circle of a given radius $R \neq Q$ through the points P_0, P_1. Then we have for the abscissa of the third intersection point (x_τ, y_τ) of $\mathbf{T}$ with $\mathbf{K}$

$$x_\tau - x_s = \frac{Q}{Q - R}(x_g - x_s) \tag{10}$$

where (x_g, y_g) is the further intersection point of the straight line P_0P_1 with $\mathbf{T}$.

Proof. For $(x_i, y_i) \in \mathbf{T}$ there holds the identity

$$y(x - x_s) = Q(x - x_i)\{x(x + x_i - A - B) + (x_i - A)(x_i - B)\} + y_i(x_i - x_s). \tag{11}$$

For the isotropic circle $\mathbf{K}$ we have

$$\mathbf{K} : y = y_0 + R(x - x_0)(x - x_1) + \frac{y_0 - y_1}{x_0 - x_1}(x - x_0), \;\; R \neq Q$$

and from this in conjunction with (9)

$$(R - Q)(x_\tau - x_s)(x_0 - x_1) = y_1 - y_0 + Q(x_0 - x_1)\{x_0 + x_1 + x_s - A - B\}.$$

The identity (11) gives

$$(x_0 - x_s)(x_1 - x_s)(y_1 - y_0) =$$
$$Q(x_1 - x_0)\{x_0 x_1(x_0 + x_1 - A - B) - x_s[(x_0^2 + x_0 x_1 + x_1^2) - (A + B)(x_0 + x_1) + AB]\}$$

and therefore

$$x_\tau - x_s = \frac{Q}{R - Q} \cdot \frac{x_s(x_s - A)(x_s - B)}{(x_0 - x_s)(x_1 - x_s)} \, . \tag{12}$$

The application of the power theorem for tridens curves by Palman [4, p. 43f] yields (10). □

Theorem 2. *Let* $\mathbf{T}$ *denote an irreducible tridens curve of third order of the isotropic plane,* P_s *an arbitrary point on the isotropic asymptote* a_s *of* $\mathbf{T}$*, and* $\Delta(P_1P_2P_3)$ *the triangle defined by the points of contact of the non-isotropic tangents of* $\mathbf{T}$ *from* P_s*. Then the centroid of* Δ *is on* a_s *and the radius* $R(\Delta)$ *of the circumscribed circle of* Δ *is 3/2 times the radius of the asymptotic circle of* $\mathbf{T}$*.*

Proof. We can assume (9) to be the equation of $\mathbf{T}$. The points of contact of the tangents through P_s of $\mathbf{T}$ are points of the *polar cone* $\mathbf{H}(P_s)$ of P_s with respect to $\mathbf{T}$ (see f.i. [1, p. 42]). By (9) $\mathbf{H}(P_s)$ is the *special hyperbola*

$$y(x - x_s) + xy_s - Q\{(3x_s - A - B)x^2 + 2x[AB - (A + B)x_s] + ABx_s\} = 0.$$

So the abscissas x_i of the points of contact P_i of the non-isotropic tangents of $\mathbf{T}$ through P_s are the solutions of

$$x^3 - 3x_s x^2 + \left[\frac{y_s}{Q} - AB + 2(A + B)x_s\right] x - \left[\frac{y_s}{Q} + AB\right] x_s = 0. \tag{13}$$

Thus we obtain the result for the centroid of Δ. Let

$$y = Rx^2 + \alpha x + \beta,$$

be the equation of the circumscribed circle of Δ. For the abscissas of its intersection points with $\mathbf{T}$ we deduce from (9) that

$$x^3 + \frac{x^2}{Q - R}[Rx_s - \alpha - Q(A + B)] + \frac{x}{Q - R}[\alpha x_s + QAB - \beta] + \frac{\beta}{Q - R}x_s = 0. \tag{14}$$

Comparing the equations (13) and (14) we find the compatibility relation

$$(3Q - 2R)(x_s - A)(x_s - B) = 0. \qquad \square$$

Corollary. *Every irreducible tridens curve* **T** *of third order of the isotropic plane can be regarded as the locus of the inflection points of a one parametric cubic parabola pencil.*

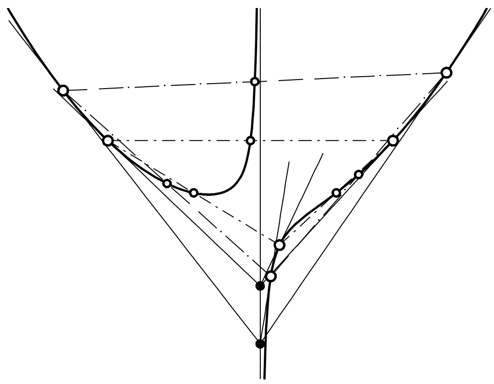
Proof. Let (9) be the equation of **T**. For the triangle $\Delta(P_1P_2P_3)$ we infer from Theorem 2 that $R(\Delta) = 3/2$ Q. Denote by x_{jk} the abscissa of the third intersection point of the straight line P_jP_k with **T**. By the above lemma we have for the abscissas x_i of the points P_i

$$x_i - x_s = \frac{Q}{Q - R(\Delta)}(x_{jk} - x_s),\ \{i,j,k\} = \{1,2,3\}. \tag{15}$$

The fact that the centroid of Δ lies on the asymptote of **T** (Theorem 2) yields with (15) that $x_{ij} + x_{jk} + x_{ki} = 3x_s$ and consequently

$$x_i + x_k = -\frac{Q}{Q - R(\Delta)}x_{ik} + \frac{3Q - 2R(\Delta)}{Q - R(\Delta)}x_s.$$

Using $R(\Delta) = 3/2\,Q$ we see that **T** passes through the side midpoints of $\Delta(P_1P_2P_3)$. A tridens curve of third order is uniquely determined by the vertices and side midpoints of an admissible triangle. The triangle $\Delta(P_1P_2P_3)$ is admissible by construction. So the statement of the corollary is a consequence of Theorem 1. $\square$



Remark 2. Reality questions of such inscribed triangles $\Delta(P_1P_2P_3)$ for a given irreducible tridens curve **T** of third order can be answered with the aid of (13). Another approach is the following: We choose an arbitrary point $P_1(x_1, y_1)$ on **T**. From (10) and (12) it follows that

$$x_1 + x_2 - 2x_s = 2(x_{12} - x_s) = -2\frac{x_s(x_s - A)(x_s - B)}{(x_1 - x_s)(x_2 - x_s)} = -(x_3 - x_s),$$

i.e., the abscissas x_1, x_2 are solutions of the quadratic equations

$$(x - x_s)^2 + (x - x_s)(x_1 - x_s) = -2\frac{x_s(x_s - A)(x_s - B)}{x_1 - x_s}. \tag{16}$$

Real, admissible inscribed triangles Δ of $\mathbf{T}$, whose side midpoints are on $\mathbf{T}$ exist exactly for

$$1/4(x_1 - x_s)^2 - 2(x_1 - x_s)^{-1} x_s(x_s - A)(x_s - B) > 0.$$

3. In this final section we briefly study the envelope $\mathbf{W}$ of the inflection point tangents of the cubic parabola pencil $\mathbf{P}$. From (2) we deduce that for the tangent at the point $P(x^*, y^*)$ of the cubic parabola $p(\mu) \in \mathbf{P}$

$$y - y^* = \{\mu x^*(x^* - a) + (2x^* - a)[\lambda^{-1}(\lambda b - a)^{-1} + \mu(x^* - \lambda b)]\}(x - x^*),$$

and using (3) and (4) we find for the desired tangents

$$\lambda(\lambda b - a)(-3x_w + a + \lambda b)y = x_w^3 + x(-3x_w^2 + 3ax_w - a^2). \tag{17}$$

A short calculation yields the *parametric representation of the envelope* $\mathbf{W}$ of the inflection point tangents of the pencil $\mathbf{P}$, namely

$$x = \frac{x_w^2(2x_w - (a + \lambda b))}{3x_w^2 - 2x_w(a + \lambda b) + a\lambda b}, \qquad y = \frac{x_w^2(x_w - a)^2}{\lambda(\lambda b - a)(3x_w^2 - 2x_w(a + \lambda b) + a\lambda b)}. \tag{18}$$

To gain an implicit equation of $\mathbf{W}$ we note that

$$\frac{y}{x} - \frac{1}{\lambda} = \frac{(x_w - \lambda b)^2}{\lambda(\lambda b - a)(2x_w - a - \lambda b)}, \qquad \frac{y}{x - a} - \frac{b}{\lambda b - a} = \frac{(x_w - \lambda b)^2}{\lambda(\lambda b - a)(2x_w - \lambda b)} \tag{19}$$

(the left sides equal to zero are the straight lines A_1A_3, A_2A_3 of the basic triangle Δ). So we obtain

$$2x_w = \frac{\lambda y(\lambda b - a)(\lambda b + a - x) - xa(x - a)}{\lambda y(\lambda b - a) - x(x - a)}.$$

From this and (19) we see that the desired equation of $\mathbf{W}$ can be written as

$$[\lambda y(\lambda b - a)(\lambda b + a - x) - xa(x - a)]^2 + 4\lambda y(\lambda b - a)^2(\lambda y - x)[\lambda y(\lambda b - a) - x(x - a)] = 0. \tag{20}$$

Differentiating (18) we obtain

$$\dot{x} = \frac{2x_w(x_w - a)(x_w - \lambda b)(3x_w - a - \lambda b)}{[3x_w^2 - 2x_w(a + \lambda b) + a\lambda b]^2}, \quad \dot{y} = \frac{2x_w(x_w - a)(x_w - \lambda b)(3x_w^2 - 3ax_w + a^2)}{\lambda[3x_w^2 - 2x_w(a + \lambda b) + a\lambda b]^2(\lambda b - a)}$$

and deduce for the tangents of the envelope $\mathbf{W}$ the equation

$$\lambda(\lambda b - a)(3x_w - a - \lambda b)y - [3x_w^2 - 3ax_w + a^2]x + x_w^3 = 0. \tag{21}$$

Theorem 3. *Let $\Delta(A_1A_2A_3)$ denote an admissible triangle of the isotropic plane and $\mathbf{P}$ the circumscribed pencil of the cubic parabolas. The envelope $\mathbf{W}$ of the inflection point tangents of the parabolas of $\mathbf{P}$ is an algebraic circular curve of order 4 and class 3. $\mathbf{W}$ has three nodes. They are the vertices of Δ. The node tangents are copunctal. They are the tangents of the tridens curve $\mathbf{T}(\boldsymbol{\Delta})$ at the vertices of Δ.*

Outline of the proof. The envelope **W** given by the equation (18) has as nodes the vertices of Δ. From the general theory in [3, p. 109] follows the copunctality of the node tangents. The equation (21) shows that the node tangents at the vertices $A_1(x_w = 0)$, $A_2(x_w = a)$ and $A_3(x_w = \lambda b)$ coincide with the tangents $t(A_1)$, $t(A_2)$ and $t(A_3)$ from (6). **W** touches the absolute line f in the absolute point F. □

The curve **W** can be obtained by means of the quadratic transformation

$$x^* = \frac{1}{2}\frac{\lambda(\lambda b - a)y(\lambda b + a - x) - ax(x-a)}{\lambda(\lambda b - a)y - x(x-a)}, \quad y^* = \frac{1}{4}\frac{\lambda(\lambda b - a)y(2b - y) - bx(x-a)}{\lambda(\lambda b - a)y - x(x-a)}$$

from the inscribed circle of Δ given by

$$(x^* - a + \lambda b)^2 + 4\lambda(a - \lambda b)y^* = 0.$$

The vertices of Δ are the fundamental points. The envelope **W** has no inflection point [3, p. 102] and exactly one double tangent. The parameter values of their conjugate complex points of contact are the solutions of

$$x_w^2 - \frac{ab\lambda(a + b\lambda)}{a^2 - ab\lambda + b^2\lambda^2}x_w - \frac{a^2b^2\lambda^2}{(a^2 - ab\lambda + b^2\lambda^2)^2} = 0.$$

W has exactly one isotropic focal point, which is determined by (18) with $3x_w = a + \lambda b$. Equation (21) together with the always real parameter values x_w satisfying $3x_w^2 - 2(a + \lambda b)x_w + ab\lambda = 0$ gives the asymptotics of **W**.

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