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# $\mathfrak{X}$C-Elements in Groups and Dietzmann Classes*

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**Abstract.** Group classes $\mathfrak{D}$ are investigated for which the following implication is generally true:

$$G/\mathbf{C}_G(x^G) \in \mathfrak{D} \text{ and } \langle x \rangle \in \mathfrak{D} \implies \langle x^G \rangle \in \mathfrak{D}.$$

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The classical Dietzmann Lemma and the theory of FC-groups, originally developed by R. Baer and B. H. Neumann (see [2],[10]) form the basis of our studies. Let $\mathfrak{X}$ be any group class (which by definition is supposed to contain the unit group $\{1\}$ and is closed under isomorphisms). We say an element $x$ of a group $G$ has *$\mathfrak{X}$-conjugacy class in $G$*, or is an *$\mathfrak{X}C$-element of $G$*, if $G$ induces on its normal closure $N = \langle x^G \rangle$ an $\mathfrak{X}$-group of automorphisms, that is, $G/\mathbf{C}_G(x^G) \in \mathfrak{X}$, where $\mathbf{C}_G(x^G) = \cap_{g \in G}\mathbf{C}_G(x^g)$ is the centralizer of the conjugacy class $x^G$ of $x$ in $G$. The set of all $\mathfrak{X}C$-elements of $G$, the *$\mathfrak{X}C$-center* of $G$ we denote by $\mathbf{Z}_{\mathfrak{X}}(G)$. If $\mathbf{Z}_{\mathfrak{X}}(G) = G$, the group $G$ is called an *$\mathfrak{X}C$-group.* We also write $\mathfrak{X}C$ to indicate the class of all $\mathfrak{X}C$-groups.

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For the class $\mathfrak{F}$ of finite groups, the $\mathfrak{F}C$-elements are exactly those with only finitely many conjugates, since $G/\mathbf{C}_G(x^G)$ is a finite group if and only if $|x^G| = |G : \mathbf{C}_G(x)|$ is finite. The classical *Dietzmann Lemma* states that the normal closure of an $\mathfrak{F}C$-element of finite order in every group is finite (see [4], [8, p. 3, Prop. 1.A.3], [13, p. 45, Cor. 2], [15, p. 3, Lemma 1.3]). We want to extend the Dietzmann Lemma to other group classes. For this purpose we say, a class $\mathfrak{D}$ is a *Dietzmann class*, if $\mathfrak{D}$ is closed under subgroups and quotient groups and if for every group $G$, the normal closure $N = \langle x^G \rangle$ is a $\mathfrak{D}$-group, whenever $x$ is a $\mathfrak{D}C$-element of $G$ such that $\langle x \rangle \in \mathfrak{D}$. Besides $\mathfrak{F}$, the trivial class $\mathfrak{I} = \{\{1\}\}$ as well as the universal class $\mathfrak{U}$ of all groups are Dietzmann classes. So every group class $\mathfrak{X}$ lies in a unique smallest Dietzmann class $\bar{\mathfrak{X}}$, the *Dietzmann closure of* $\mathfrak{X}$. We suppose that Dietzmann classes are subgroup and quotient group closed ({S,Q}-closed). This is good for the purpose of our studies, avoids abundance of notions and will tacitly (and sometimes fully) be used in the proofs.

It seems that certain extension operators, defined on the universe of all $\{S, Q\}$-closed group classes are of interest for the study of Dietzmann classes: First of all, the operator $Z$ which to every group class $\mathfrak{X}$ associates the class $Z\mathfrak{X}$ of all *central-by-*$\mathfrak{X}$-groups plays a fundamental role: The $Z$-closed classes (that is, $Z\mathfrak{X} = \mathfrak{X}$) among the {S,Q}-closed, we call the *centrally saturated classes*. They are Dietzmann classes for obvious reasons. Extensions of centrally saturated classes by arbitrary {S,Q}-closed ones remain centrally saturated and are therefore Dietzmann classes.

The iterated application of $Z$ to an {S,Q}-closed group class $\mathfrak{X}$ generates an ascending chain of classes which has as its union the centrally saturated closure, the smallest centrally saturated class which contains $\mathfrak{X}$.

Of particular interest, however, are the Dietzmann classes which are not centrally saturated - e.g. the class $\mathfrak{F}_\pi$ of finite $\pi$-groups for any set $\pi$ of prime numbers or the class $\mathfrak{U}_\aleph$ of groups the cardinality of which doesn't exceed a given cardinal number $\aleph \geq \aleph_0$. The operator $Z$ maps Dietzmann classes to Dietzmann classes. So every Dietzmann class $\mathfrak{D}$ is the starting point of an infinite ascending chain of Dietzmann classes, namely the classes $Z^n\mathfrak{D}$ of groups which are $\mathfrak{D}$-groups over the $n$-th term of its upper central series for $n = 0, 1, 2, \ldots$. In particular, the classes $\mathfrak{N}_n$ of all nilpotent groups of nilpotency class $\leq n$ are (non-centrally saturated) Dietzmann classes for all $n = 0, 1, 2, \ldots$.

There are other extension operators which map all or some Dietzmann classes to Dietzmann classes: If $\mathfrak{D}$ is a Dietzmann class, the class $L\mathfrak{D}$ of all local $\mathfrak{D}$-groups is Dietzmann. It follows that every Dietzmann class $\mathfrak{D}$ is the starting point of the 2-fold infinite chain of Dietzmann classes $Z^m L Z^n \mathfrak{D}$ $(m, n = 0, 1, 2, \ldots)$.
If $\mathfrak{D}$ is closed under the product of two normal subgroups or if $\mathfrak{D}$ contains all cyclic groups, the *Fitting class* $F\mathfrak{D}$ over $\mathfrak{D}$ is again Dietzmann. In particular, $L\mathfrak{F}_\pi$ and $F\mathfrak{F}_\pi$ are Dietzmann classes. This permits generalizations of classical results on $\mathfrak{F}C$-groups to $(L\mathfrak{F}_\pi)C$-groups.

Let $C$ denote the operator which to every group class $\mathfrak{X}$ associates the class $\mathfrak{X}C$ of all $\mathfrak{X}C$-groups. Is $\mathfrak{D}C$ always a Dietzmann class if $\mathfrak{D}$ is Dietzmann? We show that $\mathfrak{F}C$ - more general: $\mathfrak{F}_\pi C$ - is a Dietzmann class. $\mathfrak{D}C$ is also Dietzmann if $\mathfrak{D}$ contains all cyclic groups or if $\mathfrak{D}$ is residually closed. It remains open in general.

In contrast to these results, there are important group classes which are not Dietzmann classes: For example, the classes

1) $\mathfrak{T}$ of all torsion groups,
2) $\hat{\mathfrak{M}}$ of all groups with maximum condition on subgroups as well as $\mathfrak{P}$ of polycyclic and $\mathfrak{P}\mathfrak{F}$ of polycyclic-by-finite groups,
3) $\check{\mathfrak{M}}$ of groups with minimum condition on subgroups and $\check{\mathfrak{C}}$ of Černikov groups,
4) $\mathfrak{S}_2$ of metabelian groups. In particular, Dietzmann-by-Dietzmann need not be Dietzmann.

This paper is conceived as an attempt to give a preliminary introduction to the subject of Dietzmann classes and has no claim to completeness. On the contrary, the interesting part lies perhaps in the questions which one may still ask and try to answer: There are a lot of further examples of group classes for which one has to decide whether they have the Dietzmann property or not. A challenging task seems to be: How can one describe the Dietzmann closure $\bar{\mathfrak{X}}$ of a group class $\mathfrak{X}$? Is there a canonical way to do so? We settle the cases when $\mathfrak{X}$ is a set of finite abelian groups or when $\mathfrak{X}$ consists of a nonabelian group of order $pq$ only.

As complementary literature for research of different approach in related directions we may cite for example [6] and [9].

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## Generalities

The following concrete group classes will appear in our considerations:

| | | | |
|---|---|---|---|
| $\mathfrak{A}$ : | *abelian* | $\mathfrak{P}$ : | *polycyclic* |
| $\mathfrak{B}$ : | *Baer* | $\mathfrak{S}$ : | *solvable* |
| $\check{\mathfrak{C}}$ : | *Černikov* | $\mathfrak{S}_2$ : | *metabelian* |
| $\mathfrak{F}$ : | *finite* | $\mathfrak{T}$ : | *torsion* |
| $\mathfrak{N}$ : | *nilpotent* | $\mathfrak{T}_\pi$ : | $\pi$-groups for a set $\pi$ of prime numbers |
| $\mathfrak{N}_n$ : | *nilpotent of class* $\leq n$ | $\mathfrak{Z}$ : | *cyclic* |

$\mathfrak{U}$ is the universal class of all groups, $\mathfrak{U}_\aleph$ the class of all groups of cardinality $\leq \aleph$ for a given cardinal number $\aleph$ and $\mathfrak{I} = \{\{1\}\}$ means the unitary class. By $\mathfrak{D}, \mathfrak{X}, \mathfrak{Y}, \mathfrak{W}$ we indicate variable classes. Moreover, $\mathfrak{X}_\pi = \mathfrak{X} \cap \mathfrak{T}_\pi$, in particular, $\mathfrak{F}_\pi$ is the class of finite $\pi$-groups. $\mathfrak{X}\mathfrak{Y}$ is the class of $\mathfrak{X}$-by-$\mathfrak{Y}$-groups, that is: $G \in \mathfrak{X}\mathfrak{Y} \iff \exists N \trianglelefteq G$ such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$.

A group class $\mathfrak{X}$ is *S-closed, Q-closed, $N_0$-closed* or *$D_0$-closed* if it is closed under *subgroups, quotient groups, product of two normal subgroups* or *direct product of two groups*, respectively.

**Definition.** *Let $\mathfrak{X}$ be a group class, $G$ a group. An element $x \in G$ is an $\mathfrak{X}C$-*element of $G$ *if $G/\mathbf{C}_G(x^G) \in \mathfrak{X}$. The set $\mathbf{Z}_\mathfrak{X}(G)$ of all $\mathfrak{X}C$-elements of $G$ is called the $\mathfrak{X}C$-center of $G$. If $\mathbf{Z}_\mathfrak{X}(G) = G$, the group $G$ is called an $\mathfrak{X}C$-group. The class of all $\mathfrak{X}C$-groups we denote again by $\mathfrak{X}C$.*

In this way, we can interpret $C$ as a right operator $C : \mathfrak{X} \to \mathfrak{X}C$, which to every group class $\mathfrak{X}$ associates the class of all $\mathfrak{X}C$-groups. Inductively we may form $\mathfrak{X}C^{n+1} = (\mathfrak{X}C^n)C$. Nevertheless, it won't cause confusion if we write other classical extension operators (e. g. $Z, L, F, \ldots$) on the left.

**Proposition 1.**

a) *For every group class $\mathfrak{X}$ and every group $G$ we have that $\mathbf{Z}_{\mathfrak{X}}(G)$ is a characteristic subset of $G$. If $\mathfrak{X} \subseteq \mathfrak{Y}$, then*

$$\mathbf{Z}(G) = \mathbf{Z}_{\mathfrak{I}}(G) \subseteq \mathbf{Z}_{\mathfrak{X}}(G) \subseteq \mathbf{Z}_{\mathfrak{Y}}(G) \subseteq \mathbf{Z}_{\mathfrak{U}}(G) = G\ .$$

b) $\dfrac{G}{\mathbf{C}_G(\mathbf{Z}_{\mathfrak{X}}(G))}$ *is residually an $\mathfrak{X}$-group. In particular, if $G = \mathbf{Z}_{\mathfrak{X}}(G)$ is an $\mathfrak{X}C$-group, then $G/\mathbf{Z}(G)$ is residually an $\mathfrak{X}$-group.*

c) *If $\mathfrak{X}$ is Q-closed, a sufficient condition for $G$ being an $\mathfrak{X}C$-group is $G/\mathbf{Z}(G) \in \mathfrak{X}$.*

d) *If $\mathfrak{X}$ is $\{S, Q, D_0\}$-closed, then $\mathbf{Z}_{\mathfrak{X}}(G)$ is a subgroup and $\mathbf{Z}_{\mathfrak{X}}(G)$ is an $\mathfrak{X}C$-group.*

*Proof.* The proofs of a) and c) can be left to the reader.

b) Recall that a group has the property $\mathfrak{X}$ residually if its normal subgroups with $\mathfrak{X}$-quotient meet in the unit subgroup. Now, $\mathbf{C}_G(\mathbf{Z}_{\mathfrak{X}}(G)) = \bigcap_{x\in\mathbf{Z}_{\mathfrak{X}}(G)} \mathbf{C}_G(x^G)$ $G/\mathbf{C}_G(x^G) \in \mathfrak{X}$ $\forall x \in \mathbf{Z}_{\mathfrak{X}}(G)$.

d) Let $x, y \in \mathbf{Z}_{\mathfrak{X}}(G)$ and put $z = xy^{-1}$. Thus $\langle z^G\rangle \le \langle x^G\rangle\langle y^G\rangle$ and $N = \mathbf{C}_G(x^G) \cap \mathbf{C}_G(y^G) \le \mathbf{C}_G(z^G)$. Since $\mathfrak{X}$ is $\{S, D_0\}$-closed, we have $G/N \in \mathfrak{X}$. Therefore,
$\frac{G}{\mathbf{C}_G(z^G)} \cong \frac{G/N}{\mathbf{C}_G(z^G)/N} \in \mathfrak{X}$, as $\mathfrak{X}$ is Q-closed.
Let $x \in H = \mathbf{Z}_{\mathfrak{X}}(G)$ and $E = \mathbf{C}_G(x^G)$. Thus, $HE/E \cong \frac{H}{E\cap H} \in \mathfrak{X}$, that is,
$H/\mathbf{C}_H(x^H) \in \mathfrak{X}$, as $E \cap H \le \mathbf{C}_H(x^H)$ and $\mathfrak{X}$ is Q-closed. Therefore $H$ is an $\mathfrak{X}C$-group.

**Proposition 2.** *Let $\mathfrak{X}$ be a class, $G$ a group, $H \le G$ and $N \trianglelefteq G$. Then the following hold:*

a) *If $\mathfrak{X}$ is $\{S,Q\}$-closed, then $\mathbf{Z}_{\mathfrak{X}}(G) \cap H \subseteq \mathbf{Z}_{\mathfrak{X}}(H)$. In particular, $\mathfrak{X}C$ is S-closed.*

b) *If $\mathfrak{X}$ is Q-closed, then $\{xN \mid x \in \mathbf{Z}_{\mathfrak{X}}(G)\} \subseteq \mathbf{Z}_{\mathfrak{X}}(G/N)$. In particular, $\mathfrak{X}C$ is Q-closed.*

c) *If $\mathfrak{X}$ is $D_0$-closed, then $\mathfrak{X}C$ is $D_0$-closed.*

c') *However, $\mathfrak{F}$ is $N_0$-closed but $\mathfrak{F}C$ is not $N_0$-closed.*

*Proof.* a) Since $H/\mathbf{C}_H(x^H)$ is an image of $H\mathbf{C}_G(x^G)/\mathbf{C}_G(x^G) \le G/\mathbf{C}_G(x^G)$ $\forall x \in H$.
b) Since $\frac{G/N}{\mathbf{C}_{G/N}((xN)^{G/N})}$ is an image of $G/\mathbf{C}_G(x^G)$ $\forall x \in G$.

c) Let $G = A \times B$ and $x = ab \in G$ where $a \in A \in \mathfrak{X}C$ and $b \in B \in \mathfrak{X}C$. Then $x^G = a^A b^B$ and $G/\mathbf{C}_G(x^G)$ is isomorphic to $A/\mathbf{C}_A(a^A) \times B/\mathbf{C}_B(b^B) \in \mathfrak{X}$.

c') See Example 7.

### Dietzmann classes and presentation of some main results

We want to dedicate our attention to the following type of group classes:

**Definition.** *A group class $\mathfrak{D}$ is a Dietzmann class if $\mathfrak{D}$ is $\{S, Q\}$-closed and if for every group $G$ we have the implication*

$$x \in \mathbf{Z}_{\mathfrak{D}}(G) \quad \text{and} \quad \langle x\rangle \in \mathfrak{D} \implies \langle x^G\rangle \in \mathfrak{D}\ .$$

**Remark.** The unitary class $\mathfrak{I}$ and the universal class $\mathfrak{U}$ are Dietzmann classes. The intersection of Dietzmann classes is again Dietzmann, so that every group class $\mathfrak{X}$ is contained in a smallest Dietzmann class $\bar{\mathfrak{X}}$ which contains $\mathfrak{X}$ and which may be called the *Dietzmann closure* of $\mathfrak{X}$.
If $\mathfrak{D}$ contains the infinite cyclic group (i. e. all cyclic groups), the hypothesis $\langle x\rangle \in \mathfrak{D}$ can of course be dropped.
It should also be noticed that for a Dietzmann class $\mathfrak{D}$ there is in general no guarantee or necessity for the $\mathfrak{D}C$-center $\mathbf{Z}_{\mathfrak{D}}(G)$ to be a subgroup.

**Dietzmann's Lemma.** *The class $\mathfrak{F}$ of all finite groups is a Dietzmann class.*

*Proof.* See [15, p. 3, Lemma 1.3].

Not every group class is a Dietzmann class:

**Example 1.**
a) *The class $\mathfrak{Z}$ of all cyclic groups is not a Dietzmann class.*
b) *The class $\mathfrak{S}_2$ of all metabelian as well as $\mathfrak{S}_2 \cap \mathfrak{F}$, the metabelian finite groups are not Dietzmann classes.*
c) *The class $\hat{\mathfrak{M}}$ of groups with maximum condition on subgroups, the classes $\mathfrak{P}$ of polycyclic and $\mathfrak{P}\mathfrak{F}$ of polycyclic-by-finite groups are not Dietzmann classes.*

*Proof.* a) Consider the alternating group $G$ of degree 4 and let $x \in G$ be an involution. Then $N = \langle x^G\rangle \notin \mathfrak{Z}$, but $x$ is a $\mathfrak{Z}C$-element of $G$ with $\langle x\rangle \in \mathfrak{Z}$.
b) Consider the linear group $G = \mathbf{SL}(2,3)$ and let $x \in G$ have order 3. We have $G = N = \langle x^G\rangle \notin \mathfrak{S}_2$, although $G$ is a finite $\mathfrak{S}_2C$-group and $\langle x\rangle \in \mathfrak{Z} \subseteq \mathfrak{S}_2$.
c) By the *restricted regular wreath product* $G = H \wr K$ of the groups $H$ and $K$ we mean the semidirect product $G = [B]K$ where $B$ is the group of all functions of finite support from $K$ to $H$ under pointwise multiplication, the action of $K$ on $B$ being given by: $\forall f \in B$, $\forall k \in K$: $f^k(x) = f(kx)\ \ \forall x \in K$.
Let now $p$ be a prime number and consider the restricted regular wreath product $G = C_p \wr C_\infty$ of a group of order $p$ with the infinite cyclic group. There exists $f \in B$ such that $G/\mathbf{C}_G(f^G)$ is cyclic but $\langle f^G\rangle = B$ doesn't satisfy the maximum condition.

The fact that $\mathfrak{P}\mathfrak{F}$ is not a Dietzmann class is interesting in view of the following result due to Franciosi, de Giovanni and Tomkinson:

**Result 1.** (See [5, p. 37, Thm.2.2])

$$G \in (\mathfrak{P}\mathfrak{F})C \iff \langle x^G\rangle \in \mathfrak{P}\mathfrak{F} \quad \forall x \in G$$

(in particular, if every $x \in G$ has $\mathfrak{P}\mathfrak{F}$-conjugacy class, then every $x$ generates a $\mathfrak{P}\mathfrak{F}$ normal subgroup).

Easy to see, but of some surprise is

**Example 2.** *The class $\mathfrak{A}$ of all abelian groups is a Dietzmann class.*

*Proof.* Let $G$ be a group and $x \in \mathbf{Z}_{\mathfrak{A}}(G)$. Clearly $\langle x \rangle \in \mathfrak{A}$. We have to show that $\langle x^G \rangle$ is abelian. Let $N = \langle x^G \rangle$ and $E = \mathbf{C}_G(N)$. Thus $G/E \in \mathfrak{A}$, whence $E = \mathbf{C}_G(x) \trianglelefteq G$. It follows that $x^{g_1} \in \mathbf{C}_G(x^{g_2})$ $\forall g_1, g_2 \in G$. So any two generators of $N$ commute and $N$ has to be abelian.

Combining Example 2 with Dietzmann's Lemma, we see that also $\mathfrak{A} \cap \mathfrak{F}$, the finite abelian groups form a Dietzmann class. Example 1 c) shows, however, that $\mathfrak{A} \cap \hat{\mathfrak{M}}$, the class of finitely generated abelian groups is not a Dietzmann class.
In particular, sub- or upper classes of Dietzmann classes may lose the Dietzmann property. Also Dietzmann-by-Dietzmann (for example abelian-by-abelian) need not be Dietzmann.

A slightly more concealed result is:

**Theorem A.** *The class of all $\mathfrak{F}C$-groups is a Dietzmann class.*

The proof of this theorem, which simply states the general truth of the implication

$$G/\mathbf{C}_G(x^G) \in \mathfrak{F}C \implies \langle x^G \rangle \in \mathfrak{F}C$$

for an arbitrary group $G$ and any $x \in G$, is a combination of the classical Baer result "$\mathfrak{F}C$-groups are torsion groups over its center" [2], with the following

**Fundamental Lemma.** *Let $G$ be a group and $x \in G$. We put $N = \langle x^G \rangle$ and $E = \mathbf{C}_G(N)$. Let $K/E = \mathbf{Z}(G/E)$. Then $K \cap \langle x \rangle \leq \mathbf{Z}(N)$.*

We shall prove this crucial remark below.

Other main results which should be emphasized in this section show that classical extension operators on group classes respect in general or in part the eventual Dietzmann property of a class. Namely

**Theorem B.** *Let $\mathfrak{D}$ be any Dietzmann class. Then the classes*
- a) *$L\mathfrak{D}$ of the local $\mathfrak{D}$-groups and*
- b) *$Z\mathfrak{D}$ of the central-by-$\mathfrak{D}$-groups*

*are Dietzmann classes again.*

**Theorem C.** *Let $\mathfrak{D}$ be a Dietzmann class.*
- a) *If $\mathfrak{Z} \subseteq \mathfrak{D}$ or if $\mathfrak{D}$ is $N_0$-closed, then the Fitting class $F\mathfrak{D}$ over $\mathfrak{D}$ is again a Dietzmann class.*
- b) *If $\mathfrak{Z} \subseteq \mathfrak{D}$ or $\mathfrak{D}$ is residually closed, then the class of $\mathfrak{D}C$-groups is Dietzmann.*

As a consequence we can state:

**Theorem D.** *The following group classes are Dietzmann classes:*
- a) *For every $n \geq 0$, the class $\mathfrak{N}_n$ of nilpotent groups of nilpotency class $\leq n$ .*
- b) *The locally finite groups $L\mathfrak{F}$.*
- c) *The locally normal finite groups $F\mathfrak{F}$*
  *(= the torsion groups among the $\mathfrak{F}C$-groups).*

d) *The locally solvable torsion groups* $L\mathfrak{S} \cap \mathfrak{T}$.

However, the class $\mathfrak{T}$ of all torsion groups is not a Dietzmann class.

In the next section we first want to settle a type of "large" group classes which have the Dietzmann property for obvious reasons.

## The extension operator $Z$ and centrally saturated classes

**Definition.**

a) *For every* $\{S, Q\}$*-closed group class* $\mathfrak{X}$ *we denote by*

$$Z\mathfrak{X} = \{G \mid G/\mathbf{Z}(G) \in \mathfrak{X}\},$$

*the class of groups which are* $\mathfrak{X}$*-groups over its center. Obviously* $Z\mathfrak{X}$ *is again* $\{S, Q\}$*-closed and for every nonnegative integer* $n$*, the* $n$*-fold application of* $Z$ *associates to every* $\mathfrak{X}$ *the class*

$$Z^n\mathfrak{X} = \{G \mid G/\mathbf{Z}_n(G) \in \mathfrak{X}\},$$

*the groups which are* $\mathfrak{X}$*-groups over the* $n$*-th term of its upper central series.*

b) *An* $\{S, Q\}$*-closed group class* $\mathfrak{X}$ *is centrally saturated if* $Z\mathfrak{X} = \mathfrak{X}$*, i.e. if it is also* $Z$*-closed in the usual sense* $(G/\mathbf{Z}(G) \in \mathfrak{X} \iff G \in \mathfrak{X})$.

Examples of centrally saturated classes are $\mathfrak{U}$, $\mathfrak{N}$ and $\mathfrak{S}$. But also each of the following classes is centrally saturated:

1) The groups with normalizer condition ( $\forall S < G$ we have $S < \mathbf{N}_G(S)$),

2) the groups which have all subgroups subnormal,

3) the Baer groups ($\langle x \rangle \trianglelefteq G \ \forall x \in G$),

4) the hypercentral groups ( $\forall N \triangleleft G\ \exists x \in G \setminus N$ such that $[G, x] \leq N$),

5) the hyper-$\mathfrak{X}$-groups ( $\forall N \triangleleft G\ \exists N < X \trianglelefteq G$ such that $X/N \in \mathfrak{X}$ ); here $\mathfrak{X}$ is any $\{S, Q\}$-closed class with $\mathfrak{Z} \subseteq \mathfrak{X}$. In particular, when $\mathfrak{X} = \mathfrak{Z}$, we get

6) the hypercyclic groups (= the supersolvable groups in the general sense of Baer [3]).

Every class $\mathfrak{X}$ is contained in a smallest centrally saturated class, the *centrally saturated closure of* $\mathfrak{X}$ which is obtained in an obvious way:

**Proposition 3.** *Let* $\mathfrak{X}$ *be an* $\{S, Q\}$*-closed group class.*

a) $\mathfrak{X} = Z^0\mathfrak{X} \subseteq Z\mathfrak{X} = Z^1\mathfrak{X} \subseteq Z^2\mathfrak{X} \subseteq \cdots \subseteq Z^n\mathfrak{X} \subseteq \cdots$
*is an ascending chain of* $\{S, Q\}$*-closed group classes.*

b) $Z^\infty\mathfrak{X} = \bigcup_{n=1}^{\infty} Z^n\mathfrak{X} = \{G \mid G/\mathbf{Z}_n(G) \in \mathfrak{X} \text{ for some } n\}$ *is the centrally saturated closure of* $\mathfrak{X}$.

*Proof.* Can be left to the reader.

**Example 3.** For the unitary class $\mathfrak{I}$ we obtain:

$Z^n\mathfrak{I} = \mathfrak{N}_n$ is the class of nilpotent groups of nilpotency class $\leq n$ and
$Z^\infty\mathfrak{I} = \mathfrak{N}$ is the class of all nilpotent groups.

For the centrally saturated classes we have:

**Proposition 4.** *Let $\mathfrak{W}$ be a centrally saturated class. Then:*
a) $\mathfrak{N} \subseteq \mathfrak{W}$, *in particular* $\mathfrak{Z} \subseteq \mathfrak{W}$.
b) $\mathfrak{W}$ *is a Dietzmann class.*
c) $\mathfrak{W}\mathfrak{X}$ *is centrally saturated for every class* $\mathfrak{X}$ *which is* $\{S, Q\}$*-closed. In particular,* $\mathfrak{W}\mathfrak{X}$ *is Dietzmann.*

*Proof.* a) Certainly $\{1\} \in \mathfrak{W}$. Let $G$ be a nilpotent group of class $\leq n$. By induction, $G/\mathbf{Z}(G) \in \mathfrak{W}$. So $G \in Z\mathfrak{W} = \mathfrak{W}$.
b) By definition, $\mathfrak{W}$ is $\{S, Q\}$-closed. Let $G$ be any group and $x \in \mathbf{Z}_{\mathfrak{W}}(G)$. We put $N = \langle x^G \rangle$ and $E = \mathbf{C}_G(N)$. Thus, $G/E \in \mathfrak{W}$ and therefore $N/\mathbf{Z}(N) \cong EN/E \in \mathfrak{W}$. Hence $N \in \mathfrak{W}$.
c) Certainly $\mathfrak{W}\mathfrak{X}$ is $\{S, Q\}$-closed. Suppose $G/\mathbf{Z}(G) \in \mathfrak{W}\mathfrak{X}$. There exists $N/\mathbf{Z}(G) \trianglelefteq G/\mathbf{Z}(G)$ such that $N/\mathbf{Z}(G) \in \mathfrak{W}$ and $G/N \in \mathfrak{X}$. From $N/\mathbf{Z}(N) \in \mathfrak{W}$ it follows $N \in Z\mathfrak{W} = \mathfrak{W}$. So $G \in \mathfrak{W}\mathfrak{X}$.

**Caution.** The fact that arbitrary extensions of a centrally saturated class remain centrally saturated and are therefore Dietzmann classes, should not mislead one to the conclusion that an $\{S, Q\}$-closed class which simply contains a centrally saturated class is Dietzmann or centrally saturated. This is wrong:
Consider $\mathfrak{X} = \mathfrak{N} \cup \mathfrak{F}$ and $G = \langle a \rangle \times F$, where $a$ has infinite order and $F$ is a finite nonabelian simple group. Let $x = (a, y_0) \in G$ with $y_0 \neq 1$. We have $G/\mathbf{Z}(G) \cong F \in \mathfrak{F} \subseteq \mathfrak{X}$ and $\langle x \rangle \in \mathfrak{Z} \subseteq \mathfrak{N} \subseteq \mathfrak{X}$, but $\langle x^G \rangle \notin \mathfrak{X}$, since $\langle x^G \rangle$ is neither finite nor nilpotent. So this $\mathfrak{X}$ is not Dietzmann, although it contains the centrally saturated class $\mathfrak{N}$.

**Consequence 1.** *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be any $\{S, Q\}$-closed classes of groups. Then*

$$(Z^\infty\mathfrak{X})\mathfrak{Y},$$

*the class of $(Z^\infty\mathfrak{X})$-by-$\mathfrak{Y}$-groups, is a centrally saturated and therefore a Dietzmann class.*

**Remark.** We always have $Z^\infty(\mathfrak{X}\mathfrak{Y}) \subseteq (Z^\infty\mathfrak{X})\mathfrak{Y}$. But equality doesn't hold in general: For $\mathfrak{X} = \mathfrak{I}$ and $\mathfrak{Y} = \mathfrak{F}_{\{2\}}$ we have $Z^\infty(\mathfrak{X}\mathfrak{Y}) = Z^\infty(\mathfrak{I}\mathfrak{F}_{\{2\}}) = \mathfrak{N} \subset \mathfrak{N}\mathfrak{F}_{\{2\}} = (Z^\infty\mathfrak{I})\mathfrak{F}_{\{2\}} = (Z^\infty\mathfrak{X})\mathfrak{Y}$, as $G \in \mathfrak{N}\mathfrak{F}_{\{2\}} \setminus \mathfrak{N}$ when $G$ is the symmetric group of degree 3.

### The operator $Z$ and non-centrally saturated Dietzmann classes

$\mathfrak{F}$ is a non-centrally saturated Dietzmann class. First of all, Dietzmann's Lemma can be extended as:

**Proposition 5.**
a) *For every cardinal number $\aleph \geq \aleph_0$, the class $\mathfrak{U}_\aleph$ of all groups of cardinality $\leq \aleph$, is a Dietzmann class.*

b) *For every set $\pi$ of prime numbers, $\mathfrak{F}_\pi$ is a Dietzmann class.*

*Proof.* a) We have $\mathfrak{Z} \subseteq \mathfrak{U}_\aleph$. Let $G$ be a group and $x \in G$ such that $G/\mathbf{C}_G(x^G) \in \mathfrak{U}_\aleph$. Then of course also $N = \langle x^G \rangle \in \mathfrak{U}_\aleph$, as the generating system $x^G$ of $N$ is of cardinality $|x^G| = |G : \mathbf{C}_G(x)| = |\{\mathbf{C}_G(x)y \mid y \in G\}| \leq |G/\mathbf{C}_G(x^G)| \leq \aleph$.

b) Let $G$ be a group, $x \in G$ with $\langle x \rangle \in \mathfrak{F}_\pi$ and suppose $G/E \in \mathfrak{F}_\pi$ where $E = \mathbf{C}_G(N)$ and $N = \langle x^G \rangle$. By Dietzmann's Lemma, $N$ is finite. Moreover, $N/\mathbf{Z}(N) \cong EN/E$ is a $\pi$-group. So the $\pi$-elements of $N$ form a characteristic subgroup of $N$ which has to contain the generators $x^g$ of $N$ for all $g \in G$. Therefore $N$ is a $\pi$-group.

We now come to the proof of our

**Fundamental Lemma.** *Let $G$ be a group and $x \in G$. We put $N = \langle x^G \rangle$ and $E = \mathbf{C}_G(N)$. Let $K/E = \mathbf{Z}(G/E)$. Then $K \cap \langle x \rangle \leq \mathbf{Z}(N)$.*

*Proof.* We have $E \cap N = \mathbf{Z}(N)$. We put $Y = K \cap \langle x \rangle$. Now $YE/E \leq \mathbf{Z}(G/E)$ and therefore $YE \trianglelefteq G$. Thus $Y\mathbf{Z}(N) = N \cap YE \trianglelefteq G$. For all $g \in G$ we have therefore $Y \leq Y\mathbf{Z}(N) = (Y\mathbf{Z}(N))^g = Y^g\mathbf{Z}(N) \leq \langle x^g \rangle \mathbf{Z}(N) \leq N$. Since $\langle x^g \rangle \mathbf{Z}(N)$ is an abelian group, we conclude that $N = \langle x^G \rangle$ centralizes $Y$. So $Y \leq E \cap \langle x \rangle \leq \mathbf{Z}(N)$.

As a first application of the Fundamental Lemma we provide a proof of our Theorem B b) which states that the operator $Z$ is generally "Dietzmann-friendly":

**Proposition 6.** *Let $\mathfrak{D}$ be a Dietzmann class. Then $Z\mathfrak{D}$ is a Dietzmann class.*

*Proof.* $Z\mathfrak{D}$ is closed under subgroups and homomorphic images. Let $G$ be a group and $x \in \mathbf{Z}_{Z\mathfrak{D}}(G)$. Clearly $\langle x \rangle \in \mathfrak{Z} \subseteq Z\mathfrak{D}$. We have to show that $\langle x^G \rangle \in Z\mathfrak{D}$. We put $N = \langle x^G \rangle$ and $E = \mathbf{C}_G(N)$. Thus $E \cap N = \mathbf{Z}(N)$ and $G/E \in Z\mathfrak{D}$. Let $K/E = \mathbf{Z}(G/E)$, so that $G/K \in \mathfrak{D}$. With $Y = \langle x \rangle \cap K$ we have $\langle x \rangle / Y \cong K\langle x \rangle / K \in \mathfrak{D}$, as $\mathfrak{D}$ is subgroup-closed. By the Fundamental Lemma, $Y \leq E \cap \langle x \rangle$. Since $\langle x \rangle / Y \in \mathfrak{D}$, the cyclic group $\langle xE \rangle = \langle x \rangle E/E \cong \langle x \rangle / E \cap \langle x \rangle$ is also a $\mathfrak{D}$-group. Since $G/E$ is a $\mathfrak{D}C$-group and $\mathfrak{D}$ is a Dietzmann class, this means $N/\mathbf{Z}(N) \cong NE/E = \langle (xE)^{G/E} \rangle \in \mathfrak{D}$. Therefore $N \in Z\mathfrak{D}$.

**Corollary.** *If $\pi$ is a set of prime numbers, for the Dietzmann class $\mathfrak{F}_\pi$ of all finite $\pi$-groups we have the proper chain of Dietzmann classes*

$$\mathfrak{F}_\pi \subset Z\mathfrak{F}_\pi \subset \cdots \subset Z^n\mathfrak{F}_\pi \subset \cdots .$$

*In particular, if $\pi$ is empty, that is, $\mathfrak{F}_\emptyset = \mathfrak{I}$, this chain becomes*

$$\mathfrak{I} = \mathfrak{N}_0 \subset \mathfrak{N}_1 = \mathfrak{A} \subset \mathfrak{N}_2 \subset \cdots \subset \mathfrak{N}_n \subset \cdots$$

*and we see that for each $n \geq 0$, the classes $\mathfrak{N}_n$ of nilpotent groups of nilpotency class $\leq n$ form Dietzmann classes.*

So Theorem D a) is proved.

**The local operator $L$**

**Definition.** *Let $\mathfrak{X}$ be an $\{S,Q\}$-closed group class. We denote by $L\mathfrak{X}$ the class of local $\mathfrak{X}$-groups, that is,*

$$G \in L\mathfrak{X} \iff \langle S\rangle \in \mathfrak{X} \quad \textit{for every finite subset } S \subseteq G.$$

(See [8], [13]). Also the operator $L$ is generally Dietzmann-friendly: Our Theorem B a) is identical with

**Proposition 7.** *Let $\mathfrak{D}$ be a Dietzmann class. Then $L\mathfrak{D}$ is a Dietzmann class.*

*Proof.* Clearly $L\mathfrak{D}$ is closed under subgroups and homomorphic images. Let $G$ be a group and $x \in \mathbf{Z}_{L\mathfrak{D}}(G)$ such that $\langle x\rangle \in L\mathfrak{D}$, that is, $\langle x\rangle \in \mathfrak{D}$. Let $N = \langle x^G\rangle$ and $E = \mathbf{C}_G(N)$. Thus $G/E \in L\mathfrak{D}$. We have to show that $N \in L\mathfrak{D}$ :
Let $S$ be a finite subset of $N$. Then $\langle S\rangle \leq \langle x, x^{g_1}, x^{g_2}, \ldots, x^{g_n}\rangle$ for certain $g_1, g_2, \ldots, g_n \in G$. We consider $G_0 = \langle E; x, g_1, \ldots, g_n\rangle$ and have that the finitely generated subgroup $G_0/E$ of $G/E$ is a $\mathfrak{D}$-group. Put $N_0 = \langle x^{G_0}\rangle$. Since $E \leq E_0 = \mathbf{C}_{G_0}(N_0)$, we see that $G_0/E_0 \in \mathfrak{D}$ which means that $x$ is a $\mathfrak{D}C$-element of $G_0$. Since $\langle x\rangle \in \mathfrak{D}$ and $\mathfrak{D}$ is Dietzmann, we conclude that $N_0 \in \mathfrak{D}$. But $\langle S\rangle \leq \langle x, x^{g_1}, \ldots, g^{g_n}\rangle \leq N_0$, so that $\langle S\rangle \in \mathfrak{D}$. This means $N \in L\mathfrak{D}$.

By Theorem B b), $Z^n\mathfrak{D}$ $(n \geq 0)$, are Dietzmann classes for every Dietzmann class $\mathfrak{D}$. Moreover, combining a) and b) of Theorem B, we see that every Dietzmann class is the starting point of a 2-fold infinite series of Dietzmann classes:

**Consequence 2.** *For every Dietzmann class $\mathfrak{D}$, all members in the scheme*

$$\begin{array}{ccccccccc}
L\mathfrak{D} & \subseteq & ZL\mathfrak{D} & \subseteq & \cdots & \subseteq & Z^mL\mathfrak{D} & \subseteq \cdots \\
\cap\!\! | & & \cap\!\! | & & & & \cap\!\! | & \\
LZ\mathfrak{D} & \subseteq & ZLZ\mathfrak{D} & \subseteq & \cdots & \subseteq & Z^mLZ\mathfrak{D} & \subseteq \cdots \\
\cap\!\! | & & \cap\!\! | & & & & \cap\!\! | & \\
\vdots & \cdots & \vdots & & \cdots & & \vdots & \cdots \\
\cap\!\! | & & \cap\!\! | & & & & \cap\!\! | & \\
LZ^n\mathfrak{D} & \subseteq & ZLZ^n\mathfrak{D} & \subseteq & \cdots & \subseteq & Z^mLZ^n\mathfrak{D} & \subseteq \cdots \\
\cap\!\! | & & \cap\!\! | & & & & \cap\!\! | & \\
\vdots & \cdots & \vdots & & \cdots & & \vdots & \cdots
\end{array}$$

*are Dietzmann classes.*

**Remark.** We have for all $\{S,Q\}$-closed classes $\mathfrak{X}$ : $Z^nL\mathfrak{X} \subseteq LZ^n\mathfrak{X}$. Since $L$ is idempotent, we get therefore $LZ^nL\mathfrak{X} \subseteq LZ^n\mathfrak{X}$. But $L\mathfrak{X} \supseteq \mathfrak{X}$ implies also $LZ^nL\mathfrak{X} \supseteq LZ^n\mathfrak{X}$. Thus $LZ^nL = LZ^n$. It follows $Z^mLZ^{n_1}LZ^{n_2}L\cdots Z^{n_r}L = Z^mLZ^{n_1+n_2+\ldots+n_r}$ for all nonnegative exponents $m, n_1, \ldots, n_r$.
The classes $Z^nL\mathfrak{X}$ and $LZ^n\mathfrak{X}$ are distinct in general. For example, $ZL\mathfrak{F} \subset LZ\mathfrak{F}$:

Let $2 < p_1, p_2, \ldots, p_k, \ldots$ be a sequence of distinct prime numbers. Let $H = C_{p_1} \times C_{p_2} \times \cdots \times C_{p_k} \times \cdots$ denote the restricted direct product of groups of order $p_k$. Let $\varepsilon_k$ be a primitive $(p_k - 1)th$ root of unity mod $p_k$. For $(x_1, x_2, \ldots, x_k, \ldots) \in H$ define $(x_1, x_2, \ldots, x_k, \ldots)^a = (x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_k^{\varepsilon_k}, \ldots)$. Then $a$ is an automorphism of $H$ of infinite order. For the group $G = [H]\langle a \rangle$ we have $\mathbf{Z}(G) = 1$ and it follows $G \in LZ\mathfrak{F} \setminus ZL\mathfrak{F}$.

### The Fitting operator $F$

**Definition.** *For every $\{S, Q\}$-closed class $\mathfrak{X}$ we denote by $F\mathfrak{X}$ the Fitting class over $\mathfrak{X}$, that is,*

$$G \in F\mathfrak{X} \quad \Longleftrightarrow \quad \langle x^G \rangle \in \mathfrak{X} \quad \forall x \in G.$$

**Remark.** For $G \in F\mathfrak{X}$ we say that $G$ is a *Fitting-$\mathfrak{X}$-group.* If $\mathfrak{X}$ is also $N_0$-closed, every finite set of elements of a group in $F\mathfrak{X}$ generates a normal $\mathfrak{X}$-subgroup. In this case $F\mathfrak{X} \subseteq L\mathfrak{X}$ and the groups in $F\mathfrak{X}$ are known in the literature as the "locally normal (and) $\mathfrak{X}$-groups" ([13, p. 36], [8, p. 3/4]). The Fitting-$\mathfrak{N}$-groups are exactly the groups which are generated by their nilpotent normal subgroups and are known as the "Fitting groups".

Theorem C a) is identical with

**Proposition 8.** *Let $\mathfrak{D}$ be a Dietzmann class. Suppose that $\mathfrak{D}$ is $N_0$-closed or $\mathfrak{Z} \subseteq \mathfrak{D}$. Then $F\mathfrak{D}$ is a Dietzmann class.*

*Proof.* Clearly $F\mathfrak{D}$ is $\{S, Q\}$-closed. Let $G$ be a group and $x \in \mathbf{Z}_{F\mathfrak{D}}(G)$ such that $\langle x \rangle \in F\mathfrak{D}$, that is, $\langle x \rangle \in \mathfrak{D}$. We have to show that $\langle x^G \rangle \in F\mathfrak{D}$. We put $N = \langle x^G \rangle$ and $E = \mathbf{C}_G(N)$. Since $G/E \in F\mathfrak{D}$, we see that $N/\mathbf{Z}(N) \cong NE/E = \langle (xE)^{G/E} \rangle \in \mathfrak{D}$. In particular, $N$ is a $\mathfrak{D}C$-group. Let $y \in N$.
(i) If $\mathfrak{Z} \subseteq \mathfrak{D}$, then $\langle y \rangle \in \mathfrak{D}$ and we conclude $\langle y^N \rangle \in \mathfrak{D}$ by the Dietzmann property of $\mathfrak{D}$.
(ii) Let $\mathfrak{D}$ be $N_0$-closed. Since $\langle x \rangle \in \mathfrak{D}$, we conclude $\langle x^N \rangle \in \mathfrak{D}$. So $N \trianglerighteq \langle x^N \rangle^g \in \mathfrak{D}$ for all $g \in G$. There exist $g_1, g_2, \ldots, g_n \in G$ such that $y \in \prod_{i=1}^{n} \langle x^N \rangle^{g_i} \in \mathfrak{D}$, as $\mathfrak{D}$ is $N_0$-closed. Therefore $\langle y \rangle \in \mathfrak{D}$ and it follows that $\langle y^N \rangle \in \mathfrak{D}$.
So $N \in F\mathfrak{D}$ in both cases.

**Consequence 4.** *Let $\mathfrak{D}$ be any Dietzmann class which is $N_0$-closed or which contains all cyclic groups. Then all members in the chains*

$$F\mathfrak{D} \subseteq FZ\mathfrak{D} \subseteq \cdots \subseteq FZ^n\mathfrak{D} \subseteq \cdots$$

*and*

$$F\mathfrak{D} \subseteq ZF\mathfrak{D} \subseteq \cdots \subseteq Z^nF\mathfrak{D} \subseteq \cdots$$

*are Dietzmann classes again.*

### Some Dietzmann classes of torsion groups

**Proposition 9.** *Let $\pi$ be a set of prime numbers. The following classes of torsion groups have the Dietzmann property:*

a) *The class* $L\mathfrak{F}_\pi$ *of locally finite* $\pi$*-groups.*

b) *The Fitting class* $F\mathfrak{F}_\pi$ *over the finite* $\pi$*-groups*
*(= the locally normal finite* $\pi$*-groups = the* $\pi$*-groups among the* $\mathfrak{F}C$*-groups).*

c) *The class* $L\mathfrak{S} \cap \mathfrak{T}_\pi$ *of locally solvable* $\pi$*-groups.*

*Proof.* For a) it suffices to remark that $\mathfrak{F}_\pi$ is a Dietzmann class by Proposition 5 b). Thus Theorem B a) applies.
b) $\mathfrak{F}_\pi$ is also $N_0$-closed. So Theorem C a) applies.
c) Since locally solvable torsion groups are locally finite, we have that $L\mathfrak{S} \cap \mathfrak{T}_\pi = L\mathfrak{S} \cap L\mathfrak{F} \cap \mathfrak{T}_\pi = L\mathfrak{S} \cap L\mathfrak{F}_\pi$ is the intersection of two Dietzmann classes and therefore Dietzmann.

In particular, for $\pi = \{\textit{all primes}\}$, Theorem D b) c) d) is shown.

The classes in Proposition 9 are special classes of torsion groups. This is remarkable in view of the fact that the class $\mathfrak{T}$ of all torsion groups is not a Dietzmann class. The following example was pointed out to us by Prof. P. Shumyatsky:

**Example 4.** *The class* $\mathfrak{T}$ *of all torsion groups is not a Dietzmann class.*

*Proof.* There exists a nonabelian torsionfree group $H$ for which $H/\mathbf{Z}(H)$ is a torsion group (see [1, Ch. VII, §1, p. 269 ff.]). Let $x$ be a nontrivial inner automorphism of $H$ and consider the semidirect product $G = [H]\langle x\rangle$. Then $\langle x\rangle \in \mathfrak{T}$ and $G/\mathbf{C}_G(x^G) \in \mathfrak{T}$, but $\langle x\rangle < \langle x^G\rangle = \langle x\rangle(H \cap \langle x^G\rangle) \notin \mathfrak{T}$.

Also other important classes of torsion groups are not Dietzmann classes:

**Example 5.** *The class* $\check{\mathfrak{M}}$ *of groups with minimum condition on subgroups as well as the class* $\check{\mathfrak{C}}$ *of the Černikov groups are not Dietzmann classes.*

*Proof.* Recall that a group is a Černikov group if it is abelian-by-finite with the minimum condition on its subgroups, that is, $\check{\mathfrak{C}} = \check{\mathfrak{M}} \cap \mathfrak{A}\mathfrak{F}$. Let $p$ be a prime number. In the restricted regular wreath product $G = C_p \wr C_{p^\infty}$, of a group of order $p$ with the $p$-Prüfer group, there exists an element $f \in B$ of order $p$ such that $G/\mathbf{C}_G(f^G) \cong C_{p^\infty} \in \check{\mathfrak{C}}$, while $\langle f^G\rangle = B$ doesn't satisfy the minimum condition.

Also this example (compare with Example 1 c)) is interesting in view of the following Polovickii-Theorem:

**Result 2.** (See [12], [13, p. 127, Thm.4.36])

$$G/\mathbf{C}_G(x^G) \in \check{\mathfrak{C}} \;\; \forall x \in G \implies \langle x^G\rangle \in \check{\mathfrak{C}} \;\; \textit{for every torsion element } x \textit{ of } G$$

(i.e. if all elements of $G$ have Černikov conjugacy classes, then all torsion elements generate Černikov normal subgroups).

The same example also shows that $\mathfrak{A} \cap \check{\mathfrak{M}}$, the abelian groups with minimum condition do not form a Dietzmann class, whereas the torsion abelian groups $\mathfrak{A} \cap \mathfrak{T} = \mathfrak{A} \cap L\mathfrak{F}$ do.

The fact that $L\mathfrak{F}_\pi$ is a Dietzmann class, permits the generalization of classical results on $\mathfrak{F}C$-groups for $(L\mathfrak{F}_\pi)C$-groups (see [10, p. 184, 5.1]). Recall here that Dietzmann's Lemma can be interpreted by the equality $\mathfrak{F}C \cap \mathfrak{T} = F\mathfrak{F}$.

**Proposition 10.** *Let $\pi$ be a set of prime numbers, $G$ a group and $\mathbf{T}_\pi(G)$ the set of the $\pi$-elements of $G$. Then*

a) *$\mathbf{T}_\pi(G) \cap \mathbf{Z}_{L\mathfrak{F}_\pi}(G)$, the $\pi$-elements of the $(L\mathfrak{F}_\pi)C$-center of $G$, form a characteristic subgroup of $G$. In particular, if $G$ is an $(L\mathfrak{F}_\pi)C$-group, then $\mathbf{T}_\pi(G)$ is a subgroup.*

b) *$(L\mathfrak{F}_\pi)C \cap \mathfrak{T}_\pi = L\mathfrak{F}_\pi$.*

c) *However, if $\pi \neq \emptyset$, we have $(F\mathfrak{F}_\pi)C \cap \mathfrak{T}_\pi \supset F\mathfrak{F}_\pi$.*

*Proof.* a) Let $x, y \in \mathbf{T}_\pi(G) \cap \mathbf{Z}_{L\mathfrak{F}_\pi}(G)$. Thus $\langle x^G \rangle$ and $\langle y^G \rangle$ are locally finite $\pi$-groups as $L\mathfrak{F}_\pi$ is Dietzmann. Since $L\mathfrak{F}_\pi$ is $N_0$-closed (see [8, p. 2, 1.A.2], [13, p. 35, Th. 1.45]), $xy^{-1} \in \langle x^G \rangle \langle y^G \rangle \subseteq \mathbf{T}_\pi(G) \cap \mathbf{Z}_{L\mathfrak{F}_\pi}(G)$.

b) "$\supseteq$" is clear. For $x_1, \dots, x_r \in G \in (L\mathfrak{F}_\pi)C \cap \mathfrak{T}_\pi$, we have $x_1, \dots, x_r \in \langle x_1^G \rangle \cdots \langle x_r^G \rangle \in L\mathfrak{F}_\pi$, so that $\langle x_1, \dots, x_r \rangle \in \mathfrak{F}_\pi$. This means $G \in L\mathfrak{F}_\pi$.

c) Certainly $(F\mathfrak{F}_\pi)C \cap \mathfrak{T}_\pi \supseteq F\mathfrak{F}_\pi$. We have to construct an example of a group $G \in (F\mathfrak{F}_\pi)C \cap \mathfrak{T}_\pi \setminus F\mathfrak{F}_\pi$: For $p \in \pi$ consider $G = [A]\langle y \rangle$ where $A$ is any infinite direct product of cyclic groups of order $p^2$ and $a^y = a^{p+1}$ $\forall a \in A$. Then $G \in (F\mathfrak{F}_\pi)C \cap \mathfrak{T}_\pi$, but $G \notin F\mathfrak{F}_\pi$. Indeed we have that $y$ is of order $p$ so that $G \in \mathfrak{T}_\pi$. Moreover $\mathbf{Z}(G) = \{a^p \mid a \in A\} = G' = [A, y]$. Therefore, $G/\mathbf{Z}(G) \in \mathfrak{A} \cap \mathfrak{T}_\pi \subseteq F\mathfrak{F}_\pi$, whence $G \in (F\mathfrak{F}_\pi)C$. But $\langle y^G \rangle = [A, y]\langle y \rangle \notin \mathfrak{F}$, so that $G \notin F\mathfrak{F}_\pi$.

We also have

**Proposition 11.** *For every $(L\mathfrak{F})C$-group $G$, the commutator subgroup is locally finite and the set $\mathbf{T}(G)$ of all torsion elements forms a characteristic subgroup with abelian factorgroup.*

*Proof.* By Proposition 10 a), looking at the set $\pi$ of all prime numbers, we see that $\mathbf{T}(G)$ is a subgroup. Moreover, for any $x_1, \dots, x_n \in G'$, we have $x_1, \dots, x_n \in H'$ for a finitely generated subgroup $H \leq G$. So $H \in Z\mathfrak{F}$ and it follows that $H'$ has finite order, whence $\langle x_1, \dots, x_n \rangle$ is finite. So $G'$ is locally finite and $G' \leq \mathbf{T}(G)$.

**Remark.** Example 4 shows that, in $\mathfrak{T}C$-groups, the set $\mathbf{T}(G)$ need not be a subgroup.

### The Dietzmann class of all $\mathfrak{F}_\pi C$-groups

Also the Dietzmann property of the classes $\mathfrak{F}_\pi C$ follows by our Fundamental Lemma together with the following extension of a result of Baer [2]:

**Lemma.** *Let $\pi$ be a set of prime numbers. If $G$ is an $\mathfrak{F}_\pi C$-group, then $G/\mathbf{Z}(G)$ is a torsion $\pi$-group.*

*Proof.* By [2] (see [15, p. 4, Thm. 1.4]), $G/\mathbf{Z}(G) \in \mathfrak{T}$. But we must have $G/\mathbf{Z}(G) \in \mathfrak{T}_\pi$ , since $\mathbf{Z}(G) = \bigcap\limits_{x \in G} \mathbf{C}_G(x^G)$ and $G/\mathbf{C}_G(x^G) \in \mathfrak{F}_\pi$.

**Proposition 12.** *Let $\pi$ be a set of prime numbers.*

a) *For every group $G$ and $x \in \mathbf{Z}_{\mathfrak{F}_\pi C}(G)$ we have $\langle x^G \rangle \in Z\mathfrak{F}_\pi$.*

b) *$\mathfrak{F}_\pi C$ is a Dietzmann class.*

For $\pi = \{all\ primes\}$, Theorem A follows.

*Proof.* a) Let $G$ be a group and $x \in G$ an $(\mathfrak{F}_\pi C)C$-element. Put $N = \langle x^G \rangle$ and $E = \mathbf{C}_G(N)$. Thus $E \cap N = \mathbf{Z}(N)$ and $G/E \in \mathfrak{F}_\pi C$. Let $K/E = \mathbf{Z}(G/E)$ and $Y = \langle x \rangle \cap K$. By the Lemma, $G/K$ is a torsion $\pi$-group. Therefore $\langle x \rangle / Y \cong K\langle x \rangle / K \in \mathfrak{F}_\pi$. The Fundamental Lemma yields $Y \leq E$ so that $xE$ is an element of $G/E$ of finite $\pi$-order. Since $G/E$ is an $\mathfrak{F}_\pi C$-group, Proposition 5 b) shows that $N/\mathbf{Z}(N) \cong NE/E = \langle (xE)^{G/E} \rangle$ is a finite $\pi$-group. Therefore $N \in Z\mathfrak{F}_\pi$.
b) $\mathfrak{F}_\pi C$ is $\{S, Q\}$-closed and for every group $G$ and all $x \in \mathbf{Z}_{\mathfrak{F}_\pi C}(G)$ we have by a): $\langle x^G \rangle \in Z\mathfrak{F}_\pi \subseteq \mathfrak{F}_\pi C$.

## The operator C and the question on the Dietzmann property of $\mathfrak{D}C$

Recall the definition of the class of all $\mathfrak{X}C$-groups:

$$G \in \mathfrak{X}C \iff G/\mathbf{C}_G(x^G) \in \mathfrak{X} \ \ \forall x \in G.$$

Clearly $\mathfrak{X} \subseteq Z\mathfrak{X} \subseteq \mathfrak{X}C$ for every $Q$-closed class of groups. If $\mathfrak{X}$ is also residually closed, then $Z\mathfrak{X} = \mathfrak{X}C$ (see Proposition 1 b)). For example, for the residually closed classes $\mathfrak{N}_n$ we have $Z\mathfrak{N}_n = \mathfrak{N}_n C = \mathfrak{N}_{n+1}$.
$\mathfrak{X}$ is *$C$-closed*, if $\mathfrak{X}C = \mathfrak{X}$. $\{S, Q, C\}$-closed classes are also centrally saturated and therefore Dietzmann classes.

**Example 6.**

a) *The class $\mathfrak{B}$ of all Baer groups is $C$-closed.*

b) *The class $\mathfrak{N}$ of all nilpotent groups is centrally saturated but not $C$-closed.*

*Proof.* a) If $G \in \mathfrak{B}C$ and $x \in G$, then $\langle x \rangle E/E = \langle xE \rangle \trianglelefteq \trianglelefteq G/E \in \mathfrak{B}$, where $E = \mathbf{C}_G(x^G)$. But $\langle x \rangle \trianglelefteq \langle x \rangle E$, so that $\langle x \rangle \trianglelefteq \trianglelefteq G$. This means $G \in \mathfrak{B}$.
b) Let $G_n \in \mathfrak{N}_n \setminus \mathfrak{N}_{n-1}$ for $n = 1, 2, \ldots$. For the restricted direct product $H$ of the $G_n$ we have $H \in \mathfrak{N}C \setminus \mathfrak{N}$.

Some extensions of centrally saturated classes are even C-closed:

**Proposition 13.** *Let $\mathfrak{W}$ be a centrally saturated class. Then*

a) *$F\mathfrak{W}$ is $C$-closed.*

b) *If $\mathfrak{W}$ is $D_0$-closed, then $L\mathfrak{W}$ is $C$-closed.*

*In particular, these classes are centrally saturated and therefore Dietzmann.*

*Proof.* a) Let $G \in (F\mathfrak{W})C$. We have to show $G \in F\mathfrak{W}$: For every $x \in G$ we have $G/E \in F\mathfrak{W}$, where $E = \mathbf{C}_G(N)$ and $N = \langle x^G \rangle$. It follows $\langle (xE)^{G/E} \rangle \in \mathfrak{W}$, so that $N/\mathbf{Z}(N) \in \mathfrak{W}$. But then $N = \langle x^G \rangle \in Z\mathfrak{W} = \mathfrak{W}$. Since $x$ was arbitrary, this means that $G \in F\mathfrak{W}$.
b) Let $G \in (L\mathfrak{W})C$, $S \subseteq G$ a finite subset. Let $E = \bigcap_{x \in S} \mathbf{C}_G(x^G)$ and $H = \langle S \rangle$. Since $S$ is

finite and $\mathfrak{W}$ is $D_0$-closed, we see that $G/E \in L\mathfrak{W}$. Therefore $HE/E \cong H/H \cap E \in \mathfrak{W}$. We conclude $H/\mathbf{Z}(H) \in \mathfrak{W}$, whence $H \in Z\mathfrak{W} = \mathfrak{W}$. This means $G \in L\mathfrak{W}$. So $L\mathfrak{W}$ is C-closed.

So the locally nilpotent, the locally normal nilpotent (= the Fitting groups), the locally solvable and the locally normal solvable groups are further examples of $C$-closed classes.

We already know that $\mathfrak{F}_\pi C$ are Dietzmann classes for every set $\pi$ of primes. What about $\mathfrak{D}C$ in general?
Theorem C b) is identical with:

**Proposition 14.** *Let $\mathfrak{D}$ be a Dietzmann class and suppose that $\mathfrak{Z} \subseteq \mathfrak{D}$ or that $\mathfrak{D}$ is residually closed. Then $\mathfrak{D}C$ is a Dietzmann class.*

*Proof.* (i) If $\mathfrak{D}$ is residually closed, then $\mathfrak{D}C = Z\mathfrak{D}$ and by Theorem B b) $\mathfrak{D}C$ is again Dietzmann.
(ii) Let $\mathfrak{Z} \subseteq \mathfrak{D}$. Given any group $G$ and $x \in \mathbf{Z}_{\mathfrak{D}C}(G)$, we put $N = \langle x^G \rangle$ and $E = \mathbf{C}_G(N)$. Thus $G/E \in \mathfrak{D}C$. Since $\langle xE \rangle \in \mathfrak{Z} \subseteq \mathfrak{D}$, we get $N/\mathbf{Z}(N) \cong EN/E = \langle (xE)^{G/E} \rangle \in \mathfrak{D}$. So $N \in Z\mathfrak{D} \subseteq \mathfrak{D}C$. This means that $\mathfrak{D}C$ is a Dietzmann class.

**Consequence 5.** *All terms in the chains*

$$\mathfrak{F}_\pi \subset \mathfrak{F}_\pi C \subset \mathfrak{F}_\pi C^2 \subset \cdots \subset \mathfrak{F}_\pi C^n \subset \cdots ,$$

$$L\mathfrak{F}_\pi \subset L(\mathfrak{F}_\pi C) \subset \cdots \subset L(\mathfrak{F}_\pi C^n) \subset \cdots$$

*and*

$$F\mathfrak{F}_\pi \subset F(\mathfrak{F}_\pi C) \subset \cdots \subset F(\mathfrak{F}_\pi C^n) \subset \cdots$$

*are Dietzmann classes.*

*Proof.* The first chain: For $n = 0$ , this is Proposition 5 b). For $n = 1$ this is Proposition 12. For $n \geq 2$, this is a consequence of Theorem C b), since $\mathfrak{Z} \subseteq \mathfrak{F}_\pi C$.
The second chain: By Theorem B a) we may apply the local operator $L$ to the first chain.
The third chain: $F\mathfrak{F}_\pi$ is Dietzmann by Theorem C a) as $\mathfrak{F}_\pi$ is $N_0$-closed. For $n \geq 1$, $F(\mathfrak{F}_\pi C^n)$ is Dietzmann as $\mathfrak{Z} \subseteq \mathfrak{F}_\pi C^n$.

**Question.** Is it perhaps true that $\mathfrak{D}C$ is always a Dietzmann class whenever $\mathfrak{D}$ is Dietzmann? We know it for $\mathfrak{Z} \subseteq \mathfrak{D}$ or for $\mathfrak{D} = \mathfrak{F}_\pi$ or when $\mathfrak{D}$ is residually closed.
What about $(L\mathfrak{F})C$ – is it a Dietzmann class? The proof pattern of Proposition 12 which shows the Dietzmann property for $\mathfrak{F}C$ does not work for $(L\mathfrak{F})C$, since $(L\mathfrak{F})C$-groups may very well have non-torsion center quotient group (see [11, p. 385]):
Let $H = C_{p^\infty} \times C_{p^\infty}$ for some prime $p$ and consider the automorphism $x$ of $H$ defined by $(a, b)^x = (a, ab)$ $\forall (a, b) \in H$. Put $G = [H]\langle x \rangle$. We have $\mathbf{Z}(G) = \{(1, b) \mid b \in C_{p^\infty}\}$. So $G/\mathbf{Z}(G) \notin \mathfrak{T}$ as $x$ has infinite order. But $G/\mathbf{C}_G(x^G) \cong C_{p^\infty}$ and $H < \mathbf{C}_G((a, b)^G)$ for all $(a, b) \in H$. This shows $G \in (L\mathfrak{F})C$ (in fact $G \in \check{\mathfrak{C}}C$).
If we could replace in our Fundamental Lemma the center $K/E$ by $L/E = \mathbf{C}_{G/E}(NE/E)$ and obtained $L \cap \langle x \rangle \leq \mathbf{Z}(N)$, we would be done with the general proof of the Dietzmann property for $\mathfrak{D}C$. But this "altered fundamental Lemma" is definitively false, as is shown by

the dihedral group of order 16:
Let $G = \langle x, t \mid x^2 = (xt)^2 = 1 = t^8 \neq t^4 \rangle$ with $N = \langle x^G \rangle = \langle t^2, x \rangle$, $E = \mathbf{C}_G(N) = \mathbf{Z}(N) = \mathbf{Z}(G) = \langle t^4 \rangle$. We have $L/E = \mathbf{C}_{G/E}(NE/E) = N/E$ and so $L \cap \langle x \rangle = \langle x \rangle \not\leq E$.

But, the same proof like that of Proposition 12 gives:

**Proposition 15.**

a) *For every group $G$ and $x \in G$ with $G/\mathbf{C}_G(x^G) \in (L\mathfrak{F}_\pi)C \cap Z\mathfrak{T}_\pi$ we have $\langle x^G \rangle \in ZL\mathfrak{F}_\pi \subseteq (L\mathfrak{F}_\pi)C \cap Z\mathfrak{T}_\pi$.*

b) *$(L\mathfrak{F}_\pi)C \cap Z\mathfrak{T}_\pi$ is a Dietzmann class.*

Another question is, whether Proposition 12 a) can be generalized for more than one $(\mathfrak{F}C)C$-element. That is, if $G$ is a group and $x_1, x_2, \ldots, x_r \in \mathbf{Z}_{\mathfrak{F}C}(G)$, is it true that $N = \langle x_1^G \rangle \ldots \langle x_r^G \rangle$ is central-by-finite or at least an $\mathfrak{F}C$-group when $r \geq 2$ ? That this is not so is shown by the following example (which also shows that $\mathfrak{F}C$ is not $N_0$-closed):

**Example 7.** Let $G$ be the group of $3 \times 3$ upper triangular matrices over the integers with 1 in the diagonal and consider the elements

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

in $G$. We have that $G = \langle x^G \rangle \langle y^G \rangle$ is the product of the abelian normal subgroups $\langle x^G \rangle$ and $\langle y^G \rangle$ of $G$. It follows that $x$ and $y$ are $\mathfrak{A}$C-elements (in particular $(\mathfrak{F}C)$C-elements) of $G$. But $N = G$ is not an $\mathfrak{F}C$-group, since these elements have infinitely many conjugates in $G$.

But the following remains true:

**Proposition 16.** *Let $G$ be a group, $\pi$ a set of prime numbers and $x_1, x_2, \ldots, x_r \in \mathbf{Z}_{\mathfrak{F}_\pi C}(G)$ elements of finite $\pi$-orders. Then*

$$N = \langle x_1^G \rangle \langle x_2^G \rangle \cdots \langle x_r^G \rangle \in Z\mathfrak{F}_\pi,$$

*in particular, $N$ is an $\mathfrak{F}_\pi C$-group.*

*Proof.* Let $E = \bigcap_{i=1}^{r} \mathbf{C}_G(x_i^G)$. Thus $G/E$ is an $\mathfrak{F}_\pi$C-group as $\mathfrak{F}_\pi C$ is $\{S, Q, D_0\}$-closed. Since the $x_i E$ are of finite $\pi$-orders by hypothesis, Proposition 5 b) shows that $N/\mathbf{Z}(N) \cong NE/E = \langle x_1 E \rangle^{G/E} \langle x_2 E \rangle^{G/E} \cdots \langle x_r E \rangle^{G/E}$ is a finite $\pi$-group.

## Examples of Dietzmann closures

We want to finish our studies with the construction of the Dietzmann closure in some of the simplest cases. First we prove:

**Proposition 17.** *Let $\mathfrak{X}$ be a class containing the cyclic group $C_n$. Then for any $m$, the $m$-fold direct product $C_n \times C_n \times \cdots \times C_n$ lies in the Dietzmann closure $\bar{\mathfrak{X}}$.*

*Proof.* We induct on $m$, the case $m = 1$ being clear. We think of $C_n$ in additive notation. Let already $K = C_n \times \cdots \times C_n \in \bar{\mathfrak{X}}$ when $K$ has $m-1$ factors. Let $N = K \times C_n$ and consider the semidirect product $G = [N]A$ where

$$A = \left\{ \begin{pmatrix} 1 & a_2 & a_3 & \cdots & a_m \\ 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \mid a_k \in C_n \right\} \cong K \ .$$

For $x = (1, 0, \ldots, 0)$ we have $\langle x \rangle \in \mathfrak{X}$ and $N = \langle x^G \rangle$ as well as $N = \mathbf{C}_G(N)$, so that by induction, $K \cong G/\mathbf{C}_G(N) \in \bar{\mathfrak{X}}$. We conclude $N \in \bar{\mathfrak{X}}$, as $\bar{\mathfrak{X}}$ is a Dietzmann class.

**Example 8.** *Let $n_1, n_2, \ldots, n_k, \ldots$ be a sequence of natural numbers. We fix any finite abelian groups $F_{n_1}, F_{n_2}, \ldots, F_{n_k} \ldots$ of exponents exactly $n_k$ and put $\mathfrak{X} = \{F_{n_1}, F_{n_2}, \ldots\}$. We claim that the Dietzmann closure of $\mathfrak{X}$ is*

$$\bar{\mathfrak{X}} = \bigcup_{k=1}^{\infty} \mathfrak{A}_{n_k} \ ,$$

*where $\mathfrak{A}_{n_k} = \{Y \in \mathfrak{A} \cap \mathfrak{F} \mid x^{n_k} = 1 \ \forall x \in Y\}$ is the class of finite abelian groups of exponent dividing $n_k$.*

*Proof.* Let $\mathfrak{D}_0 = \bigcup_{k=1}^{\infty} \mathfrak{A}_{n_k}$. Clearly $\mathfrak{X} \subseteq \mathfrak{D}_0$.

(i) $\mathfrak{D}_0$ is a Dietzmann class: $\mathfrak{D}_0$ is $\{S, Q\}$-closed. Let $G$ be a group and $x \in \mathbf{Z}_{\mathfrak{D}_0}(G)$ with $\langle x \rangle \in \mathfrak{D}_0$. So $G/\mathbf{C}_G(x^G) \in \mathfrak{A}_{n_k}$ and $\langle x \rangle \in \mathfrak{A}_{n_l}$ for some $k$ and $l$. Since $\mathfrak{A} \cap \mathfrak{F}$ is a Dietzmann class, it follows $N = \langle x^G \rangle \in \mathfrak{A}_{n_l} \subseteq \mathfrak{D}_0$.

(ii) Let $\mathfrak{D}$ denote any Dietzmann class with $\mathfrak{X} \subseteq \mathfrak{D}$. Thus $C_{n_k} \in \mathfrak{D}$. By Proposition 17, every $m$-fold direct product $C_{n_k} \times \cdots \times C_{n_k} \in \mathfrak{D}$. But for every $Y \in \mathfrak{D}_0$, there exist $k$ and $m$ such that $Y \leq C_{n_k} \times \cdots \times C_{n_k}$. Therefore $Y \in \mathfrak{D}$, whence $\mathfrak{D}_0 \subseteq \mathfrak{D}$.

(i) and (ii) show that $\mathfrak{D}_0 = \bar{\mathfrak{X}}$.

**Remark.** Proposition 17 should not mislead the reader to conclude that Dietzmann classes are closed under direct products. This is wrong: For $\mathfrak{X} = \{C_2, C_3\}$ the Dietzmann closure $\bar{\mathfrak{X}} = \mathfrak{A}_2 \cup \mathfrak{A}_3$ does not contain $C_2 \times C_3$ .

**Example 9.** *Let $p, q$ be prime numbers such that $q$ divides $p-1$ and let $X$ denote the nonabelian group of order $pq$. Then the Dietzmann closure of $\mathfrak{X} = \{X\}$ is*

$$\bar{\mathfrak{X}} = \mathfrak{A}_p \cup \mathfrak{A}_q \cup \{X\} \ .$$

*Proof.* Certainly $\{C_p, C_q, X\} \subseteq \bar{\mathfrak{X}}$ so that, using Example 8 and Proposition 5 b), $\mathfrak{A}_p \cup \mathfrak{A}_q \cup \{X\} \subseteq \bar{\mathfrak{X}} \subseteq \mathfrak{F}_{\{p,q\}}$. We have to show that $\mathfrak{A}_p \cup \mathfrak{A}_q \cup \{X\}$ is a Dietzmann class. Let $G$ be a group, $x \in G$ such that $G/E$ and $\langle x \rangle$ are in $\mathfrak{A}_p \cup \mathfrak{A}_q \cup \{X\}$, where $N = \langle x^G \rangle$ and

$E = \mathbf{C}_G(N)$. Now $x$ is of order $p$ or $q$. Since we may assume that $N$ is nonabelian, we have $N/\mathbf{Z}(N) \cong X \cong G/E$ so that $NE = G$. Therefore $N = \langle x^N \rangle$ and $N$ is a finite $\{p, q\}$-group with abelian Sylow subgroups and normal Sylow $p$-subgroup $P$. Hence $x$ is of order $q$ and $N$ cannot have a proper $p$-quotient. Since $P = [P, N] \times \mathbf{C}_P(N)$ (see [7, p. 177, Thm. 2.3]), we conclude $[P, N] = P$ whence $\mathbf{C}_P(N) = 1$. So $\mathbf{Z}(N)$ is a $q$-group and $|P| = p$. With $K = [P]\langle x \rangle$ we have $N = \mathbf{Z}(N) \times K$ with $x \in K \cong X$. Thus $K = \langle x^N \rangle = N \cong X$. This shows $\bar{\mathfrak{X}} = \mathfrak{A}_p \cup \mathfrak{A}_q \cup \{X\}$.

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