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Curve Shortening Flow and the Banchoff-Pohl Inequality on Surfaces of Nonpositive Curvature

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Abstract. In this paper the classical Banchoff-Pohl inequality, an isoperimetric inequality for nonsimple closed curves in the Euclidean plane, involving the square of the winding number, is generalized to nonpositively curved surfaces. The proof uses the well-known curve shortening flow.

1. Introduction

In [3], T.F. Banchoff and W.F. Pohl presented an isoperimetric inequality for nonsimple closed curves in the Euclidean plane, where the area of the enclosed domain is replaced by the sum of the areas into which the curve divides the plane, each weighted with the square of the winding number. Their proof uses integral geometric methods. Here we generalize this inequality for curves on surfaces with variable, nonpositive curvature.

Theorem. *Let M be a smooth, two dimensional, simply connected, complete Riemannian manifold with Gauss curvature $K \leq 0$.*

Then for every closed C^2-curve C in M

$$L^2 \geq 4\pi \iint\limits_M w^2(x)dA - K_0 \left(\iint\limits_M |w(x)|dA \right)^2 \tag{1}$$

where L is the length of C, $w(x)$ the winding number of $x \in M \setminus C$ with respect to C, dA the area element, and $K_0 = \max_{x \in \operatorname{conv} C} K(x)$ with $\operatorname{conv} C$ the convex hull of C.

Equality holds if and only if C is a circle traversed in the same direction a number of times in an M with $K \equiv K_0$ on $\operatorname{conv} C$.

For the Euclidean plane $M = E^2$ we have $K_0 = 0$ and (1) is the original Banchoff-Pohl inequality, presented in [3]. If C is simple and convex, (1) represents the classical isoperimetric inequality on surfaces, see e.g. [11, Theorem 4.3].

Inequality (1) has also been proved recently and independently by R. Howard ([9, Theorem 2]). There it occurs as a special case of a new Sobolev inequality.

An earlier result is due to E. Teufel, he gave a generalization of the Banchoff-Pohl inequality in the hyperbolic plane ([12]). For similar results by L. Gysin compare [7], [8], and for a nice overview see [11].

We prove the theorem by curve shortening, i.e. we take the curve C as initial value at time $t = 0$ for the curve shortening problem on surfaces, described by the partial differential equation

$$X_t = kN \tag{2}$$

where X is a parametrization of the evolving family of curves, N the inward unit normal vector, k the geodesic curvature, and the subscript t denotes partial derivative by the time t.

Curve shortening on surfaces has been studied in a number of papers, cf. [1], [2], [4], [5], [6] and [10].

We show that the isoperimetric deficit

$$L^2 - 4\pi \iint\limits_M w^2(x)dA + K_0 \left(\iint\limits_M |w(x)|dA \right)^2 \tag{3}$$

is decreasing during the evolution, and nonnegative at final time T. This proves the inequality (1) for the initial curve C. Although singularities may occur during the flow, it can be continued until the limit curve is trivial, by creating a generalized solution of the curve shortening flow ([2, ch. 7]). "Whiskers" (redundant parts of the limit curves at singular times, which have to be removed in order to continue the flow) affect the length, but not the winding number, and hence cause only negative jumps of the isoperimetric deficit at singular times.

We use flow by geodesic curvature because kN is the negative gradient of L, therefore this flow decreases the positive term L^2 in the expression (3) in the most efficient way.

The organization of this paper is as follows. In section 2 we prove an estimate of the total absolute curvature for curves in M, which will be essential to show the monotonicity of the isoperimetric deficit, in section 3 we determine the time derivatives of the areas, and eventually in Section 4 we prove the theorem.

I would like to thank Michael Gage and Eberhard Teufel for many helpful discussions during the investigations of this problem, and the Department of Mathematics of the University of Rochester for their hospitality, where much of this work was done.

2. The total absolute curvature estimate

Proposition. *Let C be a closed C^2-curve with a finite number of transversal, nonmultiple self-intersections (points in M traversed by C exactly twice) in an M as above. Then*

$$\int\limits_C |k|ds \geq 2\pi\mu - K_0 \iint\limits_M |w(x)|dA \tag{4}$$

where k is the geodesic curvature of C in M, ds the arclength element, $\mu \geq \max_{x \in M \setminus C} |w(x)|$ a positive integer to be determined in the proof, and K_0 as above.

Proof. By the Theorem of Hadamard, M is globally diffeomorphic to the Euclidean plane E^2; more precisely the exponential map $\exp_x$ in any $x \in M$ is a diffeomorphism. Therefore w can be defined along geodesics in the usual way.

Since C is a closed C^2-curve, $M \setminus C$ consists of a finite number of disjoint domains. Exactly one of these domains is unbounded, called M_1. We define C_1 as the boundary of $\tilde{G}_1 := M \setminus \operatorname{cl} M_1$, where $\operatorname{cl} M_1$ is the topological closure of M_1. Then we consider $M \setminus \operatorname{cl}(C \setminus C_1)$, which consists of a finite number of disjoint domains as well. We set C_2 as the boundary of $\tilde{G}_2 := M \setminus \operatorname{cl} M_2$, where M_2 is the only unbounded domain at this step. At step i, $i \in \{1, \ldots, n\}$, we define C_i as the boundary of $\tilde{G}_i := M \setminus \operatorname{cl} M_i$, where M_i is the only unbounded domain among $M \setminus \operatorname{cl}(C \setminus (C_1 \cup \ldots \cup C_{i-1}))$. We proceed until $C_1 \cup \ldots \cup C_i = C$. n is finite, since $M \setminus C$ consisted only of a finite number of domains. From each C_i we obtain a positively oriented curve $\tilde{C}_i$ by changing the orientation of each arc of C_i to be positive, e.g., the normal vector along the arc must point inwardly, to $\tilde{G}_i$, after changing. Each connected part of every $\tilde{C}_i$ is a closed, positively oriented, piecewise C^2-curve with a finite number of corners. Although it may have self-intersections at its corners (cf. a figure-eight curve), we consider it as simple, since each point in $\tilde{G}_i$ has winding number 1 with respect to $\tilde{C}_i$.

We define a different winding number $\tilde{w}(x)$ as the winding number of $x \in M \setminus C$ with respect to $\tilde{C} := \tilde{C}_1 \cup \ldots \cup \tilde{C}_n$. Obviously

$$\tilde{w}(x) = 0, \qquad x \in M_1, \tag{5}$$

$$\tilde{w}(x) = i, \qquad x \in \tilde{G}_i \setminus \operatorname{cl} \tilde{G}_{i+1},\ 1 \leq i \leq n-1, \tag{6}$$

$$\tilde{w}(x) = n, \qquad x \in \tilde{G}_n, \tag{7}$$

$$\tilde{G}_i = (\tilde{w} \geq i) := \{x \in M \setminus C \mid \tilde{w}(x) \geq i\},\ 1 \leq i \leq n \tag{8}$$

and

$$\tilde{w}(x) \geq |w(x)| \text{ for every } x \in M \setminus C. \tag{9}$$

We apply the Gauss-Bonnet theorem to each connected part of $\tilde{C}_i$ and get

$$\int_{\tilde{C}_i} \tilde{k} ds = 2\pi\mu_i - \iint_{\tilde{G}_i} K dA + \gamma_i - \delta_i$$

where $\tilde{k}$ is the curvature of $\tilde{C}$, μ_i is the number of connected parts of $\tilde{C}_i$ and hence a positive integer, γ_i the sum of internal angles, and δ_i the sum of external angles of $\tilde{C}_i$, each angle nonnegative and taken as shown in Fig. 1,2, such that each connected part of $\tilde{C}_i$ is traversed as one, connected curve. Self-intersections are not crossed, but traversing of $\tilde{C}_i$ there is continued with the arc belonging to $\tilde{C}_i$ (Fig. 1), or the right-handed arc (Fig. 2), respectively.

With this we obtain

$$\begin{aligned} \int_{C_i} |k| ds &= \int_{\tilde{C}_i} |\tilde{k}| ds \geq \int_{\tilde{C}_i} \tilde{k} ds = 2\pi\mu_i - \iint_{\tilde{G}_i} K dA + \gamma_i - \delta_i \\ &\geq 2\pi\mu_i - \iint_{\tilde{G}_i} K_0 \, dA + \gamma_i - \delta_i = 2\pi\mu_i - K_0 \tilde{A}_i + \gamma_i - \delta_i \end{aligned} \tag{10}$$

with $\tilde{A}_i := \operatorname{area}(\tilde{G}_i)$, and using $\tilde{G}_i \subseteq \operatorname{conv} C$.

This leads to

$$\int_{C_1} |k| ds + \ldots + \int_{C_j} |k| ds \geq$$

$$2\pi(\mu_1 + \ldots + \mu_j) - K_0(\tilde{A}_1 + \ldots + \tilde{A}_j) + \gamma_1 + \ldots + \gamma_j - \delta_1 - \ldots - \delta_j$$

where we claim that

$$\gamma_1 + \ldots + \gamma_{j-1} + \gamma_j - \delta_1 - \ldots - \delta_j \geq \gamma_j \geq 0 \text{ for } j \in \{1, \ldots, n\}. \tag{11}$$

We consider a self-intersection, where we have only two situations, because $\tilde{w}$ can change by crossing $\tilde{C}$ only by ± 1, since there are only isolated self-intersections:

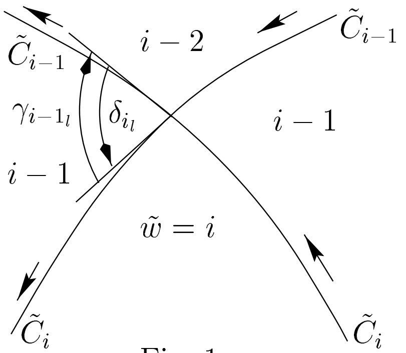

Fig. 1

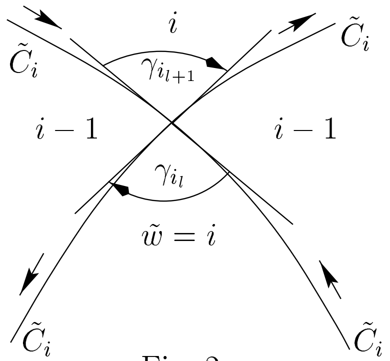

Fig. 2

From Fig. 1 we see that for each δ_{i_l}, $2 \leq i \leq j$, $1 \leq l \leq l_i$ (l_i is finite, because $\tilde{C}_i$ has only a finite number of corners), there is a γ_{i-1_l}, occuring at the same self-intersection, so that $\gamma_{i-1_l} - \delta_{i_l} = 0$. Fig. 2 shows the situation with different orientation of the arcs, there all the arcs belong to $\tilde{C}_i$ and the resulting angle at the self-intersection is $\gamma_{i_l} + \gamma_{i_{l+1}} \geq 0$, and hence $\gamma_{i-1} - \delta_i \geq 0$. Fig. 1, with $i = 2$, also shows that $\tilde{C}_1$ can not have any external angles, therefore $\delta_1 = 0$.

With (11) we obtain

$$\int_C |k| ds = \sum_{i=1}^{n} \int_{C_i} |k| ds \geq 2\pi\mu - K_0 \tilde{A} + \sum_{i=1}^{n} \gamma_i - \sum_{i=1}^{n} \delta_i \geq 2\pi\mu - K_0 \iint_M \tilde{w}(x) dA$$

where

$$\mu := \sum_{i=1}^{n} \mu_i \quad \text{ and } \quad \tilde{A} := \sum_{i=1}^{n} \tilde{A}_i = \sum_{i=1}^{n} \iint_{\tilde{G}_i} dA = \iint_M \tilde{w}(x) dA. \tag{12}$$

The assertion follows by (9) and $\mu \geq n = \max_{x \in M \setminus C} \tilde{w}(x) \geq \max_{x \in M \setminus C} |w(x)|$ (cf. (5)–(7)). $\square$

3. The different notions of area and their time derivatives

Now we set up a few notions for use throughout the reminder of the paper, for a closed C^2-curve C:

$$G_i \;:=\; (w \geq i) = \{x \in M \setminus C \mid w(x) \geq i\}, \; i \geq 1,$$

$$\begin{aligned}
G_i &:= (w \le i) = \{x \in M \setminus C \mid w(x) \le i\},\ i \le -1, \\
A_i &:= \operatorname{area}(G_i) = \iint\limits_{G_i} dA,\ A_0 := 0, \qquad (13)\\
\bar{A} &:= \iint\limits_M |w(x)| dA \\
&= -\iint\limits_{(w \le -1)} w(x) dA + \iint\limits_{(w \ge 1)} w(x) dA \\
&= -\sum_{i=-1}^{-n} \iint\limits_{(w=i)} i dA + \sum_{i=1}^{n} \iint\limits_{(w=i)} i dA \\
&= \sum_{i=-1}^{-n} \iint\limits_{(w \le i)} dA + \sum_{i=1}^{n} \iint\limits_{(w \ge i)} dA \\
&= \sum_{i=-1}^{-n} A_i + \sum_{i=1}^{n} A_i = \sum_{i=-n}^{n} A_i, \qquad (14)\\
\hat{A} &:= \iint\limits_M w^2(x) dA \\
&= \sum_{i=-1}^{-n} \iint\limits_{(w=i)} i^2 dA + \sum_{i=1}^{n} \iint\limits_{(w=i)} i^2 dA \\
&= \sum_{i=-1}^{-n} (-2i-1) \iint\limits_{(w \le i)} dA + \sum_{i=1}^{n} (2i-1) \iint\limits_{(w \ge i)} dA \\
&= \sum_{i=-1}^{-n} (-2i-1) A_i + \sum_{i=1}^{n} (2i-1) A_i. \qquad (15)
\end{aligned}$$

If $X\colon [0, L) \to M$ is an arclength parametrization for C of class C^2, we define $\hat{w}\colon [0, L) \to \mathbf{Z}$ by

$$\hat{w}(s) := \lim_{\varepsilon \to 0+} w\Big(\exp_{X(s)} \varepsilon\xi(s)\Big) + \lim_{\varepsilon \to 0-} w\Big(\exp_{X(s)} \varepsilon\xi(s)\Big) \qquad (16)$$

for a direction $\xi(s)$, not tangential to C. $\hat{w}(s)$ does not depend on $\xi(s)$. It can also be defined as $\hat{w}(s) = i^+(s) + i^-(s)$, where $i^+(s)$ is the algebraic intersection number of the geodesic ray, starting in $X(s)$ with nontangential direction $\xi(s)$, with C; $i^-(s)$ is the algebraic intersection number of the geodesic ray, starting in $X(s)$, with direction $-\xi(s)$, with C.

We set $\bar{w} : [0, L) \to \{-1, 0, 1\}$ as

$$\bar{w}(s) := \operatorname{sign} \hat{w}(s). \qquad (17)$$

w can change by crossing C at a point of C, where C is embedded, only by ± 1, hence

$$\hat{w}\,|_{\partial G_i} = 2i + 1 \text{ for } i \le -1 \text{ and } \hat{w}\,|_{\partial G_i} = 2i - 1 \text{ for } i \ge 1, \qquad (18)$$

except at self-intersections and so obviously

$$\bar{w}\,|_{\partial G_i} = -1 \text{ for } i \le -1 \text{ and } \bar{w}\,|_{\partial G_i} = 1 \text{ for } i \ge 1. \tag{19}$$

Facts we will need are that $C(t)$ has always a finite number of self-intersections during the evolution, apart from $t = 0$ and singular moments in time, and this number decreases exactly when it has a self-tangency (a nontransversal self-intersection) ([2, Theorem 1.3]). The set of moments in time at which $C(t)$ has a self-tangency is discrete. At each time, when $C(t)$ becomes singular, it looses a self-intersection, or, to be more exact, the reduced limit curve at that time has fewer self-intersections than $C(t)$ before. Also, at any nonsingular positive time, $C(t)$ has only a finite number of inflection points ([2, Theorem 1.4]). At each nonsingular time $C(t)$ is smooth ([2, Theorem 4.1] and [1, Theorem 3.1]).

Lemma. *For almost all times t the derivatives $\bar{A}'(t)$ and $\hat{A}'(t)$ for a family of curves $C(t)$ evolving according to* (2) *are*

$$\bar{A}'(t) = -\int_{C(t)} \bar{w}k\,ds, \tag{20}$$

$$\hat{A}'(t) = -\int_{C(t)} \hat{w}k\,ds. \tag{21}$$

Proof. The boundary $\partial G_i(t)$ of $G_i(t)$ consists of a finite number of C^2-arcs. For $t > 0$, the set of times, when $C(t)$ has self-tangencies, is discrete. So we have for $i \ge 0$ at almost every time t

$$\begin{aligned} A_i'(t) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[A_i(t+\varepsilon) - A_i(t)\right] = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\iint_{G_i(t+\varepsilon)} dA - \iint_{G_i(t)} dA\right] \\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\int\int_{G_i(t+\varepsilon)\setminus G_i(t)} dA - \int\int_{G_i(t)\setminus G_i(t+\varepsilon)} dA\right]. \end{aligned}$$

We assume that each $C(t)$ is parametrized by the nonglobal arclength parameter $s \in [0, L(t))$. The arcs covering $G_i(t+\varepsilon)\setminus G_i(t)$ and $G_i(t)\setminus G_i(t+\varepsilon)$, respectively, are parameterized by intervals $(s_1^{\alpha,-}(\tau),\ s_2^{\alpha,-}(\tau))$ and $(s_1^{\beta,+}(\tau),\ s_2^{\beta,+}(\tau))$, respectively, $t \le \tau \le t+\varepsilon$, with $s_1^{\alpha,-},\ s_2^{\alpha,-},\ s_1^{\beta,+},\ s_2^{\beta,+}$ depending differentiably on τ, $\alpha \in \{1,\dots,\alpha^\star\}$, $\beta \in \{1,\dots,\beta^\star\}$ ($\alpha^\star = \alpha(t)$, $\beta^\star = \beta(t)$ are finite for all t, since $C(t)$ has only a finite number of inflection points, as mentioned above), and these interval borders represent either corners or inflection points of $C(t)$. On $(s_1^{\alpha,-}, s_2^{\alpha,-})$ we have $k < 0$ and on $(s_1^{\beta,+}, s_2^{\beta,+})$ $k > 0$, respectively, for all α, β. Therefore, using a transformation formula and taking into account the negative orientation of the parameterization on $G_i(t+\varepsilon)\setminus G_i(t)$ (because of $\det(X_s, X_\tau) = k < 0$ there), we obtain

$$A_i'(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[-\sum_{\alpha=1}^{\alpha^\star}\int_t^{t+\varepsilon}\int_{s_1^{\alpha,-}(\tau)}^{s_2^{\alpha,-}(\tau)} \det(X_s, X_\tau)\,ds d\tau - \sum_{\beta=1}^{\beta^\star}\int_t^{t+\varepsilon}\int_{s_1^{\beta,+}(\tau)}^{s_2^{\beta,+}(\tau)} \det(X_s, X_\tau)\,ds d\tau\right]$$

$$= \; -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\sum_{\alpha=1}^{\alpha^\star}\int_t^{t+\varepsilon}\int_{s_1^{\alpha,-}(\tau)}^{s_2^{\alpha,-}(\tau)} k(s,\tau)\, ds d\tau + \sum_{\beta=1}^{\beta^\star}\int_t^{t+\varepsilon}\int_{s_1^{\beta,+}(\tau)}^{s_2^{\beta,+}(\tau)} k(s,\tau) ds d\tau\right].$$

With a mean value theorem we arrive at

$$A_i'(t) \;=\; -\lim_{\varepsilon\to 0}\left[\sum_{\alpha=1}^{\alpha^\star}\int_{s_1^{\alpha,-}(\tau^{\alpha,-})}^{s_2^{\alpha,-}(\tau^{\alpha,-})} k(s,\tau^{\alpha,-})\, ds + \sum_{\beta=1}^{\beta^\star}\int_{s_1^{\beta,+}(\tau^{\beta,+})}^{s_2^{\beta,+}(\tau^{\beta,+})} k(s,\tau^{\beta,+})\, ds\right]$$

wherein $t \leq \tau^{\alpha,-}, \tau^{\beta,+} \leq t+\varepsilon$ for every α, β. Eventually

$$A_i'(t) = -\sum_{\alpha=1}^{\alpha^\star}\int_{s_1^{\alpha,-}(t)}^{s_2^{\alpha,-}(t)} k(s,t)\, ds - \sum_{\beta=1}^{\beta^\star}\int_{s_1^{\beta,+}(t)}^{s_2^{\beta,+}(t)} k(s,t)\, ds = -\int_{\partial G_i(t)} k(s,t) ds.$$

For $i \leq -1$ we have

$$A_i'(t) = \int_{\partial G_i(t)} k(s,t) ds$$

since $\partial G_i(t)$ is negatively oriented.

With (14), (19) and taking into account that the set of self-intersections is discrete in each $C(t)$ for almost every t, we obtain

$$\bar{A}'(t) = \sum_{i=-1}^{-n}\int_{\partial G_i(t)} k ds - \sum_{i=1}^{n}\int_{\partial G_i(t)} k ds = -\int_{C(t)} \bar{w} k ds$$

and with (15), (18)

$$\hat{A}'(t) = \sum_{i=-1}^{-n}(-2i-1)\int_{\partial G_i(t)} k ds - \sum_{i=1}^{n}(2i-1)\int_{\partial G_i(t)} k ds = -\int_{C(t)} \hat{w} k ds. \qquad \square$$

4. Proof of the Theorem

The following proof is valid for curves which are not multiple covers of an image; for such curves the theorem follows easily from the case of curves covered only once.

The fact that a family of curves $C(t)$ stays within the convex hull $C(0)$ of the initial curve $C(0)$ during the evolution, follows from [2, Theorems 1.3, 1.4], or can be shown directly by applying the strong maximum principle for parabolic equations to the evolution equation of k (cf. [1, eq. (4.2), p. 462]).

We have for the length $L(t)$ of $C(t)$

$$L'(t) = \frac{d}{dt}L(t) = -\int_{C(t)} k^2 ds \tag{22}$$

[1, p. 462] and hence

$$\frac{1}{2}\frac{d}{dt}L^2(t) = -L(t)\int\limits_{C(t)} k^2 ds \leq -\left(\int\limits_{C(t)} |k| ds\right)^2 \leq -4\pi^2$$

using the Cauchy-Schwarz inequality and (4). This implies that the lifespan between singularities is always finite. So there can be only a finite number of singular moments $0 = t_0 < t_1 < \ldots < t_m = T < \infty$, and, with [2, Corollary to Theorem 7.1], $C(T)$ has an empty reduced limit curve $C_{red}(T)$.

Now we prove the monotonicity of the isoperimetric deficit, i.e.

$$\frac{d}{dt}\left(L^2 - 4\pi\hat{A} + K_0\bar{A}^2\right)(t) \leq 0$$

for nonsingular times, at which $C(t)$ has only transversal, nonmultiple self-intersections. t is kept fixed while doing the following estimates, and for simplicity, we omit it.

To give a general idea, we deal first with the case $K_0 = 0$ (which includes the Euclidean case $M = E^2$), where this part is somewhat easier. There we have

$$\begin{aligned}\frac{1}{2}\left(L^2 - 4\pi\hat{A}\right)' &= -L\int\limits_C k^2 ds + 2\pi\int\limits_C \hat{w} k ds \\ &\leq -\left(\int\limits_C |k| ds\right)^2 + 2\pi\int\limits_C |\hat{w}||k| ds\end{aligned}$$

with (22), (21) and the Cauchy-Schwarz inequality.

Along C_i, $1 \leq i \leq n$, we have $|\hat{w}(s)| \leq \lim_{\varepsilon\to 0+} |w(\exp_{X(s)} \varepsilon\xi(s))| + \lim_{\varepsilon\to 0-} |w(\exp_{X(s)} \varepsilon\xi(s))| \leq \lim_{\varepsilon\to 0+} \tilde{w}(\exp_{X(s)} \varepsilon\xi(s)) + \lim_{\varepsilon\to 0-} \tilde{w}(\exp_{X(s)} \varepsilon\xi(s)) = 2i - 1$ using (16), (9) and (5)–(7). This leads to

$$\begin{aligned}\frac{1}{2}\left(L^2 - 4\pi\hat{A}\right)' \leq &-\left(\sum_{i=1}^n \int\limits_{C_i} |k| ds\right)^2 + 2\pi\sum_{i=1}^n (2i-1)\int\limits_{C_i} |k| ds = \\ &\int\limits_{C_n} |k| ds\left(-2\int\limits_{C_1\cup\ldots\cup C_{n-1}} |k| ds - \int\limits_{C_n} |k| ds + 2\pi(2n-1)\right) + \\ &\int\limits_{C_{n-1}} |k| ds\left(-2\int\limits_{C_1\cup\ldots\cup C_{n-2}} |k| ds - \int\limits_{C_{n-1}} |k| ds + 2\pi(2n-3)\right) + \ldots \\ &+\int\limits_{C_2} |k| ds\left(-2\int\limits_{C_1} |k| ds - \int\limits_{C_2} |k| ds + 6\pi\right) + \\ &\int\limits_{C_1} |k| ds\left(-\int\limits_{C_1} |k| ds + 2\pi\right) \leq\end{aligned}$$

$$2\pi(-2\mu_1 - \ldots - 2\mu_{n-1} - \mu_n + 2n - 1)\int\limits_{C_n} |k|ds +$$
$$2\pi(-2\mu_1 - \ldots - 2\mu_{n-2} - \mu_{n-1} + 2n - 3)\int\limits_{C_{n-1}} |k|ds + \ldots$$
$$+ 2\pi(-2\mu_1 - \mu_2 + 3)\int\limits_{C_2} |k|ds + 2\pi(-\mu_1 + 1)\int\limits_{C_1} |k|ds \leq 0$$

with (10), (11) and $\mu_i \geq 1$ for all $i \in \{1, \ldots, n\}$.

Now we treat the situation with general M, where we keep in mind that K_0 always satisfies $K_0 \leq 0$ and $-K_0 \geq 0$. Similar to above, we get

$$\begin{aligned}
\frac{1}{2}\left(L^2 - 4\pi\hat{A} + K_0\bar{A}^2\right)' &= -L\int\limits_C k^2 ds + 2\pi\int\limits_C \hat{w}kds - K_0\bar{A}\int\limits_C \bar{w}kds \\
&\leq -\left(\int\limits_C |k|ds\right)^2 + 2\pi\int\limits_C |\hat{w}||k|ds - K_0\bar{A}\int\limits_C |\bar{w}||k|ds \\
&\leq -\left(2\pi\mu - K_0\bar{A}\right)\int\limits_C |k|ds + 2\pi\int\limits_C |\hat{w}||k|ds - K_0\bar{A}\int\limits_C |k|ds \\
&\leq -2\pi n\int\limits_C |k|ds + 2\pi\int\limits_C |\hat{w}||k|ds \qquad (23)
\end{aligned}$$

with $|\bar{w}| \leq 1$, (4), and $\mu \geq n$.

From now on we consider the case where n is even and continue with

$$\begin{aligned}
\frac{1}{2}\left(L^2 - 4\pi\hat{A} + K_0\bar{A}^2\right)' \leq\ & 2\pi\Bigg[(1-n)\int\limits_{C_1} |k|ds + (3-n)\int\limits_{C_2} |k|ds + \ldots \\
& - \int\limits_{C_{\frac{n}{2}}} |k|ds + \int\limits_{C_{\frac{n}{2}+1}} |k|ds + \ldots \\
& +(n-3)\int\limits_{C_{n-1}} |k|ds + (n-1)\int\limits_{C_n} |k|ds\Bigg]. \qquad (24)
\end{aligned}$$

by using the estimate for $|\hat{w}|$ as in the Euclidean case.

Here two cases occur: Either (24) is nonpositive (in this case we are done), or it is not. The latter situation can be written as

$$\int\limits_{C_{\frac{n}{2}+1}} |k|ds + 3\int\limits_{C_{\frac{n}{2}+2}} |k|ds + \ldots + (n-1)\int\limits_{C_n} |k|ds \geq$$
$$(n-1)\int\limits_{C_1} |k|ds + (n-3)\int\limits_{C_2} |k|ds + \ldots + 3\int\limits_{C_{\frac{n}{2}-1}} |k|ds + \int\limits_{C_{\frac{n}{2}}} |k|ds \geq$$
$$(n-1)\left(2\pi\mu_1 - K_0\tilde{A}_1 + \gamma_1\right) + (n-3)\left(2\pi\mu_2 - K_0\tilde{A}_2 + \gamma_2 - \delta_2\right) + \ldots$$

$$+3\left(2\pi\mu_{\frac{n}{2}-1} - K_0\tilde{A}_{\frac{n}{2}-1} + \gamma_{\frac{n}{2}-1} - \delta_{\frac{n}{2}-1}\right) + 2\pi\mu_{\frac{n}{2}} - K_0\tilde{A}_{\frac{n}{2}} + \gamma_{\frac{n}{2}} - \delta_{\frac{n}{2}} \geq$$
$$2\pi\Big((n-1)+(n-3)+\ldots+3+1\Big) -$$
$$K_0\left((n-1)\tilde{A}_1 + (n-3)\tilde{A}_2 + \ldots + 3\tilde{A}_{\frac{n}{2}-1} + \tilde{A}_{\frac{n}{2}}\right) +$$
$$(n-3)(\gamma_1-\delta_2) + (n-5)(\gamma_2-\delta_3) + \ldots + 3(\gamma_{\frac{n}{2}-2} - \delta_{\frac{n}{2}-1}) + \gamma_{\frac{n}{2}-1} - \delta_{\frac{n}{2}}$$
$$+2(\gamma_1 + \ldots + \gamma_{\frac{n}{2}-1}) + \gamma_{\frac{n}{2}}$$

using (10) and $\mu_i \geq 1$ for all i. The formula $(n-1)+(n-3)+\ldots+3+1 = \frac{1}{4}n^2$ and (11) lead to

$$\begin{aligned}
&\int\limits_{C_{\frac{n}{2}+1}} |k|ds + 3\int\limits_{C_{\frac{n}{2}+2}} |k|ds + \ldots + (n-1)\int\limits_{C_n} |k|ds \geq \\
&2\pi\frac{n^2}{4} - K_0\left((n-1)\tilde{A}_1 + (n-3)\tilde{A}_2 + \ldots + 3\tilde{A}_{\frac{n}{2}-1} + \tilde{A}_{\frac{n}{2}}\right) \\
&+2(\gamma_1 + \ldots + \gamma_{\frac{n}{2}-1}) + \gamma_{\frac{n}{2}}.
\end{aligned} \tag{25}$$

With (25), (10) as well as (11) and $\mu_i \geq 1$ again, we establish a new inequality for the total absolute curvature of C:

$$\begin{aligned}
(n-1)\int\limits_C |k|ds \;=\;& (n-1)\int\limits_{C_1\cup\ldots\cup C_{\frac{n}{2}}} |k|ds + (n-2)\int\limits_{C_{\frac{n}{2}+1}} |k|ds \\
&+ (n-4)\int\limits_{C_{\frac{n}{2}+2}} |k|ds + \ldots + 2\int\limits_{C_{n-1}} |k|ds \\
&+ \int\limits_{C_{\frac{n}{2}+1}} |k|ds + 3\int\limits_{C_{\frac{n}{2}+2}} |k|ds + \ldots + (n-1)\int\limits_{C_n} |k|ds \geq \\
&(n-1)\left(2\pi(\mu_1 + \ldots + \mu_{\frac{n}{2}}) - K_0\left(\tilde{A}_1 + \ldots + \tilde{A}_{\frac{n}{2}}\right) + \gamma_{\frac{n}{2}}\right) + \\
&(n-2)\left(2\pi\mu_{\frac{n}{2}+1} - K_0\tilde{A}_{\frac{n}{2}+1} + \gamma_{\frac{n}{2}+1} - \delta_{\frac{n}{2}+1}\right) \\
&+ (n-4)\left(2\pi\mu_{\frac{n}{2}+2} - K_0\tilde{A}_{\frac{n}{2}+2} + \gamma_{\frac{n}{2}+2} - \delta_{\frac{n}{2}+2}\right) + \ldots \\
&+ 2\left(2\pi\mu_{n-1} - K_0\tilde{A}_{n-1} + \gamma_{n-1} - \delta_{n-1}\right) \\
&+ 2\pi\frac{n^2}{4} - K_0\left((n-1)\tilde{A}_1 + (n-3)\tilde{A}_2 + \ldots + 3\tilde{A}_{\frac{n}{2}-1} + \tilde{A}_{\frac{n}{2}}\right) \\
&+ \; 2(\gamma_1 + \ldots + \gamma_{\frac{n}{2}-1}) + \gamma_{\frac{n}{2}} \geq \\
&2\pi\left[(n-1)\frac{n}{2} + (n-2) + (n-4) + \ldots + 2 + \frac{n^2}{4}\right] \\
&- K_0\Big((2n-2)\tilde{A}_1 + (2n-4)\tilde{A}_2 + \ldots \\
&\quad +(n+2)\tilde{A}_{\frac{n}{2}-1} + n\tilde{A}_{\frac{n}{2}} + (n-2)\tilde{A}_{\frac{n}{2}+1} + \ldots + 2\tilde{A}_{n-1}\Big) \\
&+ (n-2)(\gamma_{\frac{n}{2}} - \delta_{\frac{n}{2}+1}) + (n-4)(\gamma_{\frac{n}{2}+1} - \delta_{\frac{n}{2}+2}) + \ldots + 2(\gamma_{n-2} - \delta_{n-1}) \\
&+ \gamma_{\frac{n}{2}} + 2(\gamma_{\frac{n}{2}+1} + \ldots + \gamma_{n-1}) + 2(\gamma_1 + \ldots + \gamma_{\frac{n}{2}-1}) + \gamma_{\frac{n}{2}}
\end{aligned}$$

$$\geq 2\pi(n^2-n) - K_0\Big((2n-2)\tilde{A}_1 + (2n-4)\tilde{A}_2 + \dots + (n+2)\tilde{A}_{\frac{n}{2}-1} + n\tilde{A}_{\frac{n}{2}} + (n-2)\tilde{A}_{\frac{n}{2}+1} + \dots + 2\tilde{A}_{n-1}\Big) + 2(\gamma_1 + \dots + \gamma_{n-1})$$

with $(n-2)+(n-4)+\dots+2 = \frac{1}{4}(n-2)n$, and eventually

$$(n-1)\int_C |k|ds \geq 2\pi(n-1)n - (n-1)K_0\tilde{A} + 2\gamma$$

by using $\gamma := \gamma_1 + \dots + \gamma_{n-1} \geq 0$ and $(n-1)\tilde{A}_1 \geq (n-1)\tilde{A}_n$, $(n-3)\tilde{A}_2 \geq (n-3)\tilde{A}_{n-1}, \dots, 3\tilde{A}_{\frac{n}{2}-1} \geq 3\tilde{A}_{\frac{n}{2}+2}$, $\tilde{A}_{\frac{n}{2}} \geq \tilde{A}_{\frac{n}{2}+1}$. So we have

$$\int_C |k|ds \geq 2\pi n - K_0\tilde{A} + \frac{2\gamma}{n-1}. \tag{26}$$

For almost any time (26) is sharper than

$$\int_C |k|ds \geq 2\pi n - K_0\bar{A}$$

what we used to get (23), since the set of moments in time, where $C = C(t)$ has self-tangencies, is discrete, therefore $\gamma = \gamma(t) > 0$ for almost all t. We write (26) different as

$$\int_C |k|ds = 2\pi n - K_0\tilde{A} + \vartheta \text{ with } \vartheta \geq \frac{2\gamma}{n-1} \geq 0 \tag{27}$$

and use (27) to estimate the derivative of the isoperimetric deficit again, and get

$$\frac{1}{2}\left(L^2 - 4\pi\hat{A} + K_0\bar{A}^2\right)' \leq -2\pi n\int_C |k|ds + 2\pi\int_C |\hat{w}||k|ds - \vartheta\int_C |k|ds$$

similar to (23). Again, we can write this as

$$\begin{aligned}\frac{1}{2}\left(L^2 - 4\pi\hat{A} + K_0\bar{A}^2\right)' \;\leq\; 2\pi\Bigg[&(1-n)\int_{C_1} |k|ds + (3-n)\int_{C_2} |k|ds + \dots \\ &- \int_{C_{\frac{n}{2}}} |k|ds + \int_{C_{\frac{n}{2}+1}} |k|ds + \dots \\ &+(n-3)\int_{C_{n-1}} |k|ds + (n-1)\int_{C_n} |k|ds\Bigg] - \vartheta\int_C |k|ds\end{aligned} \tag{28}$$

and consider the case, where the above sum is nonnegative, otherwise we are finished. This leads to

$$\int_{C_{\frac{n}{2}+1}} |k|ds + 3\int_{C_{\frac{n}{2}+2}} |k|ds + \dots + (n-1)\int_{C_n} |k|ds \geq$$

$$(n-1)\int_{C_1} |k|ds + (n-3)\int_{C_2} |k|ds + \dots + 3\int_{C_{\frac{n}{2}-1}} |k|ds + \int_{C_{\frac{n}{2}}} |k|ds + \frac{\vartheta}{2\pi}\int_C |k|ds.$$

Then the same calculation as above yields

$$(n-1)\int_C |k|ds \geq 2\pi(n-1)n - (n-1)K_0\tilde{A} + 2\gamma + \frac{\vartheta}{2\pi}\int_C |k|ds.$$

With this we can establish another inequality for the total absolute curvature of C :

$$\int_C |k|ds \geq 2\pi n - K_0\tilde{A} + \frac{2\gamma}{n-1} + \frac{\vartheta}{2\pi(n-1)}\int_C |k|ds$$

and, with $\gamma \geq 0$ and $\int_C |k|ds \geq 2\pi n$,

$$\int_C |k|ds \geq 2\pi n - K_0\tilde{A} + \frac{n}{n-1}\vartheta,$$

which contradicts (27).

For $n \geq 3$ odd the argumentation is completely analogous. The situation with $n = 1$ includes the case where C can be simple, which means $\gamma = 0$ for a nondiscrete set of times. But this situation is easy to rule out, we have

$$\begin{aligned}\frac{1}{2}\left(L^2 - 4\pi\hat{A} + K_0\bar{A}^2\right)' &\leq -\left(\int_C |k|ds\right)^2 + 2\pi\int_C |k|ds - K_0\bar{A}\int_C |k|ds \\ &= \left(-\int_C |k|ds + 2\pi - K_0\bar{A}\right)\int_C |k|ds \leq 0\end{aligned}$$

with $|\hat{w}| = |\bar{w}| \equiv 1$ in this case and (4).

So we have $(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)'(t) \leq 0$ for almost all t as long as C has only transversal, nonmultiple self-intersections. A curve with multiple self-intersections (points traversed more than twice), which may occur during the evolution, can be approximated by a family of curves with only nonmultiple self-intersections (in the sense of a smooth homotopy), for which $(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)' \leq 0$ and for the approximated curve as well.

Removal of the redundant parts of the limit curve at a singular time $t_j, 1 \leq j \leq m$, means decreasing of the length, while the areas $\bar{A}$ and $\hat{A}$ remain unchanged, since winding numbers do not change by removing. At times, where self-tangencies occur; $L, \bar{A}$ and $\hat{A}$ do not jump either.

Hence, we have $(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)'(t) \leq 0$ almost everywhere and negative jumps or continuity at the remaining set of times.

With $C_{red}(T) = \emptyset$ it follows that $(L^2_{red} - 4\pi\hat{A}_{red} + K_0\bar{A}^2_{red})(T) = 0$.
This asserts $(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)(0) \geq 0$.

In the case of equality, i.e. $(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)(0) = 0$, we have $(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)(t) = 0$ for every t and hence $(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)'(t) = 0$ for all t as well. This implies equality in the Cauchy-Schwarz inequality $L(t)\int_{C(t)} k^2 ds \geq (\int_{C(t)} |k|ds)^2$ and therefore $|k(\cdot, t)|$ to be constant for each t. Also, we have equality in (4), which means $K \equiv K_0$ on the domain G covered by $C(t),\ 0 \leq t \leq T,\ G \subseteq \operatorname{conv} C(0)$. Therefore, each $C(t), 0 \leq t < T$, is a circle, traversed in the same direction a constant number of times, in an M with $K \equiv K_0$ on the disc $\operatorname{cl} \tilde{G}_1 = \operatorname{conv} C(0)$.

On the other hand, starting with a multiply covered circle, or, for simplicity, with a circle traversed only once, and $K \equiv K_0$ on its enclosed disc, yields

$$\begin{aligned} \frac{1}{2}(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)'(t) &= -L(t) \int\limits_{C(t)} k^2 ds + 2\pi \int\limits_{C(t)} kds - K_0\bar{A}(t) \int\limits_{C(t)} kds \\ &= \Big(-L(t)k(t) + 2\pi - K_0\bar{A}(t)\Big) \int\limits_{C(t)} kds = 0, \end{aligned}$$

for all t, using $L(t)k(t) = \int_{C(t)} kds$ and the Gauss-Bonnet theorem. A circle converges to a point without developing a singularity [2, Theorem 8.1], hence $(L^2 - 4\pi\hat{A} + K_0\tilde{A}^2)(T) = 0$ and so $(L^2 - 4\pi\hat{A} + K_0\bar{A}^2)(0) = 0$. □

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