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Geometrical Probabilities for Convex Test Bodies

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Abstract. The purpose of the present paper is to resolve problems of the Buffon type for any convex test body and a lattice the elementary cell of which is a triangle or a regular hexagon. Finally the independence question for three lattices of parallel lines of different distances is studied.

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0. In [1] A. Aleman, M. Stoka and T. Zamfirescu considered Buffon's problem for an arbitrary convex test body $\mathbf{K}$ and a lattice of lines whose fundamental cell is a parallelogram and computed the probability p that the test body $\mathbf{K}$ dropped at random intersects one of the lines of the lattice (see also [5]). If $a,\ b$ are the lengths of the sides of the parallelogram, $\alpha \in [0, \frac{\pi}{2}]$ and $\pi - \alpha$ are its angles, $\mathcal{L}$ is the length of the boundary of $\mathbf{K}$ and $L(\varphi)$ is the width of the body in the direction φ, then for the case when

$$\max_{0\leq\varphi\leq\pi} L(\varphi) < \min(a\sin\alpha,\ b\sin\alpha)$$

it was proved that

$$p = \frac{(a+b)\mathcal{L}}{\pi ab\sin\alpha} - \frac{1}{\pi ab\sin^2\alpha}\int\limits_0^\pi L(\varphi)L(\varphi+\alpha)d\varphi\ . \tag{1}$$

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Here, as usual, φ is defined as the angle between a fixed line in the plane and a given direction related to the body.

In this paper we consider two further types of lattices in the euclidean plane $\mathcal{R}_3 = \mathcal{R}_3(a,b,c)$ and $\mathcal{R}_6 = \mathcal{R}_6(a)$: the fundamental cell of $\mathcal{R}_3$ is a triangle with sides of length a, b and c, and the one of $\mathcal{R}_6$ is a regular hexagon with side-length a. As a test body we consider an arbitrary convex body $\mathbf{K}$ with centroid S and length of the boundary $\mathcal{L}$. As pointed out above, we fix a direction d related to $\mathbf{K}$ and we will specify this whenever necessary by writing $(\mathbf{K}, d)$. When $\mathbf{K}$ is tangent to an oriented line g, then S_g will denote the orthogonal projection of S onto g, and if φ is the angle between d and $\overline{SS_g}$, we set $pr(\varphi) := |\overline{SS_g}|$, the distance from S to g. The 2π-periodic extension $pr : \mathbf{R} \to \mathbf{R}$ of this function will be called the *support function* with respect to $(\mathbf{K}, d)$. The *width* of $(\mathbf{K}, d)$ in the direction φ is then given by $L : \mathbf{R} \to \mathbf{R}$, $L(\varphi) := pr(\varphi) + pr(\varphi + \pi)$. The purpose of the present paper is to compute the probabilities $p_{\mathrm{K}}^{(3)}$ and $p_{\mathrm{K}}^{(6)}$ that the test body dropped at random intersects the lattices $\mathcal{R}_3$ and $\mathcal{R}_6$ respectively under the assumption that the support function pr (and hence L) is known.

We will also need some natural assumptions concerning the distribution of the random variables involved here. Let $\mathcal{K}$ be the set of all convex bodies that are congruent to $\mathbf{K}$ and have centroids inside the fundamental cell. We are going to consider these convex bodies as uniformly distributed, that is, their centroid S, regarded as a 2-dimensional random variable, is uniformly distributed in the fundamental cell, the random variable φ is uniformly distributed in the intervall $[0, 2\pi]$, and S and φ are stochastically independent.

1. Let $\mathbf{T}$ be a fixed cell of $\mathcal{R}_3$; the vertices of $\mathbf{T}$ and also the sizes of the corresponding angles will be denoted A, B and C. Further, we let $|\overline{AB}| = c, |\overline{BC}| = a$, $|\overline{CA}| = b$, and R be the radius of the circle circumscribed to $\mathbf{T}$. The outward normals to the sides $\overline{BC}, \overline{CA}$ and $\overline{AB}$ of $\mathbf{T}$ are denoted by n_a, n_b and n_c. If φ is the angle between d and n_a, then the angle between d and n_b, or d and n_c is $\varphi - C + \pi$, $\varphi + B + \pi$ respectively. If φ is fixed, there is a unique triangle $\mathbf{T}(\mathbf{K}, \varphi)$, whose sides are parallel to those of $\mathbf{T}$, have the same outward normal vectors, and are also tangent to $\mathbf{K}$. Clearly, $\mathbf{T}(\mathbf{K}, \varphi)$ contains $\mathbf{K}$, and for this given value of φ it is independent (up to translations) of the position of $\mathbf{K}$. Let A_φ, B_φ and C_φ be the vertices of $\mathbf{T}(\mathbf{K}, \varphi)$ and $t(\mathbf{K}, \varphi)$ its area (which can be expressed in terms of the support function). Finally, let t be the area of $\mathbf{T}$, that is $t = 2R^2 \sin A \sin B \sin C$. We have sketched this situation below (Figure 1). With the above notations one has

Theorem 1. *If the diameter of* $\mathbf{K}$ *is less than* $\frac{4t}{a+b+c}$ *then*

$$p_{\mathbf{K}}^{(3)} = \frac{\mathcal{L}}{4\pi R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}} - \frac{\int\limits_0^{2\pi} t(\mathbf{K}, \varphi) d\varphi}{4\pi R^2 \sin A \sin B \sin C} \, . \tag{2}$$

Proof. Let $\mathcal{M}$ denote the set of test bodies $\mathbf{K}$ (i.e. the set of bodies obtained from the fixed body $\mathbf{K}$ by means of translations and rotations) whose centroids lie in $\mathbf{T}$, and $\mathcal{N}$ the set of test bodies completely contained in $\mathbf{T}$. Using the elementary kinematic measure in the plane (see [4], page 126), that is $d\mathbf{K} = dx \wedge dy \wedge d\varphi$, where x and y are the coordinates of S, we

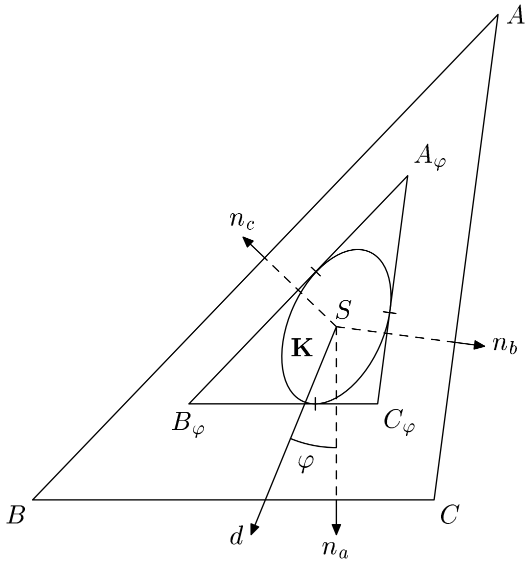

Figure 1

see that $\mathcal{M}$ and $\mathcal{N}$ can be interpreted as subsets of $\mathbf{R}^3$. Moreover, if μ denotes the Lebesgue measure in $\mathbf{R}^3$, we get

$$p_{\mathbf{K}}^{(3)} = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})} \ . \tag{3}$$

We first note that

$$\mu(\mathcal{M}) = \int_0^{2\pi} d\varphi \iint_{\{(x,y)\in\mathbf{T}\}} dxdy = 2\pi t = 4\pi R^2 \sin A \sin B \sin C \ . \tag{4}$$

Suppose that $\mathbf{K}$ is tangent to $\overline{BC}$ and $\overline{BA}$ ($\overline{CB}$ and $\overline{CA}$, or $\overline{AC}$ and $\overline{AB}$) such that the angle between d and n_a is φ, and denote by B'_φ (C'_φ, or A'_φ) the position of S (always in the interior of $\mathbf{T}$). As it is shown in Figure 2 below we denote by $\mathbf{T}'(\varphi)$ the triangle with vertices $A'_\varphi, B'_\varphi, C'_\varphi$ and by B_a, C_a the projections of B'_φ, C'_φ on BC, by B_c, A_c the projections of B'_φ, A'_φ on $\overline{AB}$, etc.

One has $|\overline{B'_\varphi B_a}| = |\overline{C'_\varphi C_a}| = pr(\varphi)$, $|\overline{C'_\varphi C_b}| = |\overline{A'_\varphi A_b}| = pr(\varphi - C + \pi)$ and $|\overline{A'_\varphi A_c}| = |\overline{B'_\varphi B_c}| = pr(\varphi + B + \pi)$. To simplify the notations we set

$$pr(\varphi) =: pr(\varphi, a),\ pr(\varphi - C + \pi) =: pr(\varphi, b),\ pr(\varphi + B + \pi) =: pr(\varphi, c) \ .$$

From $|\overline{BB_a}| = \frac{1}{\sin B}\Big[pr(\varphi, c) + pr(\varphi, a)\cos B\Big], |\overline{CC_a}| = \frac{1}{\sin C}\Big[pr(\varphi, b) + pr(\varphi, a)\cos C\Big]$ we get the length $a(\varphi)$ of $|\overline{B'_\varphi C'_\varphi}|$:

$$a(\varphi) = a - \frac{1}{\sin B \sin C}\ \Big[pr(\varphi, a)\sin A + pr(\varphi, b)\sin B + pr(\varphi, c)\sin C\Big] \ .$$

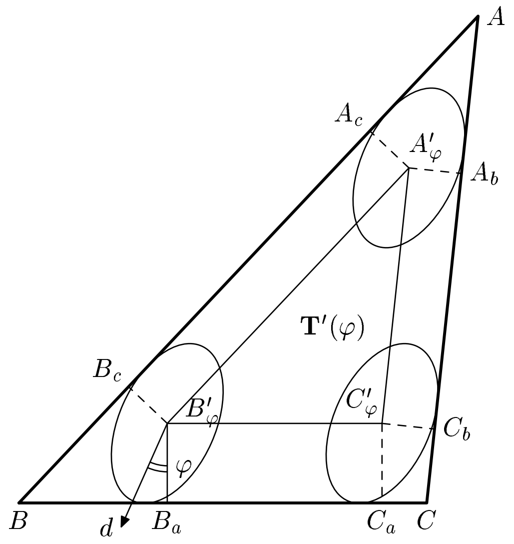

Figure 2

If $E(\mathbf{K},\mathbf{T},\varphi) := pr(\varphi,a)\sin A + pr(\varphi,b)\sin B + pr(\varphi,c)\sin C$, then for the area $t(\varphi)$ of $\mathbf{T}'(\varphi)$ we have

$$t(\varphi) = \frac{[a(\varphi)]^2\sin B\sin C}{2\sin A} = \frac{a^2\sin B\sin C}{2\sin A} - \frac{aE(\mathbf{K},\mathbf{T},\varphi)}{\sin A} + \frac{E(\mathbf{K},\mathbf{T},\varphi)^2}{2\sin A\sin B\sin C}\ .$$

This gives

$$\begin{aligned}\mu(\mathcal{N}) &= \int_0^{2\pi} d\varphi \iint_{\{(x,y)\in \mathbf{T}'_\varphi\}} dxdy = \int_0^{2\pi} \text{area } (\mathbf{T}\ominus\varphi\mathbf{K})d\varphi = \int_0^{2\pi} t(\varphi)d\varphi \\ &= 2\pi t - \tfrac{a}{\sin A}\int_0^{2\pi} E(\mathbf{K},\mathbf{T},\varphi)d\varphi + \tfrac{1}{2\sin A\sin B\sin C}\int_0^{2\pi} E(\mathbf{K},\mathbf{T},\varphi)^2 d\varphi.\end{aligned} \tag{5}$$

Using now the Cauchy formula for convex bodies, i.e. $\int_0^{2\pi} pr(\varphi)d\varphi = \mathcal{L}$, and the fact that pr is periodic we obtain

$$\begin{aligned}\int_0^{2\pi} E(\mathbf{K},\mathbf{T},\varphi)d\varphi &= \sin A\int_0^{2\pi} pr(\varphi,a)d\varphi + \sin B\int_0^{2\pi} pr(\varphi,b)d\varphi + \sin C\int_0^{2\pi} pr(\varphi,c)d\varphi \\ &= \sin A\int_0^{2\pi} pr(\varphi)d\varphi + \sin B\int_{\pi-C}^{3\pi-C} pr(\varphi)d\varphi + \sin C\int_{\pi+B}^{3\pi+B} pr(\varphi)d\varphi \\ &= (\sin A+\sin B+\sin C)\ \int_0^{2\pi} pr(\varphi)d\varphi \\ &= (\sin A+\sin B+\sin C)\ \mathcal{L} = 4\ \mathcal{L}\cos\tfrac{A}{2}\cos\tfrac{B}{2}\cos\tfrac{C}{2}\ .\end{aligned}$$

Let R_φ be the radius of the circle circumscribed to the triangle $A_\varphi B_\varphi C_\varphi$. Since the sizes of the angles in A_φ, B_φ and C_φ are A, B and C respectively, by the sine theorem it follows that:

$$\begin{aligned}E(\mathbf{K},\mathbf{T},\varphi)^2 &= \left(\tfrac{|\overline{B_\varphi C_\varphi}|pr(\varphi,a)+|\overline{C_\varphi A_\varphi}|pr(\varphi,b)+|\overline{A_\varphi B_\varphi}|pr(\varphi,c)}{2R_\varphi}\right)^2\\ &= \left(\tfrac{2t(\mathbf{K},\varphi)}{2R_\varphi}\right)^2 = \left(\tfrac{2R_\varphi^2\sin A\sin B\sin C}{R_\varphi}\right)^2\\ &= 2(2R_\varphi^2\sin A\sin B\sin C)\sin A\sin B\sin C = 2t(\mathbf{K},\varphi)\sin A\sin B\sin C.\end{aligned}$$

Thus,

$$\mu(\mathcal{N}) = 2\pi t - 8R\,\mathcal{L}\cos\frac{A}{2}\cos\frac{B}{2}\,cos\frac{C}{2} + \int_0^{2\pi} t(\mathbf{K},\varphi)d\varphi\,. \tag{6}$$

and from (3), (4) and (6) we obtain the equality (2). □

Remarks.

1) In the case when $\mathbf{T}$ is equilateral with side of length a, (2) becomes

$$\begin{aligned}p_{\mathbf{K}}^{(3)} &= \frac{2\sqrt{3}\,\mathcal{L}}{\pi a} - \frac{2}{\pi\sqrt{3}\,a^2}\int_0^{2\pi} t(\mathbf{K},\varphi)d\varphi\\ &= \frac{2\sqrt{3}}{\pi a}\Big(\mathcal{L} - \frac{1}{a}\int_0^{\frac{2\pi}{3}} t(\mathbf{K},\varphi)d\varphi\Big)\,.\end{aligned}\tag{2'}$$

In the last equality above we have used the fact that $\frac{2\pi}{3}$ is a period (in general the smallest) of the function $\varphi \mapsto t(\mathbf{K},\varphi)$.

2) In the case when the test body is a disc $\mathbf{C}(r)$ with radius $r < \frac{2t}{a+b+c}$, then $t(\mathbf{C}(r),\varphi)$ equals r^2 ctan $\frac{A}{2}$ ctan $\frac{B}{2}$ ctan $\frac{C}{2}$, and from (2) we obtain

$$p_{\mathbf{C}(r)}^{(3)} = \frac{r}{2R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} - \frac{r^2}{16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}\,. \tag{7}$$

3) If $\mathbf{K}$ is a needle $\mathbf{N}$ of length l then $p(\varphi) = \frac{l}{2}\,|\sin\varphi|$, and (2) gives

$$p_{\mathbf{N}}^{(3)} = \frac{l}{2\pi R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} - \frac{l^2 f(A,B,C)}{8\pi R^2\sin^2 A\sin^2 B\sin^2 C}\,, \tag{8}$$

where we have denoted

$$f(A,B,C) := A\sin^2 A + B\sin^2 B + C\sin^2 C + 3\sin A\sin B\sin C\,.$$

This result has been proved by M. Pettineo in [3].

2. The lattice $\mathcal{R}_3(a,b,c)$ is the union of three families of equidistant parallel lines. If t is the area of the fundamental cell $\mathbf{T}$ of the lattice $\mathcal{R}_3(a,b,c)$, then the distance between the parallel lines is $\frac{2t}{a}$, $\frac{2t}{b}$, $\frac{2t}{c}$ respectively. We denote the three generating families of equidistant parallel lines by $\mathcal{R}_1\left(\frac{2t}{a}\right)$, $\mathcal{R}_1\left(\frac{2t}{b}\right)$, $\mathcal{R}_1\left(\frac{2t}{c}\right)$ respectively.

Theorem 2. *The probability p that the convex body $\mathbf{K}$ with diameter less than $\frac{4t}{a+b+c}$ dropped at random intersects exactly three lines of the lattice $\mathcal{R}_3(a,b,c)$ is given by the equation*

$$8\pi R^2 p \sin A \sin B \sin C = \int_0^{2\pi} \left\{ L(\varphi)\left[\frac{L(\varphi+A)}{\sin A} + \frac{L(\varphi+B)}{\sin B} + \frac{L(\varphi+C)}{\sin C}\right] - 2t(\mathbf{K},\varphi)\right\} d\varphi \,. \tag{9}$$

Proof. Let us denote by I_a the event that $\mathbf{K}$ intersects a line of $\mathcal{R}_1\left(\frac{2t}{a}\right)$. Analogously, we define the events I_b and I_c. Our probability is then $p = P(I_a \cap I_b \cap I_c)$. It can be easily seen that the probability $P(I_a)$ to intersect a line of $\mathcal{R}_1(\frac{2t}{a})$ is the same for the random body $\mathbf{K}$, whether it is generated with respect to the lattice $\mathcal{R}_3(a,b,c)$ or with respect to the lattice $\mathcal{R}_1(\frac{2t}{a})$. Therefore, the well-known formula of Barbier [2] gives

$$P(I_a) = \frac{\mathcal{L}}{\pi \frac{2t}{a}} = \frac{\mathcal{L}\sin A}{2\pi R \sin A \sin B \sin C} \tag{10}$$

and analogous expressions for $P(I_b)$ und $P(I_c)$. The union $\mathcal{R}_1\left(\frac{2t}{a}\right) \cup \mathcal{R}_1\left(\frac{2t}{b}\right)$ is a lattice of lines whose fundamental cell is a parallelogram with sides of length a and b and angles C and $\pi - C$. Again, the event $I_a \cup I_b$ has the same probability for a random body $\mathbf{K}$ in the case of the triangular lattice $\mathcal{R}_3(a,b,c)$ as in the case of the parallelogram lattice $\mathcal{R}_1(\frac{2t}{a}) \cup \mathcal{R}_1(\frac{2t}{b})$. Hence, from (1) it follows

$$P(I_a \cup I_b) = \frac{(\sin A + \sin B)\mathcal{L}}{2\pi R \sin A \sin B \sin C} - \frac{1}{4\pi R^2 \sin A \sin B \sin C} \int_0^{\pi} \frac{L(\varphi)L(\varphi+C)}{\sin C} d\varphi. \tag{11}$$

Similarly, one computes $P(I_b \cup I_c)$ and $P(I_c \cup I_a)$. Finally, the value $P(I_a \cup I_b \cup I_c) = p_{\mathbf{K}}^{(3)}$ is given in Theorem 1, and formula (9) is obtained by replacing the values computed above in the well known relation

$$P(I_a \cap I_b \cap I_c) = P(I_a) + P(I_b) + P(I_c) - P(I_a \cup I_b) - P(I_b \cup I_c) - P(I_c \cup I_a) + P(I_a \cup I_b \cup I_c)\,.$$

□

Remarks.

1) If especially $\mathbf{K}$ is a disc of radius r the width L is the constant function $2r$, hence:

$$p = \frac{r^2}{R^2 \sin A \sin B \sin C}\left(\frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} - \frac{1}{2}\operatorname{ctan}\frac{A}{2}\operatorname{ctan}\frac{B}{2}\operatorname{ctan}\frac{C}{2}\right) \tag{9'}$$

2) When $\mathbf{K}$ is a needle $\mathbf{N}$ of length l such that $l < \min\left\{\frac{2t}{a}, \frac{2t}{b}, \frac{2t}{c}\right\}$, then the area $t(\mathbf{N},\varphi)$ of $\mathbf{T}(\mathbf{N},\varphi)$ is given by

$$t(\mathbf{N},\varphi) = \begin{cases} l^2 \, \frac{(1+\cos 2\varphi)\sin A}{4 \sin B \sin C} & \text{if } C - \frac{\pi}{2} \le \varphi \le \frac{\pi}{2} - B\,, \\ l^2 \, \frac{(1+\cos(2\varphi - 2C))\sin B}{4\sin A \sin C} & \text{if } \frac{\pi}{2} - B \le \varphi \le \frac{\pi}{2}\,, \\ l^2 \, \frac{(1+\cos(2\varphi + 2B))\sin C}{4 \sin A \sin B} & \text{if } \frac{\pi}{2} \le \varphi \le \frac{\pi}{2} + C\,. \end{cases}$$

Moreover, $\mathbf{T}(\mathbf{N}, \varphi)$ and $\mathbf{T}(\mathbf{N}, \varphi+\pi)$ coincide for all values of φ. This implies

$$\int_0^{2\pi} t(\mathbf{N}, \varphi) d\varphi = 2 \int_{C-\frac{\pi}{2}}^{\frac{\pi}{2}+C} t(\mathbf{N}, \varphi) d\varphi = \frac{l^2 \sin A}{4 \sin B \sin C} \left(\varphi + \frac{1}{2} \sin 2\varphi\right)\Big|_{C-\frac{\pi}{2}}^{\frac{\pi}{2}-B} +$$

$$\frac{l^2 \sin B}{4 \sin A \sin C} \left(\varphi + \frac{1}{2} \sin(2\varphi - 2C)\right)\Big|_{\frac{\pi}{2}-B}^{\frac{\pi}{2}} + \frac{l^2 \sin C}{4 \sin A \sin B} \left(\varphi + \frac{1}{2} \sin(2\varphi + 2B)\right)\Big|_{\frac{\pi}{2}}^{\frac{\pi}{2}+C}$$

$$= \frac{l^2 (A \sin^2 A + B \sin^2 B + C \sin^2 C)}{4 \sin A \sin B \sin C} + \frac{l^2}{8} \left[\frac{\sin A (\sin 2B + \sin 2C)}{\sin B \sin C} + \right.$$

$$\left. \frac{\sin B (\sin 2A + \sin 2C)}{\sin A \sin C} + \frac{\sin C (\sin 2A + \sin 2B)}{\sin A \sin B}\right]$$

$$= \frac{l^2 (A \sin^2 A + B \sin^2 B + C \sin^2 C)}{4 \sin A \sin B \sin C} + \frac{3l^2}{4}.$$

Now for the needle of length l, the width in the direction φ is the function $\varphi \mapsto l\ |\sin \varphi|$, hence

$$\begin{aligned} \int_0^{2\pi} L(\varphi) L(\varphi + A) d\varphi &= \left(\int_0^{\pi - A} \sin \varphi \sin(\varphi + A) d\varphi - \int_{\pi - A}^{\pi} \sin \varphi \sin(\varphi + A) d\varphi \right. \\ &\quad \left. + \int_{\pi}^{2\pi - A} \sin \varphi \sin(\varphi + A) d\varphi - \int_{2\pi - A}^{2\pi} \sin \varphi \sin(\varphi + A) d\varphi\right) l^2 \\ &= \Big[(\pi - 2A) \cos A + 2 \sin A\Big] l^2 , \end{aligned}$$

which leads to

$$\begin{aligned} &\int_0^{2\pi} L(\varphi) \left[\frac{L(\varphi+A)}{\sin A} + \frac{L(\varphi+B)}{\sin B} + \frac{L(\varphi+C)}{\sin C}\right] d\varphi \\ &\quad = \Big\{\Big[(\pi - 2A) \text{ctan } A + (\pi - 2B) \text{ctan } B + (\pi - 2C) \text{ctan } C\Big] + 6\Big\} l^2 . \end{aligned}$$

Thus, the probability for a needle of length $l < \min\left\{\frac{2t}{a}, \frac{2t}{b}, \frac{2t}{c}\right\}$ dropped at random to intersect simultaneously three lines of the lattice $\mathcal{R}_3(a, b, c)$ is given by

$$\begin{aligned} p = &\frac{l^2}{8\pi R^2 \sin A \sin B \sin C} \Big[(\pi - 2A) \text{ctan } A + (\pi - 2B) \text{ctan } B + (\pi - 2C) \text{ctan } C \\ &- \frac{A \sin^2 A + B \sin^2 B + C \sin^2 C}{2 \sin A \sin B \sin C} + \frac{9}{2}\Big] . \end{aligned} \tag{9''}$$

When $\mathbf{T}$ is equilateral $\left(\text{i.e. } A = B = C = \frac{\pi}{3}\right)$ we have $p = \frac{\sqrt{3}\ l^2}{2\pi R^2}$. In the limit case $l \to \frac{3R}{2}$ one gets $p \to \frac{9\sqrt{3}}{8\pi}$.

Under the assumptions in Theorem 2 and with the notations from its proof we now show:

Theorem 3. *The events I_a, I_b and I_c are independent if and only if the following relations are satisfied:*

$$\int_0^{2\pi} L(\varphi)L(\varphi+A)d\varphi = \int_0^{2\pi} L(\varphi)L(\varphi+B)d\varphi = \int_0^{2\pi} L(\varphi)L(\varphi+C)d\varphi = \frac{2\mathcal{L}^2}{\pi}\ , \tag{12}$$

$$\int_0^{2\pi} t(\mathbf{K},\varphi)d\varphi = \frac{\mathcal{L}^2}{\pi}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right) - \frac{\mathcal{L}^3}{2\pi^2 R\sin A\sin B\sin C}\ . \tag{13}$$

Proof. In [1] it is shown that the events I_a und I_b are independent if and only if:

$$\int_0^{2\pi} L(\varphi)L(\varphi+C)d\varphi = \frac{2\mathcal{L}^2}{\pi}\ ,$$

which means that I_a, I_b and I_c are pairwise independent exactly when (12) holds. In order to have all three events independent the following additional condition has to be satisfied

$$P(I_a\cap I_b\cap I_c) = P(I_a)P(I_b)P(I_c)\ .$$

Using (9) and (10) we can write this last relation as

$$\frac{1}{8\pi R^2\sin A\sin B\sin C}\int_0^{2\pi}\left\{L(\varphi)\left[\frac{L(\varphi+A)}{\sin A}+\frac{L(\varphi+B)}{\sin B}+\frac{L(\varphi+C)}{\sin C}\right]-2t(\mathbf{K},\varphi)\right\}d\varphi$$
$$= \frac{\mathcal{L}^3}{8\pi^3R^3\sin^2 A\sin^2 B\sin^2 C}\ .$$

Together with the relation (12) we obtain (13). □

3. We now consider the lattice $\mathcal{R}_6$ of lines in the euclidean plane whose fundamental cell is a regular hexagon with side of length a and the test body $(\mathbf{K}, d)$ with support function pr and width-function $\mathbf{L}$.

Theorem 4. *The probability that the test body* $\mathbf{K}$ *with diameter less than* a *dropped at random intersects the lattice* $\mathcal{R}_6$ *is*

$$p_{\mathbf{K}}^{(6)} = \frac{2\mathcal{L}}{\sqrt{3}\ \pi a} - \frac{2}{3\pi a^2}\int_0^{\frac{\pi}{3}}\left[2\sum_{k=-3}^{2} pr\left(\varphi+\frac{k\pi}{3}\right)pr\left(\varphi+\frac{(k+1)\pi}{3}\right) - \sum_{k=-3}^{2} pr^2\left(\varphi+\frac{k\pi}{3}\right)\right]d\varphi. \tag{14}$$

Proof. For a fixed fundamental cell $\mathbf{H}$ with vertices A, B, C, D, E and F consider again the set $\mathcal{M}$ of all test bodies $\mathbf{K}$ whose centroid lies in $\mathbf{H}$ and the set $\mathcal{N}$ of those test bodies which are contained completely in $\mathbf{H}$. As in the proof of Theorem 1 we have to compute $\mu(\mathcal{M})$ and

$\mu(\mathcal{N})$. For obvious symmetry reasons it is sufficient to consider only the values of φ in the intervall $\left[0, \frac{\pi}{3}\right]$. Moreover, $\mu(\mathcal{M})$ is easy to determine:

$$\mu(\mathcal{M}) = \int_0^{\frac{\pi}{3}} d\varphi \iint_{\{(x,y)\in \mathrm{H}\}} dx dy = \frac{\pi}{3} \cdot \frac{3\sqrt{3}\,\pi a^2}{2} = \frac{\sqrt{3}\,\pi a^2}{2}\,.$$

Let φ be the angle between d and the outward normal to $\overline{AB}$. Then the angles between d and the outward normals to $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$ and $\overline{FA}$ are given by $\varphi - \frac{\pi}{3}$, $\varphi - \frac{2\pi}{3}$, $\varphi - \pi$, $\varphi - \frac{4\pi}{3}$ and $\varphi - \frac{5\pi}{3}$ respectively. Let A_φ, B_φ, C_φ, D_φ, E_φ, F_φ be the positions of the centroid of $\mathbf{K}$ in the interior of $\mathbf{H}$, in the case when the test body is simultaneously tangent to $\overline{AF}$ and $\overline{AB}$, $\overline{BA}$ and $\overline{BC}$, $\overline{CB}$ and $\overline{CD}, \ldots,$ $\overline{FE}$ and $\overline{FA}$ respectively. We denote by A_2 and A_1 the orthogonal projections of A_φ onto $\overline{FA}$ and $\overline{AB}$, by B_2 and B_1 the orthogonal projections of B_φ onto $\overline{AB}$ and $\overline{BC}$, etc. (see the sketch below, Figure 3).

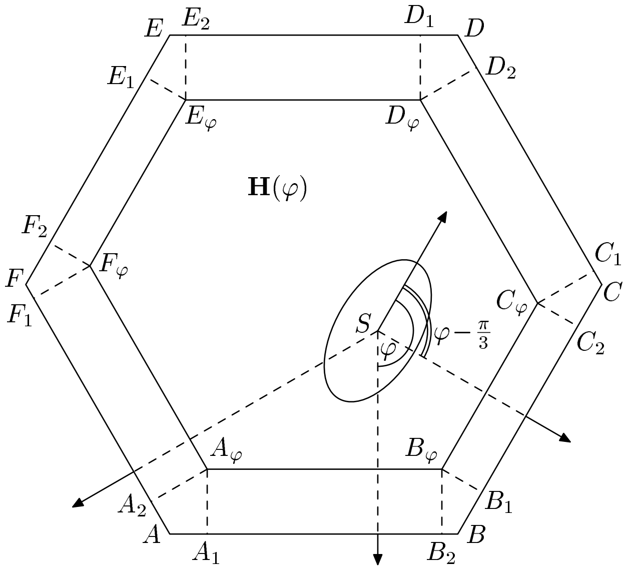

Figure 3

$|\overline{A_\varphi A_1}| = |\overline{B_\varphi B_2}| = pr(\varphi)$, $|\overline{B_\varphi B_1}| = |\overline{C_\varphi C_2}| = pr\left(\varphi - \frac{\pi}{3}\right), \ldots, |\overline{F_\varphi F_1}| = |\overline{A_\varphi A_2}| = pr\left(\varphi - \frac{5\pi}{3}\right) = pr\left(\varphi + \frac{\pi}{3}\right)$. Also, by elementary geometrical considerations:

$$|\overline{A_\varphi B_\varphi}| = a - \frac{2}{\sqrt{3}}\left[pr\left(\varphi - \frac{\pi}{3}\right) + pr\left(\varphi + \frac{\pi}{3}\right) - pr(\varphi)\right],$$

$$|\overline{B_\varphi C_\varphi}| = a - \frac{2}{\sqrt{3}}\left[pr\left(\varphi - \frac{2\pi}{3}\right) + pr(\varphi) - pr\left(\varphi - \frac{\pi}{3}\right)\right],$$

$$|\overline{C_\varphi D_\varphi}| = a - \frac{2}{\sqrt{3}}\left[pr(\varphi - \pi) + pr\left(\varphi - \frac{\pi}{3}\right) - pr\left(\varphi - \frac{2\pi}{3}\right)\right],$$

$$|\overline{D_\varphi E_\varphi}| = a - \frac{2}{\sqrt{3}}\left[pr\Big(\varphi + \frac{2\pi}{3}\Big) + pr\Big(\varphi - \frac{2\pi}{3}\Big) - pr(\varphi - \pi)\right],$$

$$|\overline{E_\varphi F_\varphi}| = a - \frac{2}{\sqrt{3}}\left[pr\Big(\varphi + \frac{\pi}{3}\Big) + pr(\varphi - \pi) - pr\Big(\varphi + \frac{2\pi}{3}\Big)\right],$$

$$|\overline{F_\varphi A_\varphi}| = a - \frac{2}{\sqrt{3}}\left[pr(\varphi) + pr\Big(\varphi + \frac{2\pi}{3}\Big) - pr\Big(\varphi + \frac{\pi}{3}\Big)\right].$$

Since all the angles of the (in general nonregular) hexagon $\mathbf{H}(\varphi)$ with vertices $A_\varphi,\ B_\varphi, \dots, F_\varphi$ have the size $\frac{2\pi}{3}$, the area of $\mathbf{H}(\varphi)$ can be computed as:

$$\text{Area } \mathbf{H}(\varphi) = \frac{3\sqrt{3}\, a^2}{2} - a\left[L(\varphi) + L\Big(\varphi - \frac{\pi}{3}\Big) + L\Big(\varphi + \frac{\pi}{3}\Big)\right]$$

$$+\frac{\sqrt{3}}{3}\left[2\sum_{k=-3}^{2} pr\Big(\varphi + \frac{k\pi}{3}\Big) pr\Big(\varphi + \frac{(k+1)\pi}{3}\Big) - \sum_{k=-3}^{2} pr^2\Big(\varphi + \frac{k\pi}{3}\Big)\right].$$

Therefore we have

$$\begin{aligned}\mu(\mathcal{N}) &= \int_0^{\frac{\pi}{3}} d\varphi \iint_{\{(x,y)\in\mathbf{H}(\varphi)\}} dx dy = \int_0^{\frac{\pi}{3}} \text{Area } \mathbf{H}(\varphi) d\varphi \\ &= \tfrac{\sqrt{3}\,\pi a^2}{2} - a\int_0^{\frac{\pi}{3}}\left[L(\varphi) + L\Big(\varphi - \tfrac{\pi}{3}\Big) + L\Big(\varphi + \tfrac{\pi}{3}\Big)\right] d\varphi \\ &\quad + \tfrac{1}{\sqrt{3}}\int_0^{\frac{\pi}{3}}\left[2\sum_{k=-3}^{2} pr\Big(\varphi + \tfrac{k\pi}{3}\Big) pr\Big(\varphi + \tfrac{(k+1)\pi}{3}\Big) - \sum_{k=-3}^{2} pr^2\Big(\varphi + \tfrac{k\pi}{3}\Big)\right] d\varphi,\end{aligned}$$

and (14) follows by replacing the values for $\mu(\mathcal{M})$ and $\mu(\mathcal{N})$ in $p^{(6)}_{\mathbf{K}} = 1 - \frac{\mu(\mathcal{N})}{\mu(\mathcal{M})}$. □

In the special case when $\mathbf{K} = \mathbf{C}(r)$ with $2r < a$ we have $pr(\varphi) = r$ and $L(\varphi) = 2r$ for all φ, and (14) becomes:

$$p^{(6)}_{\mathbf{C}(r)} = \frac{4}{\sqrt{3}}\,\frac{r}{a} - \frac{4}{3}\left(\frac{r}{a}\right)^2. \tag{15}$$

For a needle of length $l < a$ some cumbersome but straightforward computations lead to the result obtained in 1975 by Stoka (see [6]).

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