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Hermitian Harmonic Maps

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Abstract. We study the notion of Hermitian harmonicity as introduced by Jost and Yau [5] in 1993. As a parallel with harmonic maps, we define Hermitian harmonic morphisms, which we characterise as Hermitian harmonic maps satisfying a geometrical condition called biholomorphic horizontal weak conformality. Finally we completely describe maps which mix harmonicity and Hermitian harmonicity by post-composition, and give a few examples.

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1. Introduction

Harmonic maps introduced by Eells and Sampson in 1964 [2], are critical points of the Energy Functional and in this respect can be seen as a generalisation of geodesics. A well-known fact is that holomorphic maps from a non-Kählerian Hermitian manifold to a Kähler manifold need not be harmonic. The necessary conditions were obtained by Lichnerowicz in [6].

In the recent article [5], Jost and Yau introduced the notion, and proved the existence, of Hermitian harmonic maps from a Hermitian manifold to a Riemannian manifold. In contrast to the situation for harmonic maps, when the target is Kähler, holomorphic maps are always Hermitian harmonic. A rather nice feature of Hermitian harmonic maps is that they can be seen as generalising another facet of geodesics, i.e. auto-parallelism.

A class of harmonic maps with a strong geometrical flavour is the harmonic morphisms, which by definition pull back local harmonic functions to local harmonic functions and were characterised by Fuglede [3] and Ishihara [4] to be horizontally weakly conformal harmonic maps. Following the same pattern, we define Hermitian harmonic morphisms as maps pulling local Hermitian harmonic functions back to Hermitian harmonic functions, and obtain their characterisation as biholomorphic horizontal weak conformal Hermitian harmonic maps. We then consider the case of maps mixing (classical) harmonicity and Hermitian harmonicity, i.e. pulling local harmonic functions back to Hermitian harmonic functions, and show them to be precisely d'-horizontally weakly conformal Hermitian harmonic maps. Finally, we illustrate the different notions discussed with a few examples.

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2. Hermitian harmonicity

Throughout this paper we assume the Einstein convention on the summing of repeated indices. Note that lower-case letters run through unbarred indices, while upper-case letters run through both barred and unbarred indices.

Definition 1. *Let (X, γ) be a Hermitian manifold of complex dimension $dim_{\mathbb{C}}(X) = m$, equipped with the Hermitian metric γ and with local complex coordinates $(z^\alpha)_{\alpha=1,\dots,m}$, and (N, g) a Riemannian manifold of (real) dimension $dim_{\mathbb{R}}(N) = n$, with local real coordinates $(x^i)_{i=1,\dots,n}$.*

Following Jost and Yau ([5]), we define $\phi : X \to N$ to be a Hermitian harmonic map if

$$\gamma^{\alpha\overline{\beta}} \left[\frac{\partial^2 \phi^k}{\partial z^\alpha \partial z^{\overline{\beta}}} + {}^N\Gamma^k_{ij} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^j}{\partial z^{\overline{\beta}}} \right] = 0, \qquad k = 1, \dots, n. \tag{1}$$

This equation does not arise from a variational problem nor has it a divergence form.

Remark 1.

i) When (X, γ) is a Kähler manifold then (1) is the equation of a harmonic map.

ii) For $(N, g) = \mathbb{R}$ with its canonical metric, (1) is the equation of a Hermitian harmonic function $f : X \to \mathbb{R}$, that is :

$$\gamma^{\alpha\overline{\beta}} \frac{\partial^2 f}{\partial z^\alpha \partial z^{\overline{\beta}}} = 0. \tag{2}$$

iii) The existence of Hermitian harmonic maps is proved by:

Theorem 1. [5] *Let (X, γ) be a compact Hermitian manifold and (N, g) a compact Riemannian manifold of negative sectional curvature. Let $f : M \to N$ be continuous, such that f is not homotopic to a map onto a closed geodesic of N. Then there exists a Hermitian harmonic map homotopic to f.*

iv) If we denote by ∇ the Levi-Civita connection on TN the global expression of (1) is:

$$\mathrm{trace}_\gamma(\nabla_{d\phi} d\phi) = 0.$$

In this sense $d\phi$ is autoparallel.

v) Between Hermitian manifolds equation (1) becomes

$$\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2\phi^K}{\partial z^\alpha\partial z^{\overline{\beta}}}+{}^N\Gamma^K_{IJ}\frac{\partial\phi^I}{\partial z^\alpha}\frac{\partial\phi^J}{\partial z^{\overline{\beta}}}\right]=0, \tag{3}$$

where $I, J, K = 1, \overline{1}, \ldots, n, \overline{n}$.
One immediately sees from (3) that $\pm$holomorphic maps from a Hermitian manifold to a Kähler manifold are Hermitian harmonic.

Definition 2. *Let (X, J, γ) be a Hermitian manifold. The complexified tangent space $T^{\mathbb{C}}X$ can be decomposed into the direct sum*

$$T^{\mathbb{C}}X = T'X \oplus T''X,$$

where $T'X$ (resp. $T''X$) is the eigenspace associated to the eigenvalue $+i$ (resp. $-i$) of the endomorphism J, and is called the holomorphic (resp. antiholomorphic) tangent space.

Let $\phi : (X, \gamma) \to (N, h)$ be a map between Hermitian manifolds, then the complexified differential of ϕ

$$d^{\mathbb{C}}\phi : T^{\mathbb{C}}X \to T^{\mathbb{C}}N$$

decomposes into the partial differentials

$$\begin{array}{ll} d'\phi : T'X \to T^{\mathbb{C}}N & d''\phi : T''X \to T^{\mathbb{C}}N \\ \partial\phi : T'X \to T'N & \overline{\partial}\phi : T''X \to T'N \\ \partial\overline{\phi} : T'X \to T''N & \overline{\partial}\,\overline{\phi} : T''X \to T''N \end{array}$$

satisfying

$$\begin{array}{ll} d'\phi = \partial\phi + \partial\overline{\phi} & d''\phi = \overline{\partial}\phi + \overline{\partial}\,\overline{\phi} \\ \overline{\partial\phi} = \overline{\partial}\,\overline{\phi} & \overline{\partial\overline{\phi}} = \overline{\partial}\phi \end{array}$$

We can define the dual maps:

$$\begin{array}{ll} (\partial\phi)^* : T'^*N \to T'^*X & (\overline{\partial}\phi)^* : T''^*N \to T'^*X \\ (\partial\overline{\phi})^* : T'^*N \to T''^*X & (\overline{\partial}\,\overline{\phi})^* : T''^*N \to T''^*X \end{array}$$

and decompose

$$T^{\mathbb{C}}X = \mathcal{V} \oplus \mathcal{H},$$

where $\mathcal{V} = \ker d^{\mathbb{C}}\phi$ and $\mathcal{H} = \left(\ker d^{\mathbb{C}}\phi\right)^{\perp}$ and the orthogonal complement is with respect to the complex linear extension of γ.
We will denote $\mathcal{V}' = \mathcal{V}\cap T'X$ (resp. $\mathcal{H}' = \mathcal{H}\cap T'X$) and $\mathcal{V}'' = \mathcal{V}\cap T''X$ (resp. $\mathcal{H}'' = \mathcal{H}\cap T''X$) the holomorphic and antiholomorphic vertical (resp. horizontal) subbundles.

Notation. To a Hermitian metric γ on a manifold X, we associate the Hermitian scalar product (cf. [1]) Γ on $T^{\mathbb{C}}X$ such that:

$$\Gamma(X,Y) = \gamma(X,\overline{Y}) \qquad \forall X,Y \in T^{\mathbb{C}}X.$$

Note that if g is a Riemannian metric on a manifold M, then by linearity, we can extend g to $g^{\mathbb{C}}$ on $T^{\mathbb{C}}M$ and still make sense of the Hermitian scalar product

$$G(X,Y) = g^{\mathbb{C}}(X,\overline{Y}) \qquad \forall X,Y \in T^{\mathbb{C}}M.$$

Definition 3. *A Hermitian manifold (N^n, g) is called cosymplectic if in local complex coordinates $(x^i)_{i=1,\dots,n}$:*

$$g^{i\overline{j}}\,{}^{N}\Gamma^{k}_{i\overline{j}} = 0 \quad \forall k = 1,\dots,n,$$

where ${}^{N}\Gamma^{k}_{i\overline{j}}$ are the Christoffel symbols of (N,g).

Definition 4. *Let (X,γ) and (N,g) be Hermitian manifolds. We define $\phi : X \to N$ to be a Hermitian harmonic morphism if ϕ pulls back local Hermitian harmonic functions on N to local Hermitian harmonic functions on M. That is, for any open set U in N and Hermitian harmonic function $f : U \subseteq N \to \mathbb{R}$, $f \circ \phi : \phi^{-1}(U) \subseteq X \to \mathbb{R}$ is Hermitian harmonic, i.e. satisfies*

$$\gamma^{\alpha\overline{\beta}} \left[\frac{\partial^2 (f \circ \phi)}{\partial z^{\alpha} \partial z^{\overline{\beta}}} \right] = 0. \tag{4}$$

Definition 5. *Let (X,γ) and (N,g) be Hermitian manifolds. We call a map $\phi : X \to N$ biholomorphically horizontally weakly conformal if:*

i) *the maps $\partial\phi : T'X \to T'N$ and $\overline{\partial}\phi : T''X \to T'N$ are horizontally weakly conformal with respect to the Hermitian scalar products Γ and G associated to γ and g, i.e.: there exist functions μ and λ such that*

$$G(\partial\phi(X), \partial\phi(Y)) = \mu^2 \Gamma(X,Y) \quad \forall X,Y \in \mathcal{H}'$$
$$G(\overline{\partial}\phi(X), \overline{\partial}\phi(Y)) = \lambda^2 \Gamma(X,Y) \quad \forall X,Y \in \mathcal{H}''$$

ii) *the maps $\partial\phi$ and $\overline{\partial}\phi$ are orthogonal in the following sense:*

$$\gamma^* \left((\partial\phi)^*(\sigma), (\overline{\partial}\phi)^*(\omega) \right) = 0 \quad \forall \sigma,\omega \in T'^*N,$$

where γ^* is the metric dual to γ.

These conditions can be expressed in the dual bundles as:

$$\gamma^* \left((\partial\phi)^*(\sigma), (\overline{\partial}\,\overline{\phi})^*(\overline{\omega}) \right) = \mu^2 g^* (\sigma, \overline{\omega}) \tag{5}$$
$$\gamma^* \left((\overline{\partial}\phi)^*(\sigma), (\partial\overline{\phi})^*(\overline{\omega}) \right) = \lambda^2 g^* (\sigma, \overline{\omega}) \tag{6}$$
$$\gamma^* \left((\partial\phi)^*(\sigma), (\overline{\partial}\phi)^*(\omega) \right) = 0 \tag{7}$$

$$\forall \sigma,\omega \in T'^*N,$$

where γ^* and g^* are the metrics dual to γ and g.
In local complex coordinates $(z^\alpha)_{\alpha=1,\dots,m}$ on X and $(x^i)_{i=1,\dots,n}$ on N, these conditions translate as

$$\gamma^{\alpha\overline{\beta}}\frac{\partial\phi^i}{\partial z^\alpha}\frac{\partial\phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}} =\mu^2 g^{i\overline{\jmath}} \tag{8}$$

$$\gamma^{\alpha\overline{\beta}}\frac{\partial\phi^i}{\partial z^{\overline{\beta}}}\frac{\partial\phi^{\overline{\jmath}}}{\partial z^{\alpha}} =\lambda^2 g^{i\overline{\jmath}} \tag{9}$$

$$\gamma^{\alpha\overline{\beta}}\frac{\partial\phi^i}{\partial z^\alpha}\frac{\partial\phi^{j}}{\partial z^{\overline{\beta}}} =0. \tag{10}$$

Note that (8) and (9) can be rewritten as

$$\begin{aligned} g\left(\partial\phi(X),\overline{\partial}\,\overline{\phi}(\overline{Y})\right) &=\mu^2\gamma(X,\overline{Y}) \quad \forall X,Y\in\mathcal{H}' \\ g\left(\overline{\partial}\phi(X),\partial\overline{\phi}(\overline{Y})\right) &=\lambda^2\gamma(X,Y) \quad \forall X,Y\in\mathcal{H}''. \end{aligned}$$

These two conditions imply

$$\mid\overline{\partial}\phi\mid^2= n\lambda^2 \qquad \mid\partial\phi\mid^2= n\mu^2.$$

Proposition 1. *Let (X,γ) and (N,g) be Hermitian manifolds, with (N,g) cosymplectic. A map $\phi : X\to N$ is a Hermitian harmonic morphism if and only if ϕ is a biholomorphically horizontally weakly conformal Hermitian harmonic map.*

Proof. Assume the manifolds to be equipped with local complex coordinates $(z^\alpha)_{\alpha=1,\dots,m}$ on X and $(x^i)_{i=1,\dots,n}$ on N, which are orthogonal at $q\in X$ and $\phi(q)\in N$.

Lemma 1. *Let U be an open subset of N. A complex-valued function $f:U\to\mathbb{C}$ is Hermitian harmonic if and only if the real-valued functions $\Re e f$ and $\Im m f : U\to\mathbb{R}$ are Hermitian harmonic.*

Proof of Lemma 1. Suppose $f:U\to\mathbb{C}$ to be Hermitian harmonic, i.e.

$$g^{i\overline{\jmath}}\frac{\partial^2 f}{\partial x^i\partial x^{\overline{\jmath}}} = 0$$

then

$$g^{i\overline{\jmath}}\frac{\partial^2 \overline{f}}{\partial x^i\partial x^{\overline{\jmath}}} = \overline{\left(g^{i\overline{\jmath}}\frac{\partial^2 f}{\partial x^i\partial x^{\overline{\jmath}}}\right)} = 0,$$

i.e. $\overline{f}:U\to\mathbb{C}$ is Hermitian harmonic as well.

It is clear that the linear combination of two Hermitian harmonic functions is Hermitian harmonic, therefore $\Re e f=\frac{1}{2}\left(f+\overline{f}\right)$ and $\Im m f=\frac{1}{2}\left(f-\overline{f}\right)$ are real-valued Hermitian harmonic functions.

Conversely, if $\Re e f$ and $\Im m f$ are Hermitian harmonic functions then

$$\begin{aligned} 0 &= g^{i\bar{\jmath}} \frac{\partial^2 \Re e f}{\partial x^i \partial x^{\bar{\jmath}}} + \sqrt{-1} \quad g^{i\bar{\jmath}} \frac{\partial^2 \Im m f}{\partial x^i \partial x^{\bar{\jmath}}} \\ &= g^{i\bar{\jmath}} \frac{\partial^2 f}{\partial x^i \partial x^{\bar{\jmath}}}, \end{aligned}$$

that is f is Hermitian harmonic. ☐

Lemma 1 shows that Hermitian harmonic morphisms could be defined as pulling local complex-valued Hermitian harmonic functions back to local complex-valued Hermitian harmonic, instead of real-valued.

Lemma 2. *Let (N^n, g) be a Hermitian manifold and let $q \in N$. Then for any set of complex constants $(C_i, C_{\bar{\jmath}})_{i,j=1,\dots,n}$ and $(C_{i\bar{\jmath}})_{i,j=1,\dots,n}$, $(C_{i\bar{\jmath}} = C_{\bar{\jmath}i})$ such that $\sum_{i=1}^n C_{i\bar{\imath}} = 0$, then there exist a neighbourhood U of q and a complex-valued Hermitian harmonic function $f : U \to \mathbb{C}$ such that*

$$\frac{\partial f}{\partial z^i}(q) = C_i \quad \frac{\partial f}{\partial z^{\bar{\jmath}}}(q) = C_{\bar{\jmath}} \quad \frac{\partial^2 f}{\partial z^i z^{\bar{\jmath}}}(q) = C_{i\bar{\jmath}}.$$

Proof. Since the manifold N is complex, it is analytic so we can apply the Cauchy-Kowalewskaya Theorem to the operator

$$g^{i\bar{\jmath}} \frac{\partial^2 f}{\partial x^i \partial x^{\bar{\jmath}}}$$

Choose coordinates centered at the point $q \in N$, such that $q = 0$ and $g^{n\bar{n}} \neq 0$ at $q = 0$. Consider the function

$$h(z) = C_A z^A + C_{AB} z^A z^B,$$

then by the Cauchy-Kowalewskaya Theorem (cf. [7]), there exists in a neighbourhood of $q = 0$, an analytic function f, such that

$$g^{i\bar{\jmath}} \frac{\partial^2 f}{\partial x^i \partial x^{\bar{\jmath}}} = 0$$

and

$$f \equiv h, \quad \frac{\partial f}{\partial z^n} = \frac{\partial h}{\partial z^n} \quad \text{when } z^n = 0.$$

By applying $\frac{\partial}{\partial z^i}$ and $\frac{\partial}{\partial z^{\bar{\jmath}}}$, for $i, j < n$, we obtain at the point $q = 0$:

$$\begin{aligned} &\frac{\partial f}{\partial z^i}(q) = C_i, \quad \frac{\partial f}{\partial z^{\bar{\jmath}}}(q) = C_{\bar{\jmath}} \quad \forall i, j = 1, \dots, n \\ &\frac{\partial^2 f}{\partial z^i z^{\bar{\jmath}}}(q) = C_{i\bar{\jmath}} \quad \forall i, j = 1, \dots, n \text{ except for } i = j = n. \end{aligned}$$

But

$$g^{i\bar{\jmath}} \frac{\partial^2 f}{\partial x^i \partial x^{\bar{\jmath}}}(0) = 0 = g^{i\bar{\jmath}} \frac{\partial^2 h}{\partial x^i \partial x^{\bar{\jmath}}}(0),$$

yields

$$\frac{\partial^2 f}{\partial z^n z^{\overline{n}}}(0) = \frac{\partial^2 h}{\partial z^n z^{\overline{n}}}(0) = C_{n\overline{n}}.$$

$\square$

Proof of Proposition 1:
Let $\phi : X \to N$ be a biholomorphically horizontally weakly conformal Hermitian harmonic map and $f : U \subseteq N \to \mathbb{R}$ a local Hermitian harmonic function on N.
Then

$$\begin{aligned}
&\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2(f\circ\phi)}{\partial z^\alpha \partial z^{\overline{\beta}}}\right] = \\
&\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2 f}{\partial x^I \partial x^J}\frac{\partial \phi^I}{\partial z^\alpha}\frac{\partial \phi^J}{\partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^I}\frac{\partial^2 \phi^I}{\partial z^\alpha \partial z^{\overline{\beta}}}\right] = \\
&\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\partial \phi^i}{\partial z^\alpha}\frac{\partial \phi^j}{\partial z^{\overline{\beta}}} + \frac{\partial^2 f}{\partial x^{\overline{\imath}} \partial x^{\overline{\jmath}}}\frac{\partial \phi^{\overline{\imath}}}{\partial z^\alpha}\frac{\partial \phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}} + \frac{\partial^2 f}{\partial x^{\overline{\imath}} \partial x^j}\frac{\partial \phi^{\overline{\imath}}}{\partial z^\alpha}\frac{\partial \phi^j}{\partial z^{\overline{\beta}}} + \right. \\
&\quad \left. \frac{\partial^2 f}{\partial x^i \partial x^{\overline{\jmath}}}\frac{\partial \phi^i}{\partial z^\alpha}\frac{\partial \phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^i}\frac{\partial^2 \phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^{\overline{\imath}}}\frac{\partial^2 \phi^{\overline{\imath}}}{\partial z^\alpha \partial z^{\overline{\beta}}}\right] = \\
&\gamma^{\alpha\overline{\beta}}\frac{\partial \phi^{\overline{\imath}}}{\partial z^\alpha}\frac{\partial \phi^j}{\partial z^{\overline{\beta}}}\frac{\partial^2 f}{\partial x^{\overline{\imath}} \partial x^j} + \gamma^{\alpha\overline{\beta}}\frac{\partial \phi^i}{\partial z^\alpha}\frac{\partial \phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}}\frac{\partial^2 f}{\partial x^i \partial x^{\overline{\jmath}}} = \\
&\lambda^2 g^{\overline{\imath}j}\frac{\partial^2 f}{\partial x^{\overline{\imath}} \partial x^j} + \mu^2 g^{i\overline{\jmath}}\frac{\partial^2 f}{\partial x^i \partial x^{\overline{\jmath}}} = 0
\end{aligned}$$

Remark that when the target manifold is cosymplectic and ϕ is biholomorphically horizontally weakly conformal, Hermitian harmonicity is equivalent to the condition

$$\gamma^{\alpha\overline{\beta}}\frac{\partial^2 \phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}} = 0.$$

We now prove the converse.
Let $\phi : X \to N$ be a Hermitian harmonic morphism. Then for any local Hermitian harmonic function $f : U \subseteq N \to \mathbb{R}$,

$$\begin{aligned}
&\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2(f\circ\phi)}{\partial z^\alpha \partial z^{\overline{\beta}}}\right] = \\
&\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\partial \phi^i}{\partial z^\alpha}\frac{\partial \phi^j}{\partial z^{\overline{\beta}}} + \frac{\partial^2 f}{\partial x^{\overline{\imath}} \partial x^{\overline{\jmath}}}\frac{\partial \phi^{\overline{\imath}}}{\partial z^\alpha}\frac{\partial \phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}} + \frac{\partial^2 f}{\partial x^{\overline{\imath}} \partial x^j}\frac{\partial \phi^{\overline{\imath}}}{\partial z^\alpha}\frac{\partial \phi^j}{\partial z^{\overline{\beta}}} + \right. \\
&\quad \left. \frac{\partial^2 f}{\partial x^i \partial x^{\overline{\jmath}}}\frac{\partial \phi^i}{\partial z^\alpha}\frac{\partial \phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^i}\frac{\partial^2 \phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^{\overline{\imath}}}\frac{\partial^2 \phi^{\overline{\imath}}}{\partial z^\alpha \partial z^{\overline{\beta}}}\right] = \\
&\frac{\partial^2 f}{\partial x^i \partial x^j}\gamma^{\alpha\overline{\beta}}\frac{\partial \phi^i}{\partial z^\alpha}\frac{\partial \phi^j}{\partial z^{\overline{\beta}}} + \frac{\partial^2 f}{\partial x^{\overline{\imath}} \partial x^{\overline{\jmath}}}\gamma^{\alpha\overline{\beta}}\frac{\partial \phi^{\overline{\imath}}}{\partial z^\alpha}\frac{\partial \phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}} + \frac{\partial^2 f}{\partial x^{\overline{\imath}} \partial x^j}\gamma^{\alpha\overline{\beta}}\frac{\partial \phi^{\overline{\imath}}}{\partial z^\alpha}\frac{\partial \phi^j}{\partial z^{\overline{\beta}}} + \\
&\frac{\partial^2 f}{\partial x^i \partial x^{\overline{\jmath}}}\gamma^{\alpha\overline{\beta}}\frac{\partial \phi^i}{\partial z^\alpha}\frac{\partial \phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^i}\gamma^{\alpha\overline{\beta}}\frac{\partial^2 \phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^{\overline{\imath}}}\gamma^{\alpha\overline{\beta}}\frac{\partial^2 \phi^{\overline{\imath}}}{\partial z^\alpha \partial z^{\overline{\beta}}} = 0
\end{aligned} \tag{11}$$

By Lemma 2 we know that for any point $q \in N$ and constants (C_{AB}) $(C_{AB} = C_{BA})$ and (C_A) in $\mathbb{C}$, such that $\sum_i C_{i\bar{\imath}} = 0$, there exists a Hermitian harmonic function $f : U \to \mathbb{C}$ such that

$$f_{AB}(q) = C_{AB} \quad f_A(q) = C_A \quad \forall A, B = 1, \bar{1}, \dots, n, \bar{n}.$$

If we choose $C_{AB} = 0 \quad \forall A, B$ and $C_A = \delta_{Ai_0}$, equation (11) yields

$$\gamma^{\alpha\bar{\beta}} \frac{\partial^2 \phi^{i_0}}{\partial z^\alpha \partial z^{\bar{\beta}}}(q) = 0. \tag{12}$$

Similarly, for $C_{AB} = \delta^{Ai_0}_{Bj_0}$ and $C_A = 0 \quad \forall A$, we have

$$\gamma^{\alpha\bar{\beta}} \frac{\partial \phi^{i_0}}{\partial z^\alpha} \frac{\partial \phi^{j_0}}{\partial z^{\bar{\beta}}}(q) = 0.$$

For $C_{AB} = \delta^{Ai}_{B\bar{\jmath}} C_{i\bar{\jmath}}$ $(\sum_i C_{i\bar{\imath}} = 0)$ and $C_A = 0 \quad \forall A$, equation (11) becomes

$$\gamma^{\alpha\bar{\beta}} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^{\bar{\jmath}}}{\partial z^{\bar{\beta}}} C_{i\bar{\jmath}} + \gamma^{\alpha\bar{\beta}} \frac{\partial \phi^{\bar{\imath}}}{\partial z^\alpha} \frac{\partial \phi^j}{\partial z^{\bar{\beta}}} \overline{C_{i\bar{\jmath}}} = 0. \tag{13}$$

Choosing the $C_{i\bar{\jmath}}$ first real and then imaginary pure, splits (13) in two parts,

$$\gamma^{\alpha\bar{\beta}} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^{\bar{\jmath}}}{\partial z^{\bar{\beta}}} C_{i\bar{\jmath}} = 0. \tag{14}$$

and

$$\gamma^{\alpha\bar{\beta}} \frac{\partial \phi^{\bar{\imath}}}{\partial z^\alpha} \frac{\partial \phi^j}{\partial z^{\bar{\beta}}} C_{\bar{\imath}j} = 0. \tag{15}$$

We now follow an argument of Ishihara in [4], to show that equation (14) implies

$$\gamma^{\alpha\bar{\beta}} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^{\bar{\jmath}}}{\partial z^{\bar{\beta}}} = \lambda^2 g^{i\bar{\jmath}}.$$

Since the C_{ij} are chosen such that $\sum_i C_{i\bar{i}} = 0$,

$$\begin{aligned}
&\sum_{i,j} \gamma^{\alpha\bar{\beta}} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^{\bar{\jmath}}}{\partial z^{\bar{\beta}}} C_{i\bar{\jmath}} = 0 \\
&\Leftrightarrow \\
&\sum_{i \neq j} \gamma^{\alpha\bar{\beta}} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^{\bar{\jmath}}}{\partial z^{\bar{\beta}}} C_{i\bar{\jmath}} + \sum_i \gamma^{\alpha\bar{\beta}} C_{i\bar{\imath}} \left(\frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^{\bar{\imath}}}{\partial z^{\bar{\beta}}} - \frac{\partial \phi^1}{\partial z^\alpha} \frac{\partial \phi^{\bar{1}}}{\partial z^{\bar{\beta}}} \right) = 0.
\end{aligned}$$

By choosing different values for C_{ij}, we obtain

$$\begin{aligned}
\gamma^{\alpha\bar{\beta}} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^{\bar{\jmath}}}{\partial z^{\bar{\beta}}} &= 0 \quad i \neq j \\
\gamma^{\alpha\bar{\beta}} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^{\bar{\imath}}}{\partial z^{\bar{\beta}}} &= \gamma^{\alpha\bar{\beta}} \frac{\partial \phi^1}{\partial z^\alpha} \frac{\partial \phi^{\bar{1}}}{\partial z^{\bar{\beta}}},
\end{aligned}$$

that is

$$\gamma^{\alpha\overline{\beta}}\frac{\partial\phi^i}{\partial z^\alpha}\frac{\partial\phi^{\overline{\jmath}}}{\partial z^{\overline{\beta}}} = \lambda^2\delta^{ij} = \lambda^2 g^{ij}.$$

Similarly equation (15) implies

$$\gamma^{\alpha\overline{\beta}}\frac{\partial\phi^{\overline{\imath}}}{\partial z^\alpha}\frac{\partial\phi^{j}}{\partial z^{\overline{\beta}}} = \mu^2 g^{\overline{\imath}j}.$$

Recall that in Definition 5, equations (8), (9), (10) were shown to be equivalent to ϕ being biholomorphically horizontally weakly conformal and that for a biholomorphically horizontally weakly conformal map into a cosymplectic manifold, equation (12) is Hermitian harmonicity. □

Remark 2. Another possible approach would be to drop the condition of (N, g) cosymplectic and show that a map ϕ between Hermitian manifolds is a Hermitian harmonic morphism if and only if ϕ is biholomorphically horizontally weakly conformal and satisfies

$$\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2\phi^k}{\partial z^\alpha\partial z^{\overline{\beta}}} + {}^N\Gamma^k_{ij}\frac{\partial\phi^i}{\partial z^\alpha}\frac{\partial\phi^j}{\partial z^{\overline{\beta}}}\right] = 0 \qquad k = 1,\dots,n. \tag{16}$$

Note that (16) is different from equation (3), for (N, g) non-Kählerian.

As in the case of harmonic morphisms, Hermitian harmonic morphisms pull back Hermitian harmonic maps to Hermitian harmonic maps.

Proposition 2. *Let (X,γ) and (N, g) be Hermitian manifolds, with (N, g) cosymplectic, (M, h) a Riemannian manifold and $\phi : X \to N$ a Hermitian harmonic morphism. If $\psi : N \to M$ is a Hermitian harmonic map then $\psi\circ\phi : X \to M$ is a Hermitian harmonic map.*

Proof. Let X be equipped with the local complex coordinates $(z^\alpha)_{\alpha=1,\dots,m}$, N be equipped with the complex coordinates $(x^i)_{i=1,\dots,n}$ and M with the real coordinates $(y^s)_{s=1,\dots,\dim M}$. Let Γ^i_{jk} denote the Christoffel symbols of (N, g) and ${}^M\Gamma^r_{st}$ the Christoffel symbols of (M, h). Let $\psi : N \to M$ be a Hermitian harmonic map, i.e. such that

$$g^{i\overline{\jmath}}\left(\frac{\partial^2\psi^s}{\partial x^i\partial x^{\overline{\jmath}}} + {}^M\Gamma^s_{rt}\frac{\partial\psi^r}{\partial x^i}\frac{\partial\psi^t}{\partial x^{\overline{\jmath}}}\right) = 0.$$

Then

$$\begin{aligned}
&\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2(\psi\circ\phi)^s}{\partial z^\alpha\partial z^{\overline{\beta}}} + {}^M\Gamma^s_{rt}\frac{\partial(\psi\circ\phi)^r}{\partial z^\alpha}\frac{\partial(\psi\circ\phi)^t}{\partial z^{\overline{\beta}}}\right] = \\
&\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2\psi^s}{\partial x^I\partial x^J}\frac{\partial\phi^I}{\partial z^\alpha}\frac{\partial\phi^J}{\partial z^{\overline{\beta}}} + \frac{\partial\psi^s}{\partial x^I}\frac{\partial^2\phi^I}{\partial z^\alpha\partial z^{\overline{\beta}}} + {}^M\Gamma^s_{rt}\frac{\partial\psi^r}{\partial x^I}\frac{\partial\phi^I}{\partial z^\alpha}\frac{\partial\psi^t}{\partial x^J}\frac{\partial\phi^J}{\partial z^{\overline{\beta}}}\right] = \\
&\frac{\partial^2\psi^s}{\partial x^i\partial x^{\overline{\jmath}}}\mu^2 g^{i\overline{\jmath}} + \frac{\partial^2\psi^s}{\partial x^{\overline{\imath}}\partial x^j}\lambda^2 g^{\overline{\imath}j} + \mu^2 g^{i\overline{\jmath}}\,{}^M\Gamma^s_{rt}\frac{\partial\psi^r}{\partial x^i}\frac{\partial\psi^t}{\partial x^{\overline{\jmath}}} + \lambda^2 g^{\overline{\imath}j}\,{}^M\Gamma^s_{rt}\frac{\partial\psi^r}{\partial x^{\overline{\imath}}}\frac{\partial\psi^t}{\partial x^j} = \\
&(\mu^2+\lambda^2)\left(g^{i\overline{\jmath}}\left(\frac{\partial^2\psi^s}{\partial x^i\partial x^{\overline{\jmath}}} + {}^M\Gamma^s_{rt}\frac{\partial\psi^r}{\partial x^i}\frac{\partial\psi^t}{\partial x^{\overline{\jmath}}}\right)\right) = 0.
\end{aligned}$$

□

3. d'-Horizontal weak conformality

Definition 6. *Let (X, γ) be Hermitian manifold and (N, g) a Riemannian manifold. A map $\phi : X \to N$ is called a d'-horizontally weakly conformal if the map $d'\phi : T'X \to T^{\mathbb{C}}N$ is horizontally weakly conformal with respect to the Hermitian scalar products Γ on $T'X$ and G on $T^{\mathbb{C}}N$.*
In terms of the metrics γ and g, this translates as

$$g\left(d'\phi(X), d''\phi(\overline{Y})\right) = \lambda^2 \gamma\left(X, \overline{Y}\right) \qquad \forall X, Y \in \mathcal{H}'.$$

Observe that this condition is weaker than the condition of horizontal weak conformity from a Hermitian to a Riemannian manifold.

Proposition 3. *Let (X, γ) be Hermitian manifold and (N, g) a Riemannian manifold. A map $\phi : X \to N$ pulls back (real-valued) local harmonic functions on N onto local Hermitian harmonic functions on X if and only if ϕ is a d'-horizontally weakly conformal Hermitian harmonic map.*

Proof. We equip the Hermitian manifold X with the local complex coordinates $(z^\alpha)_{\alpha=1,\dots,m}$ and the Riemannian manifold N with the real coordinates $(x^i)_{i=1,\dots,n}$. We denote by Γ^i_{jk} the Christoffel symbols of the manifold (N, g). Let $\phi : X \to N$ be a d'-horizontally weakly conformal Hermitian harmonic map, i.e. satisfying

$$\gamma^{\alpha\overline{\beta}}\left(\frac{\partial^2\phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}} + \Gamma^i_{jk}\frac{\partial\phi^j}{\partial z^\alpha}\frac{\partial\phi^k}{\partial z^{\overline{\beta}}}\right) = 0$$

$$g_{ij}\frac{\partial\phi^i}{\partial z^\alpha}\frac{\partial\phi^j}{\partial z^{\overline{\beta}}} = \lambda^2\gamma_{\alpha\overline{\beta}}$$

and $f : U \subseteq N \to \mathbb{R}$ a local harmonic function, i.e. such that

$$g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) = 0.$$

By a chain rule

$$\begin{aligned}
&\gamma^{\alpha\overline{\beta}}\frac{\partial^2(f\circ\phi)}{\partial z^\alpha \partial z^{\overline{\beta}}} = \\
&\gamma^{\alpha\overline{\beta}}\left[\frac{\partial^2 f}{\partial x^i \partial x^j}\frac{\partial\phi^i}{\partial z^\alpha}\frac{\partial\phi^j}{\partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^i}\frac{\partial^2\phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}}\right] = \\
&\frac{\partial^2 f}{\partial x^i \partial x^j}\lambda^2 g^{ij} + \frac{\partial f}{\partial x^i}\left(-\gamma^{\alpha\overline{\beta}}\Gamma^i_{jk}\frac{\partial\phi^j}{\partial z^\alpha}\frac{\partial\phi^k}{\partial z^{\overline{\beta}}}\right) = \\
&\lambda^2 g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) = 0
\end{aligned}$$

Conversely, let $\phi : X \to N$ be such that $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is Hermitian harmonic whenever $f : U \subseteq N \to \mathbb{R}$ is harmonic. Then

$$\gamma^{\alpha\overline{\beta}} \frac{\partial^2 (f \circ \phi)}{\partial z^\alpha \partial z^{\overline{\beta}}} = \gamma^{\alpha\overline{\beta}} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^j}{\partial z^{\overline{\beta}}} + \frac{\partial f}{\partial x^i} \frac{\partial^2 \phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}} \right) = 0. \tag{17}$$

From the work of Ishihara [4, Lemma 4.1], we know that for any point $q \in N$ and any sets of constants (C_i) and (C_{ij}), such that $C_{ij} = C_{ji}$ and $\sum_i C_{ii} = 0$, there exists, on a neighbourhood of q, a harmonic function f such that

$$\frac{\partial f}{\partial x^i}(q) = C_i \qquad \frac{\partial^2 f}{\partial x^i \partial x^j}(q) = C_{ij}.$$

Taking $C_i = 0 \quad \forall i$, equation (17) becomes

$$\gamma^{\alpha\overline{\beta}} C_{ij} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^j}{\partial z^{\overline{\beta}}} = 0$$

which, following Ishihara's argument, implies that

$$\gamma^{\alpha\overline{\beta}} \frac{\partial \phi^i}{\partial z^\alpha} \frac{\partial \phi^j}{\partial z^{\overline{\beta}}} = \lambda^2 g^{ij}.$$

Using this condition in (17), gives us

$$\lambda^2 g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} + \gamma^{\alpha\overline{\beta}} \frac{\partial f}{\partial x^i} \frac{\partial^2 \phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}} = 0,$$

which, since f is harmonic, is equivalent to

$$\gamma^{\alpha\overline{\beta}} \left(\frac{\partial^2 \phi^i}{\partial z^\alpha \partial z^{\overline{\beta}}} + \Gamma^i_{jk} \frac{\partial \phi^j}{\partial z^\alpha} \frac{\partial \phi^k}{\partial z^{\overline{\beta}}} \right) \frac{\partial f}{\partial x^i} = 0.$$

Choosing f such that $C_{ij} = 0 \quad \forall i, j$ and $C_i = \delta_{ii_0}$ yields

$$\gamma^{\alpha\overline{\beta}} \left(\frac{\partial^2 \phi^{i_0}}{\partial z^\alpha \partial z^{\overline{\beta}}} + \Gamma^{i_0}_{jk} \frac{\partial \phi^j}{\partial z^\alpha} \frac{\partial \phi^k}{\partial z^{\overline{\beta}}} \right) = 0,$$

i.e. ϕ is Hermitian harmonic. □

Proposition 4. *Let (X, γ) be Hermitian manifold, (N, g) and (M, h) be two Riemannian manifolds. Let $\phi : X \to N$ be such that ϕ pulls back (real-valued) local harmonic functions on N onto local Hermitian harmonic functions on X. Then for any harmonic map $\psi : N \to M$, the composition $\psi \circ \phi : X \to M$ is Hermitian harmonic.*

Proof. Let X be equipped with the local complex coordinates $(z^\alpha)_{\alpha=1,\dots,m}$, N be equipped with the real coordinates $(x^i)_{i=1,\dots,n}$ and M with the real coordinates $(y^s)_{s=1,\dots,\dim M}$. Let Γ^i_{jk} denote the Christoffel symbols of (N,g) and ${}^M\Gamma^r_{st}$ the Christoffel symbols of (M,h).

If ϕ pulls back local harmonic functions to local Hermitian harmonic functions then ϕ is a d'-horizontally weakly conformal Hermitian harmonic map. Therefore, for a harmonic map $\psi : N \to M$

$$\begin{aligned}
&\gamma^{\alpha\overline{\beta}}\left(\frac{\partial^2(\psi\circ\phi)^r}{\partial z^\alpha\partial z^{\overline{\beta}}} + {}^M\Gamma^r_{st}\frac{\partial(\psi\circ\phi)^s}{\partial z^\alpha}\frac{\partial(\psi\circ\phi)^t}{\partial z^{\overline{\beta}}}\right) = \\
&\gamma^{\alpha\overline{\beta}}\left(\frac{\partial^2\psi^r}{\partial x^i\partial x^j}\frac{\partial\phi^i}{\partial z^\alpha}\frac{\partial\phi^j}{\partial z^{\overline{\beta}}} + \frac{\partial\psi^r}{\partial x^i}\frac{\partial^2\phi^i}{\partial z^\alpha\partial z^{\overline{\beta}}} + {}^M\Gamma^r_{st}\frac{\partial\psi^s}{\partial x^i}\frac{\partial\phi^i}{\partial z^\alpha}\frac{\partial\psi^t}{\partial x^j}\frac{\partial\phi^j}{\partial z^{\overline{\beta}}}\right) = \\
&\frac{\partial^2\psi^r}{\partial x^i\partial x^j}\lambda^2 g^{ij} + \frac{\partial\psi^r}{\partial x^i}\left(-\Gamma^i_{jk}\frac{\partial\phi^j}{\partial z^\alpha}\frac{\partial\phi^k}{\partial z^{\overline{\beta}}}\gamma^{\alpha\overline{\beta}}\right) + \lambda^2\,{}^M\Gamma^r_{st}g^{ij}\frac{\partial\psi^s}{\partial x^i}\frac{\partial\psi^t}{\partial x^j} = \\
&\lambda^2 g^{ij}\left[\frac{\partial^2\psi^r}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial\psi^r}{\partial x^k} + {}^M\Gamma^r_{st}\frac{\partial\psi^s}{\partial x^i}\frac{\partial\psi^t}{\partial x^j}\right] = 0,
\end{aligned}$$

i.e. $\psi\circ\phi$ is Hermitian harmonic. □

4. Examples

Example 1. Let M be a Hermitian manifold, S a Riemann surface (and therefore cosymplectic) and $\phi : M \to S$ a holomorphic map. Then ϕ is Hermitian harmonic and in local coordinates

$$\mu^2(x) = \gamma^{\alpha\overline{\beta}}\frac{\partial\phi}{\partial z^\alpha}\frac{\partial\overline{\phi}}{\partial z^{\overline{\beta}}},$$

satisfies the only non-trivial condition of biholomorphic horizontal weak conformality. Therefore ϕ is a Hermitian harmonic morphism.

Example 2. Let S be a Riemann surface of metric $\gamma\, dz\otimes d\overline{z}$ and H^2 the Hopf manifold of complex dimension 2 defined by $H^2 = (\mathbb{C}^2 - \{0\}/\Delta_\lambda)$, where Δ_λ is the group generated by the transformations $z\mapsto\lambda z,\ z\in\mathbb{C}-\{0,1\}$.
The space H^2 is a compact Hermitian manifold, though not Kähler, equipped with the metric

$$ds^2 = \frac{dz^1\otimes dz^{\overline{1}} + dz^2\otimes dz^{\overline{2}}}{\mid z^1\mid^2 + \mid z^2\mid^2}.$$

Choose $f:\mathbb{R}\to\mathbb{R}$ to be a solution of the ordinary differential equation (for example a power of x):

$$xg(x)g''(x) + g(x)g'(x) - xg'(x)^2 = 0 \tag{18}$$

and define the map

$$\begin{aligned}
&\phi : S \to H^2\\
&\quad z\mapsto \phi(z) = \left(f\left(\mid z\mid^2\right), f\left(\mid z\mid^2\right)\right).
\end{aligned}$$

The map ϕ is d'-horizontally weakly conformal, since this condition reduces to

$$\frac{\partial\phi^1}{\partial z}\frac{\partial\phi^{\overline{1}}}{\partial\overline{z}} + \frac{\partial\phi^2}{\partial z}\frac{\partial\phi^{\overline{2}}}{\partial\overline{z}} = \gamma\lambda^2 \mid \phi(z) \mid^2,$$

which is trivially satisfied for

$$\lambda^2 = \frac{\mid d\phi(z) \mid^2}{\gamma \mid \phi(z) \mid^2},$$

where $\mid \phi(z) \mid^2 = \mid \phi^1(z) \mid^2 + \mid \phi^2(z) \mid^2$.
Recall the Christoffel symbols of H^2, at the point (z^1, z^2):

$$\begin{aligned}
\Gamma^1_{11} = 2\overline{\Gamma^1_{\overline{2}\overline{2}}} = \frac{-z^{\overline{1}}}{\mid z \mid^2} \quad &, \Gamma^1_{21} = \overline{\Gamma^1_{1\overline{2}}} = \frac{-z^{\overline{2}}}{2 \mid z \mid^2} \\
\Gamma^1_{22} = \Gamma^1_{1\overline{1}} = \Gamma^1_{2\overline{1}} = \Gamma^1_{\overline{1}\overline{1}} = \Gamma^1_{\overline{1}\overline{2}} = \Gamma^1_{\overline{2}\overline{2}} = 0 \\
2\Gamma^2_{1\overline{1}} = \overline{\Gamma^2_{22}} = \frac{z^2}{\mid z \mid^2} \quad &, \Gamma^2_{12} = \overline{\Gamma^2_{\overline{1}2}} = \frac{-z^{\overline{1}}}{2 \mid z \mid^2} \\
\Gamma^2_{11} = \Gamma^2_{\overline{1}\overline{1}} = \Gamma^2_{2\overline{2}} = \Gamma^2_{1\overline{2}} = \Gamma^2_{\overline{1}\overline{2}} = \Gamma^2_{\overline{2}\overline{2}} = 0.
\end{aligned}$$

For the chosen map ϕ:

$$\begin{aligned}
\frac{\partial\phi^i}{\partial z}(z) = \overline{z}f'(\mid z \mid^2), \frac{\partial\phi^i}{\partial\overline{z}}(z) = zf'(\mid z \mid^2), \\
\frac{\partial^2\phi^i}{\partial z\partial\overline{z}}(z) = f'(\mid z \mid^2) + z\overline{z}f''(\mid z \mid^2) \quad i = 1, 2.
\end{aligned}$$

Therefore the right-hand side of equation (3) becomes:

$$\begin{aligned}
&\frac{\partial^2\phi^1}{\partial z\partial\overline{z}} + \Gamma^1_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial z}\frac{\partial\phi^\beta}{\partial\overline{z}} + \Gamma^1_{\overline{\alpha}\beta}\frac{\partial\phi^{\overline{\alpha}}}{\partial z}\frac{\partial\phi^\beta}{\partial\overline{z}} + \Gamma^1_{\alpha\overline{\beta}}\frac{\partial\phi^\alpha}{\partial z}\frac{\partial\phi^{\overline{\beta}}}{\partial\overline{z}} = \\
&\frac{\partial^2\phi^1}{\partial z\partial\overline{z}} - \frac{\phi^{\overline{1}}}{\mid \phi(z) \mid^2}\frac{\partial\phi^1}{\partial z}\frac{\partial\phi^1}{\partial\overline{z}} - \frac{\phi^{\overline{2}}}{\mid \phi(z) \mid^2}\frac{\partial\phi^1}{\partial z}\frac{\partial\phi^2}{\partial\overline{z}} - \\
&\quad \frac{\phi^2}{\mid \phi(z) \mid^2}\frac{\partial\phi^1}{\partial z}\frac{\partial\phi^{\overline{2}}}{\partial\overline{z}} + \frac{\phi^{\overline{1}}}{\mid \phi(z) \mid^2}\frac{\partial\phi^2}{\partial z}\frac{\partial\phi^{\overline{2}}}{\partial\overline{z}} = \\
&f'(x) + \mid z \mid^2 f''(x) - \mid z \mid^2 \frac{f'^2(x)}{f(x)} = 0
\end{aligned} \tag{19}$$

since f is a solution of (18).

Therefore ϕ is a d'-horizontally weakly conformal Hermitian harmonic and according to Proposition 3 pulls back local harmonic functions on H^2 onto local Hermitian harmonic functions on S.

Finally note that since, for example $\Gamma^1_{\overline{2}\overline{2}} \neq 0$, equation (19) shows that holomorphic maps from a Kähler manifold (here the Riemann surface S) to a non-Kählerian Hermitian manifold are not necessarily Hermitian harmonic.

References

[1] Besse, A. L.: *Einstein Manifolds.* A Series of Modern Surveys in Mathematics **10**, Springer-Verlag 1987.

[2] Eells, J.; Sampson, J. H.: *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86** (1964), 109–160.

[3] Fuglede, B.: *Harmonic morphisms between Riemannian manifolds.* Ann. Inst. Fourier (Grenoble), **28** (1978), 107–144.

[4] Ishihara, T.: *A mapping of Riemannian manifolds which preserves harmonic functions.* J. Math. Kyoto Univ. **19** (1979), 215–229.

[5] Jost, J.; Yau, S. T.: *Harmonic maps and superrigidity.* In R.E. Greene and S.T. Yau, editors: Differential Geometry, Proc. Sympos. Pure Math. **54**, 245–280, Amer. Math. Soc., Providence RI, 1993.

[6] Lichnerowicz, A.: *Applications harmoniques et variétés kählériennes.* Rend. Sem. Mat. Fis. Milano **39** (1969), 186–195.

[7] Trèves, F.: *Linear Partial Differential Equations with Constant Coefficients.* Mathematics and its Applications **6**, Gordon and Breach 1966.

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