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On the Affine Rectification of Convex Curves

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1. Introduction

In the following let C be a plane convex curve, i.e. a connected, nonoriented part of the boundary of an arbitrary twodimensional convex region K in the affine plane $\mathbb{A}^n$ with the support function $h(u)$ $(u \in S^1)$. The curve

$$C: \; x = x(t) \quad (\alpha \leqq t \leqq \beta) \tag{1}$$

may be either closed or an arc of bd K. We attach to C the set $\mathcal{T}(C)$ of all of its *support elements*, consisting of a point $x(t)$ together with a support line $H_{h(u)}(u)$ of K, passing through $x(t)$ $\;(u_-(t) \leqq u \leqq u_+(t))$ [1], with the outer unit normal vector u (at the different endpoints $x(\alpha)$ and $x(\beta)$ of an arc C final support elements with the support lines $H_{h(u_-)}(u_-)$ $(u_- \leqq u_+(\alpha))$ and $H_{h(u_+)}(u_+)$ $(u_-(\beta) \leqq u_+)$ may be defined).

It was the idea of G. Pick and W. Blaschke to inscribe into this *generalized tangent bundle* $\mathcal{T}(C)$ of C the union $\bigcup_{l=1}^{k} \mathcal{P}_{l-1l}$ of parabola arcs $\mathcal{P}_{l-1l}$, determined by their final support elements $\mathcal{L}_{l-1} := (x(t_{l-1}), H_{h(u_{l-1})}(u_{l-1}))$ and $\mathcal{L}_l := (x(t_l), H_{h(u_l)}(u_l))$ belonging to a certain subdivision S of $\mathcal{T}(C)$ $\;(l = 1, ..., k;\; t_0 = \alpha,\; u_0 \leqq u_+(\alpha),\; t_k = \beta,\; u_k \geqq u_-(\beta))$. This is possible if we suppose $0 < \measuredangle(u_{l-1}, u_l) < \pi$ and exclude the degenerate cases $x(t_{l-1}) \in H_{h(u_l)}(u_l)$ or $x(t_l) \in H_{h(u_{l-1})}(u_{l-1})$ where we have to replace the parabola arc by a line segment of C. The used method is the equiaffine analogue to the euclidean method of inscribing a polygon because a parabola arc has vanishing affine curvature in analogy to the fact that a line segment has vanishing euclidean curvature. Since the equiaffine arclength of $\mathcal{P}_{l-1l}$ (also in the degenerate case) equals $2V(\Delta_{l-1l})^{\frac{1}{3}}$ where $V(\Delta_{l-1l})$ is the twodimensional volume of the triangle Δ_{l-1l} with the vertices $x(t_{l-1}), x(t_l)$ and $H_{h(u_{l-1})}(u_{l-1}) \cap H_{h(u_l)}(u_l)$ (see [1], § 3) it makes sense to define

$$\overset{a}{d}(\mathcal{L}_{l-1}, \mathcal{L}_l) := 2 \cdot V(\Delta_{l-1l})^{\frac{1}{3}} \tag{2}$$

[1]We identify the points u of S^1 with their polar angles.

as the *affine distance* of the final support elements $\mathcal{L}_{l-1}$ and $\mathcal{L}_l$ of $\mathcal{P}_{l-1l}$. The so defined affine distance fulfils the (inverse) affine triangle inequality

$$\overset{a}{d}(\mathcal{L}_1, \mathcal{L}_2) \geqq \overset{a}{d}(\mathcal{L}_1, \mathcal{L}_3) + \overset{a}{d}(\mathcal{L}_3, \mathcal{L}_2) \tag{3}$$

if $\mathcal{L}_3$ is an intermediate support element of the support elements $\mathcal{L}_1$ and $\mathcal{L}_2$ of $\mathcal{T}(C)$. Here equality holds precisely in the trivial cases $t_1 = t_2, u_1 < u_2$ or $t_1 < t_2, u_1 = u_2$ or $t_1 < t_2, u_1 = u_{12} < u_2$ resp. $u_1 < u_{12} = u_2$ where all affine distances vanish and in the nontrivial case $t_1 < t_2, u_1 < u_{12} < u_2$ for any support element $\mathcal{L}_3$ of the parabola arc $\mathcal{P}$ with the final support elements $\mathcal{L}_1$ and $\mathcal{L}_2$ (see [1], § 16; u_{12} is the outer unit normal vector of the segment from $x(t_1)$ to $x(t_2)$).

For all these reasons we introduce an equiaffinely invariant arclength $L(\mathcal{T}(C))$ for the generalized tangent bundle $\mathcal{T}(C)$ of C by

Definition 1.1. *The (always existing) nonnegative entity*

$$L(\mathcal{T}(C)) := \inf_S \left(\sum_{l=1}^{k} \overset{a}{d}(\mathcal{L}_{l-1}, \mathcal{L}_l) \right) = \inf_S \left(\sum_{l=1}^{k} 2 \cdot V(\Delta_{l-1l})^{\frac{1}{3}} \right) \tag{4}$$

is called affine arclength *of* $\mathcal{T}(C)$ [2] *(affine rectification).*

From Definition 1.1 we may derive

Proposition 1.2. a) *If $C_1 \cup C_2$ is a simple convex curve (closed or an arc) then*

$$L(\mathcal{T}(C_1 \cup C_2)) = L(\mathcal{T}(C_1)) + L(\mathcal{T}(C_2)). \tag{5}$$

b) *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are the final support elements of $\mathcal{T}(C)$ $(\measuredangle(u_1, u_2) < \pi)$ then*

$$L(\mathcal{T}(C)) \leqq \overset{a}{d}(\mathcal{L}_1, \mathcal{L}_2) \tag{6}$$

with equality only for $\overset{a}{d}(\mathcal{L}_1, \mathcal{L}_2) = 0$ or if C equals the parabola arc $\mathcal{P}$ with $\mathcal{T}(\mathcal{P})$ connecting $\mathcal{L}_1$ and $\mathcal{L}_2$.

Proof. a) The additivity property (5) is a consequence of the fact that $\mathcal{T}(C_1 \cup C_2) = \mathcal{T}(C_1) \cup \mathcal{T}(C_2)$ with $\mathcal{T}(C_1) \cap \mathcal{T}(C_2) =: \mathcal{L}_3$ for suitable chosen $\mathcal{T}(C_1)$ and $\mathcal{T}(C_2)$ at the end of C_1 and the beginning of C_2 and of Definition 1.1 since because of (3) there does not exist any restriction for the infimum in (4) (with $C = C_1 \cup C_2$) if each subdivision of $\mathcal{T}(C_1 \cup C_2)$ contains $\mathcal{L}_3$.

b) The inequality (6) is an immediate consequence of (3) and (4). Herein we deduce from the equality in

$$L(\mathcal{T}(C)) = L(\mathcal{T}(C_1)) + L(\mathcal{T}(C_2)) \leqq \overset{a}{d}(\mathcal{L}_1, \mathcal{L}_3) + \overset{a}{d}(\mathcal{L}_3, \mathcal{L}_2) \leqq \overset{a}{d}(\mathcal{L}_1, \mathcal{L}_2) \tag{7}$$

[2]Here it must be supposed that $\overset{a}{d}(\mathcal{L}_{l-1}, \mathcal{L}_l)$ is defined, i.e. $0 \leqq \measuredangle(u_{l-1}, u_l) < \pi$.

(see (5), (6) and (3)) that either $\overset{a}{d}(\mathcal{L}_1, \mathcal{L}_2) = 0$ or that the (arbitrarily chosen) intermediate support element $\mathcal{L}_3$ of C lies in the generalized tangent bundle $\mathcal{T}(\mathcal{P})$. This completes the proof of Proposition 1.2. $\square$

2. Onedimensional affine surface area for an arc of the boundary of a twodimensional plane convex region

Now we start with an arc C of bd K which also may be closed and state that it has been defined for C the notion of a (onedimensional) affine surface area $A(C)$ of C by the Lebesgue integral

$$A(C) := \int_C \kappa(x)^{\frac{1}{3}}\, d\mathcal{H}^1(x) = \lim_{\delta\to+0} c_n \cdot \frac{V(K,C,\delta)}{\delta^{\frac{2}{n+1}}} \tag{8}$$

(C. Schütt and E. Werner [8]). Here κ is the euclidean curvature existing at the normal points of C, i.e. almost everywhere (see [7], theorem 2.5.5), and $\mathcal{H}^1$ is the onedimensional Hausdorff measure on C. Further c_n is a dimension depending constant and $V(K,C,\delta)$ the volume of the shell between K_δ and K over the arc C of bd K where K_δ is the convex floating body of K to the parameter $\delta > 0$ (see [8]). According to the transformation formula

$$\int_C \kappa(x)^{\frac{1}{3}}\, d\mathcal{H}^1(x) = \int_{u(C)} \Delta(h)(u)^{\frac{2}{3}}\, d\omega(u) \tag{9}$$

of D. Hug (see [3], section 2) for the spherical map the affine surface area may also be expressed by

$$A(C) = \int_{u(C)} \Delta(h)(u)^{\frac{2}{3}}\, d\omega(u) \tag{10}$$

(Leichtweiß [4]) with the twodimensional Laplace operator Δ and the Lebesgue measure $\omega(\cdot)$ on S^1. Because of the equivalence of the representation (10) of $A(\text{bd } K) = A(K)$ with a representation of $A(K)$ by E. Lutwak as an infimum (see [5] and [2]) it follows that the functional A is upper semicontinuous in the Hausdorff sense, i.e.

$$\limsup_{\nu\to\infty} A(K_\nu) \leqq A(K) \tag{11}$$

for every sequence $\{K_\nu\}_{\nu\in\mathbb{N}}$ of plane convex regions converging to K.

3. Main result

In the classical case of a convex curve C of differentiability class $\mathcal{C}^3_+$ W. Blaschke (see [1], § 4) proved the coincidence of the result (4) of the affine rectification of C with the onedimensional surface area $A(C)$ of C (see (8). The main result of this paper says that this remains to be true if the smoothness assumption for C is omitted:

Theorem 3.1. *For every arc C of the boundary of a twodimensional convex region K in $\mathbb{A}^2$ we have the equality*

$$\inf_S \left(\sum_{l=1}^{k} \overset{a}{d}(\mathcal{L}_{l-1}, \mathcal{L}_l) \right) = \int_C \kappa(x)^{\frac{1}{3}} \, d\mathcal{H}^1(x). \; ^3 \tag{12}$$

4. Proof of the main result

We shall prove Theorem 3.1 under the stronger assumption that the final support elements $\mathcal{L}_0$ and $\mathcal{L}_k$ of the arc

$$C : x = x(t) \quad (\alpha \leqq t \leqq \beta) \tag{13}$$

have outer unit normal vectors u_- and u_+ with

$$0 \leqq \measuredangle(u_-, u_+) < \pi. \tag{14}$$

Here (14) means no loss of generality as we can get this relation by subdividing C and as both parts of the equality (12) have the additivity property (see (5)).

Now, in a first step of the proof we will show the inequality

$$\int_C \kappa(x)^{\frac{1}{3}} \, d\mathcal{H}^1(x) \leqq \inf_S \left(\sum_{l=1}^{k} \overset{a}{d}(\mathcal{L}_{l-1}, \mathcal{L}_l) \right) =: L(\mathcal{T}(C)). \tag{15}$$

After the consideration just made above it suffices to prove

$$\int_C \kappa(x)^{\frac{1}{3}} \, d\mathcal{H}^1(x) \leqq \overset{a}{d}(\mathcal{L}_0, \mathcal{L}_k) = 2 \cdot V(\Delta_{0k})^{\frac{1}{3}} \tag{16}$$

where $V(\Delta_{0k})$ is the area of the triangle with the vertices $p_0 := x(\alpha)$, $p_k := x(\beta)$ and $p_{0k} := H_{h(u_-)}(u_-) \cap H_{h(u_+)}(u_+)$ if $\mathcal{L}_0 := (x(\alpha), H_{h(u_-)}(u_-))$ and $\mathcal{L}_k := (x(\beta), H_{h(u_+)}(u_+))$ are the final support elements of C [4]. The inequality (16) is trivial if $p_0 \in H_{h(u_+)}(u_+)$ or $p_k \in H_{h(u_-)}(u_-)$ and therefore $V(\Delta_{0k}) = 0$ because then C is a line segment with $\int_C \kappa(x)^{\frac{1}{3}} \, d\mathcal{H}^1(x) = 0$. For this reason we may additionally suppose

$$p_0 \notin H_{h(u_+)}(u_+), \;\; p_k \notin H_{h(u_-)}(u_-) \tag{17}$$

and thus

$$\measuredangle(u_-, u_+) > 0. \tag{18}$$

Now the proof of (16) will be made in an indirect manner. We assume

$$\int_C \kappa(x)^{\frac{1}{3}} \, d\mathcal{H}^1(x) > 2 \cdot V(\Delta_{0k})^{\frac{1}{3}} \tag{19}$$

[3] Despite the euclidean constructions this result is purely equiaffinely invariant.
[4] In the case $u_- = u_+$ we have to read $V(\Delta_{0k}) = 0$.

and come to a contradiction. At first we approximate the arc $C: \; x = x(t) \quad (t_0 \leqq t \leqq t_k)$ with the support elements $(x(t), H_{h(u)}(u))$, where $u_-(t) \leqq u \leqq u_+(t)$ and especially $u_- \leqq u_+(t_0), \;\; u_-(t_k) \leqq u_+$, by the outer parallel arc C^ρ with the support elements $(x(t) + \rho u, H_{h(u)+\rho}(u)) \;\; (\rho > 0)$. Then the arc C^ρ of differentiability class $\mathcal{C}^1$ has $\rho(x)$-smooth points with $\rho(x) \geqq \rho$ [5], and its affine surface area has the value

$$A(C^\rho) := \int_{C^\rho} (\kappa^\rho)(x)^{\frac{1}{3}} \; d(\mathcal{H}^\rho)^1(x) = \int_{u_-}^{u_+} \Delta(h^\rho)(u)^{\frac{2}{3}} \; d\omega(u)$$

$$= \int_{u_-}^{u_+} (\Delta h + \rho \cdot 1)(u)^{\frac{2}{3}} \; d\omega(u) \quad (\rho \geqq 0) \tag{20}$$

(see (9)). But (20) implies

$$\lim_{\rho \to +0} A(C^\rho) = A(C) \tag{21}$$

by Levi's theorem on the monotone convergence such that ρ may be chosen so small that, by a continuity argument, (19) and (17) also hold for C^ρ instead of C. Therefore we go further assuming that (19) as well as (17) and (18) are valid for an arc C of class $\mathcal{C}^1$ with $\rho(x)$-smooth points where $\rho(x) \geqq \rho > 0$.

In this case the spherical map $\sigma^{(C)}$ of C is well-defined and Lipschitzian (see [4], Hilfssatz 1). If we introduce the euclidean arclength parameter σ on C by

$$\sigma(t) := \int_{x(t_0)}^{x(t)} d\mathcal{H}^1(x) = \int_{t_0}^{t} \sqrt[+]{\left(\frac{d\xi_1}{dt}\right)^2 (\tau) + \left(\frac{d\xi_2}{dt}\right)^2 (\tau)} \; d\tau \tag{22}$$

$(x = (\xi_1, \xi_2))$ and the euclidean arclength parameter Θ on the circle S^1 by

$$\Theta(u) := \arctan \frac{\upsilon_2}{\upsilon_1} \tag{23}$$

$(u = (\upsilon_1, \upsilon_2))$ the Lipschitz property of $\sigma^{(C)}$ implies its absolute continuity and therefore

$$\Theta(\sigma) = \int_0^\sigma \frac{d\Theta}{d\sigma}(\varsigma) \; d\varsigma + \Theta(0). \tag{24}$$

Here the derivative $\frac{d\Theta}{d\sigma}$ exists almost everywhere, represents a measurable function and moreover we have by [3], lemma 2.3, for the almost everywhere on C existing normal points $x(\sigma)$ the relation

$$\frac{d\Theta}{d\sigma}(\sigma) = \kappa(\sigma). \tag{25}$$

In the same time the Lipschitz property of $\sigma^{(C)}$ obviously yields

$$|\kappa(\sigma)| = \left| \frac{d\Theta}{d\sigma}(\sigma) \right| \leqq M. \tag{26}$$

[5] A boundary point x of a convex region K is called $\rho(x)$-smooth if $\rho(x) \geqq 0$ is the maximal radius of a ball $B_{\rho(x)}$ with $x \in B_{\rho(x)} \subseteqq K$.

By this way the function $\kappa(\sigma)$ determines the arc C in a unique manner by

$$\xi_1(\sigma) = \int_0^\sigma \cos\left(\Theta(\varsigma) + \frac{\pi}{2}\right)\, d\varsigma + \xi_1(0),$$

$$\xi_2(\sigma) = \int_0^\sigma \sin\left(\Theta(\varsigma) + \frac{\pi}{2}\right)\, d\varsigma + \xi_2(0) \tag{27}$$

where $\Theta(\varsigma)$ is given by (24) and (25).

It is now our aim to approximate C by a $\mathcal{C}^\infty$-smooth convex arc $\tilde{C}$ which also has the properties (17) and (19) of C. For this purpose we consider the function $\kappa(\sigma)^{\frac{1}{3}}$. It is measurable like $\kappa(\sigma)$ and bounded:

$$0 \leqq \kappa(\sigma)^{\frac{1}{3}} \leqq M^{\frac{1}{3}} \tag{28}$$

(see (26)). Hence it is an element of the function space $L^2[0,\ell]$ $(\ell := \sigma(t_k)$ equals the euclidean arclength of C). Now we apply the well-known fact that $\kappa(\sigma)^{\frac{1}{3}}$ may be approximated in the quadratic mean by a Fourier polynomial $p(\sigma)$ on the interval $[0,\ell]$ with prescribed accuracy $\delta > 0$, i.e.

$$\|\kappa^{\frac{1}{3}} - p\|_{L^2[0,\ell]} := \left(\int_0^\ell (\kappa(\sigma)^{\frac{1}{3}} - p(\sigma))^2\, d\sigma\right)^{\frac{1}{2}} < \delta \tag{29}$$

(see [6], p. 60–62). This inequality remains valid if $p(\sigma)$ will be truncated from below and above by the constant functions 0 and $M^{\frac{1}{3}}$ followed by a suitable smoothing procedure (see (28)). By this way we obtain a $\mathcal{C}^\infty$-smooth function $\tilde{p}$ on $[0,\ell]$ with the property

$$\int_0^\ell (\kappa(\sigma)^{\frac{1}{3}} - \tilde{p}(\sigma))^2\, d\sigma < \delta^2 \quad \text{where} \quad 0 \leqq \tilde{p}(\sigma) \leqq M^{\frac{1}{3}}. \tag{30}$$

Now we define the $\mathcal{C}^\infty$-smooth function $\tilde{\Theta}(\sigma)$ on $[0,\ell]$ by

$$\tilde{\Theta}(\sigma) = \int_0^\sigma \tilde{p}(\varsigma)^3\, d\varsigma + \Theta(0) \tag{31}$$

which determines the $\mathcal{C}^\infty$-smooth convex arc $\tilde{C}$, given by

$$\tilde{\xi}_1(\sigma) = \int_0^\sigma \cos\left(\tilde{\Theta}(\varsigma) + \frac{\pi}{2}\right)\, d\varsigma + \xi_1(0),$$

$$\tilde{\xi}_2(\sigma) = \int_0^\sigma \sin\left(\tilde{\Theta}(\varsigma) + \frac{\pi}{2}\right)\, d\varsigma + \xi_2(0). \tag{32}$$

Clearly σ with $\sigma \in [0,\ell]$ is also the euclidean arclength parameter for $\tilde{C}$ for which $\tilde{u}(\sigma) := (\cos\tilde{\Theta}(\sigma), \sin\tilde{\Theta}(\sigma))$ is the unit normal vector and $\tilde{p}(\sigma)^3 \geqq 0$ the curvature at the point

$(\tilde{\xi}_1(\sigma), \tilde{\xi}_2(\sigma))$. Therefore the arcs C and $\tilde{C}$ are touching at their starting points with $\sigma = 0$. They do not differ too much because of the estimation

$$|\tilde{\Theta}(\sigma) - \Theta(\sigma)| = \left| \int_0^\sigma (\tilde{p}(\varsigma)^3 - \kappa(\varsigma))\, d\varsigma \right|$$

$$= \left| \int_0^\sigma (\tilde{p}(\varsigma) - \kappa(\varsigma)^{\frac{1}{3}}) \cdot (\tilde{p}(\varsigma)^2 + \tilde{p}(\varsigma)\kappa(\varsigma)^{\frac{1}{3}} + \kappa(\varsigma)^{\frac{2}{3}})\, d\varsigma \right|$$

$$\leqq \left(\int_0^\sigma (\tilde{p}(\varsigma) - \kappa(\varsigma)^{\frac{1}{3}})^2\, d\varsigma \right)^{\frac{1}{2}} \cdot 3M^{\frac{2}{3}} \sigma^{\frac{1}{2}} \leqq 3M^{\frac{2}{3}} \delta \cdot \sigma^{\frac{1}{2}} \quad (0 \leqq \sigma \leqq \ell) \tag{33}$$

(see (31), (24), (25), (28) and (30)) which implies now in the same manner because of (32) and the corresponding relations for the original arc C of class $\mathcal{C}^1$ that for $i = 1, 2$

$$|\tilde{\xi}_i(\sigma) - \xi_i(\sigma)| \leqq \int_0^\sigma |\tilde{\Theta}(\varsigma) - \Theta(\varsigma)|\, d\varsigma \leqq 2M^{\frac{2}{3}} \delta \cdot \sigma^{\frac{3}{2}} \quad (0 \leqq \sigma \leqq \ell). \tag{34}$$

Here we have used the Lipschitz property of the functions cos and sin with the Lipschitz constant 1. Since we also have for the affine surface areas $A(\tilde{C})$ and $A(C)$ the relation

$$|A(\tilde{C}) - A(C)| = \left| \int_0^\ell (\tilde{p}(\sigma) - \kappa(\sigma)^{\frac{1}{3}})\, d\sigma \right|$$

$$\leqq \left(\int_0^\ell (\tilde{p}(\sigma) - \kappa(\sigma)^{\frac{1}{3}})^2\, d\sigma \right)^{\frac{1}{2}} \cdot \left(\int_0^\ell d\sigma \right)^{\frac{1}{2}} < \delta \cdot \ell^{\frac{1}{2}} \tag{35}$$

(see (30)) we see by (35) as well as by (33) and (34) for $\sigma = \ell$ that δ may be chosen so small that by a continuity argument (19) and (17) also hold for $\tilde{C}$ instead of C.

Now we may proceed assuming that (19) as well as (17) and (18) are valid for an arc C of class $\mathcal{C}^\infty$. For a further computation it is convenient to introduce affine coordinates $\{\xi_1, \xi_2\}$ in the plane $\mathbb{A}^2$ requiring that the affinely independent points p_0, p_k and p_{0k} (see (17) and (18) [6]) have the coordinates

$$p_{0k} = (0,0), \quad p_0 = (-1,1) \text{ and } p_k = (1,1). \tag{36}$$

Then C may be represented by

$$\xi_1(t) = t, \quad \xi_2(t) = f(t) \quad (-1 \leqq t \leqq 1) \tag{37}$$

(f of class $\mathcal{C}^\infty$) because of its convexity and the fact that the lines $\xi_1 + \xi_2 = 0$ and $\xi_1 - \xi_2 = 0$ are the tangential lines of C at its ends $t = -1$ and $t = 1$. This implies for the derivative $\dot{f}$ of f with respect to t the relations

$$\dot{f}(-1) = -1, \quad \dot{f}(1) = 1. \tag{38}$$

[6] Without loss of generality we have here additionally assumed the validity of (14) by a continuity argument.

Hence we obtain after application of the Hölder inequality

$$\int_C \kappa(x)^{\frac{1}{3}}\, d\mathcal{H}^1(x) = A(C) = \int_{-1}^{1} \begin{vmatrix} 1 & \dot{f}(t) \\ 0 & \ddot{f}(t) \end{vmatrix}^{\frac{1}{3}} dt = \int_{-1}^{1} \ddot{f}(t)^{\frac{1}{3}}\, dt$$

$$\leqq \left(\int_{-1}^{1} \ddot{f}(t)dt\right)^{\frac{1}{3}} \cdot \left(\int_{-1}^{1} dt\right)^{\frac{2}{3}} = (\dot{f}(1) - \dot{f}(-1))^{\frac{1}{3}} \cdot 2^{\frac{2}{3}} = 2 = 2 \cdot V(\Delta_{0k})^{\frac{1}{3}} \tag{39}$$

as obviously $V(\Delta_{0k})^{\frac{1}{3}}$ equals 1. But (39) contradicts the assumption (19) for C and this contradiction shows that (16) and therefore also (15) are valid.

Now in a second step we will prove the inverse inequality

$$L(\mathcal{T}(C)) \leqq \int_C \kappa(x)^{\frac{1}{3}}\, d\mathcal{H}^1(x) \tag{40}$$

for every arc $C : x = x(t) \quad (t_0 \leqq t \leqq t_k)$ of a closed convex curve $C \cup C'$ in $\mathbb{A}^2$ where $\mathcal{T}(C) \cap \mathcal{T}(C')$ is the union of the common final support elements of C and C'. Here it is the idea to interpret $\sum_{l=1}^{k} \overset{a}{d}(\mathcal{L}_{l-1}, \mathcal{L}_l) = \sum_{l=1}^{k} 2 \cdot V(\Delta_{l-1l})^{\frac{1}{3}}$ as affine surface area $A(\hat{C})$ of an arc $\hat{C}$ which results from C if we replace the subarcs C_{l-1l} of C from $x(t_{l-1})$ to $x(t_l)$ in the nondegenerate case $x(t_{l-1}) \notin H_{h(u_l)}(u_l)$ and $x(t_l) \notin H_{h(u_{l-1})}(u_{l-1})$ and thus $\measuredangle(u_{l-1}, u_l) > 0$ (compare (17)) by the parabola arcs $\mathcal{P}_{l-1l}$ determined by their final support elements $\mathcal{L}_{l-1} = (x(t_{l-1}), H_{h(u_{l-1})}(u_{l-1}))$ and $\mathcal{L}_l = (x(t_l), H_{h(u_l)}(u_l)) \quad (l = 1, ..., k)$. This is possible since we have equality in the inequality (39) for $\mathcal{P}_{l-1l}$ with the representation

$$\xi_1(t) = t, \;\; \xi_2(t) = \frac{1}{2}(t^2 + 1) \quad (-1 \leqq t \leqq 1)\ ^7 \tag{41}$$

and since we have $A(C_{l-1l}) = 0 = 2 \cdot V(\Delta_{l-1l})^{\frac{1}{3}}$ in the degenerate case.

Now we start with the definition (4) of $L(\mathcal{T}(C))$. As an infimum $L(\mathcal{T}(C))$ may also be represented as a limit in the following manner:

$$L(\mathcal{T}(C)) = \lim_{\nu\to\infty} \left(\sum_{l_\nu=1}^{k_\nu} \overset{a}{d}(\mathcal{L}_{l_\nu-1}, \mathcal{L}_{l_\nu}) \right) = \lim_{\nu\to\infty} A(\hat{C}_\nu). \tag{42}$$

By a suitable subdividing procedure for the subarcs $C_{l_\nu-1l_\nu}$ which does not cancel the infimum nature of the limit (42) because of (3) we achieve without loss of generality the additional property

$$\tan \measuredangle(u_{l_\nu-1}, u_{l_\nu}) < \frac{1}{\nu} \quad (l_\nu = 1, ..., k_\nu) \tag{43}$$

for all $\nu \in \mathbb{N}$. Now the absolute value of the difference of the support functions h resp. $\hat{h}_\nu$ of C resp. $\hat{C}_\nu$ on S^1 along the subarc $C_{l_\nu-1l_\nu}$ may be estimated from above by the height

[7]Without loss of generality we have here assumed $\measuredangle(u_{l-1}, u_l) < \pi$ (compare footnote 6).

Δ_{l_ν} of the triangle $\Delta_{l_\nu-1l_\nu}$ over its side $[p_{l_\nu-1}, p_{l_\nu}]$ because $C_{l_\nu-1l_\nu}$ and $\mathcal{P}_{l_\nu-1l_\nu}$ are contained in $\Delta_{l_\nu-1l_\nu}$:

$$|\hat{h}_\nu(u) - h(u)| \leqq \Delta_{l_\nu} \quad (u_{l_\nu-1} \leqq u \leqq u_{l_\nu};\ 1 \leqq l_\nu \leqq k_\nu).$$

Hence we have because of $\Delta_{l_\nu} \leqq |p_{l_\nu} - p_{l_\nu-1}| \cdot \tan\measuredangle(u_{l_\nu-1}, u_{l_\nu}) \leqq D(C) \cdot \frac{1}{\nu}$ where $D(C)$ is the diameter of C (see (43)), using the definition of the Hausdorff distance,

$$\delta(K, \hat{K}_\nu) \leqq D(C) \cdot \frac{1}{\nu} \quad (\nu \in \mathbb{N}) \tag{44}$$

if K resp. $\hat{K}_\nu$ are the convex bodies in $\mathbb{A}^2$ bounded by $C \cup C'$ resp. $\hat{C}_\nu \cup C'$. In other words (44) may be expressed by

$$\lim_{\nu\to\infty} \hat{K}_\nu = K \tag{45}$$

in the Hausdorff sense. But by (11)

$$\limsup_{\nu\to\infty} A(\hat{K}_\nu) = \limsup_{\nu\to\infty} A(\hat{C}_\nu) + A(C') \leqq A(K) = A(C) + A(C')$$

whence finally by (42)

$$L(\mathcal{T}(C)) = \limsup_{\nu\to\infty} A(\hat{C}_\nu) \leqq A(C) = \int_C \kappa(x)^{\frac{1}{3}}\, d\mathcal{H}^1(x)$$

follows. Thus we have shown the wanted inequality (40) which together with the inequality (15) yields the assertion (12) of Theorem 3.1. □

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