Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 1, 53-65.

About an Inequality of Kubota for Plane Convex Figures

Anita Kripfganz
Institut für Mathematik, Universität Leipzig,
Augustusplatz 10, 04109 Leipzig, Germany

Abstract. This paper is concerned with area-minimal isoperimetric and isodiametric plane convex figures. The corresponding extremal problem is a concave minimization problem in a complete metric space. Among the subset of those figures having maximal circumradius the optimal one is determined as solution of a perimeter partition problem. A certain kind of nonsymmetric inpolyeders of the Reuleaux triangle solves the problem. The corresponding optimal value function gives lower bounds for the area in terms of the perimeter and the diameter. These results allow a sharpening of Kubota's isoperimetric inequality for the considered set of convex figures.

MSC 1991: 52A40 (primary); 52A38, 90C90 (secondary)
Keywords: geometric inequalities, plane convex figures, isoperimetric and isodiametric problem, global minimization

1. Introduction

Numerous geometrical problems are intuitive in the sense of being easy to state. A special kind of such problems is concerned with extremal characteristic parameters of plane convex figures whereas some other numerical characteristics are fixed. These problems are closely related to those of finding complete systems of inequalities for finitely many parameters. A complete system contains relations between the parameters such that for any set of numbers satisfying these inequalities a figure with corresponding values of the characteristics exists. Problems of this type are summarized (e.g.) in the well-known survey of Bonnesen and Fenchel [3], in the problem book of Jaglom and Boltjanski [8], in the survey article of Santaló [20] and in collections of geometric inequalities [4, 18].
For this type of problems even some simplest questions concerning three or four characteristic parameters have not been conclusively investigated. One of such problems is the following:

What is the plane convex and compact set of least area having prescribed diameter and perimeter length?

With respect to the characteristics area, diameter and perimeter length sharp upper bounds for the area in terms of perimeter length and diameter are known. We ask for sharp lower bounds. For this we discuss the problem from the point of view of optimization theory.

A figure $\mathcal{B}$ is a plane compact set. Using the notations

$\mathcal{A}_2$ for the set of all convex figures,
$\mathcal{F}$ for the area,
$\mathcal{L}$ for the perimeter length and
$\mathcal{D}$ for the diameter

the problem is formulated as a parametric optimization problem:

$$\text{(P)}: \quad \mathcal{F}(\mathcal{B}) \to \text{Min!} \qquad \mathcal{B} \in \mathcal{A}_2 : \mathcal{L}(\mathcal{B}) = L, \ \mathcal{D}(\mathcal{B}) = D.$$

The optimal value function of this problem depends on the parameter values L and D and describes sharp lower bounds for the area.
For certain values of L and D isoperimetric and isodiametric convex figures $\mathcal{B}^*$ of least area are well-known. Namely, if $2D \le L \le 3D$ then $\mathcal{B}^*$ is an isosceles triangle (see e.g. [8]) with

$$\mathcal{F}(\mathcal{B}^*) = \frac{1}{4}(L - 2D)\sqrt{L(4D - L)}. \tag{1.1}$$

For the maximal possible perimeter length $L = \pi D$ the Reuleaux triangle has the minimal area

$$\mathcal{F}(\mathcal{B}^*) = \frac{1}{2}(\pi - \sqrt{3})D^2. \tag{1.2}$$

First proofs of this area-minimizing property of the Reuleaux triangle are due to Blaschke [1] and Lebesgue [17]. A modern approach via control theory is given by Klötzler [9]. This author transforms the geometrical problem into a problem of optimal control and evaluates necessary optimality conditions in the shape of Pontryagin's Maximum Principle.
For $3D < L < \pi D$ optimal figures are unknown up to now. The inequalities

$$F \ge \frac{1}{4}(L - 2D)\sqrt{L(4D - L)}, \tag{1.3}$$

$$F \ge \frac{1}{4}(L - 2D)\sqrt{3}D \tag{1.4}$$

were established by Kubota [14, 15] and Favard [6]. For $L \ge 3D$ the lower bound for the area in (1.4) is a sharper one than that in (1.3). Both inequalities are not sharp. Sholander [22] has shown necessary properties of an area-minimal figure, namely:

1. $\mathcal{B}^*$ is a triarc. This means that there are at most three points of $\partial\mathcal{B}^*$ such that each pair of parallel supporting lines contains at least one of these points.
2. $\mathcal{B}^*$ is a polyeder and a cutting figure of a Reuleaux-n-gon, $n \le 5$.

From this Sholander has conjectured that $\mathcal{B}^*$ is an inpolyeder of the Reuleaux triangle.

We discuss the structure of optimization problem (P) and study relations to the circumradius. This leads to the supposition that $\mathcal{B}^*$ has maximal circumradius R. For $L \in (3D, \pi D)$ and $R = D/\sqrt{3}$ we determine area-minimal isoperimetric and isodiametric convex figures. The results allow us to sharpen inequality (1.4) in the considered subset of convex figures.

2. Isoperimetric and isodiametric convex figures

To begin with, we summarize properties of the extremal problem (P) and illustrate the feasible domain by means of a diagram for the considered parameters.

2.1. The optimization problem

Together with Minkowski addition and multiplication with non-negative real numbers the set

$$\mathcal{A}_2 = \{\mathcal{B} \subseteq \mathbb{E}^2 \quad / \quad \mathcal{B} \text{ convex and compact}\}$$

is an abstract convex cone. Equipped with the Hausdorff metric $\mathcal{A}_2$ is a complete metric space. Each closed and bounded subset of this metric space is compact (see [21]). Now, we restrict us to a subset $\mathcal{A}_2^d$ of $\mathcal{A}_2$. If $\mathcal{B} \in \mathcal{A}_2^d$ then one of its diameters points to a fixed direction d. Each figure of $\mathcal{A}_2$ is congruent to one of $\mathcal{A}_2^d$. The set $\mathcal{A}_2^d$ is a convex closed cone, too. The functionals $\mathcal{L}$ and $\mathcal{D}$ are continuous and Minkowski linear (additive and positively homogeneous) on $\mathcal{A}_2^d$. Hence

$$\mathcal{A}_2(L, D) = \{\mathcal{B} \in \mathcal{A}_2^d \quad / \quad \mathcal{L}(\mathcal{B}) = L,\ \mathcal{D}(\mathcal{B}) = D\}$$

is a convex and compact subset of $\mathcal{A}_2$.

Because of the Brunn-Minkowski theorem the square root of the area is a concave functional on $\mathcal{A}_2$. This means that the considered geometrical problem corresponds to the following concave minimization problem:

$$(\mathrm{P}_1): \qquad \sqrt{\mathcal{F}(\mathcal{B})} \to \text{Min!} \qquad \mathcal{B} \in \mathcal{A}_2(L, D).$$

Optimal solutions exist. A figure $\mathcal{B} \in \mathcal{A}_2(L, D)$ which can be represented as a proper convex combination of two different ones from $\mathcal{A}_2(L, D)$ is not area-minimal. For instance, diameter and perimeter preserving symmetrizations of figures $\mathcal{B} \in \mathcal{A}_2^d$ like the centralization $C(\mathcal{B}) = \frac{1}{2}\mathcal{B} + \frac{1}{2}(-\mathcal{B})$ and the special Blaschke symmetrization $B(\mathcal{B}, L_d) = \frac{1}{2}\mathcal{B} + \frac{1}{2}\mathcal{B}^d$ with respect to a line L_d of direction d do not lead to smaller areas. In this, $\mathcal{B}^d$ denotes the reflection of $\mathcal{B}$ in L_d. Furthermore, a local minimum need not be the global one.

2.2. The *L-D-R*-diagram

We take into consideration the circumradius of a convex figure as an additional characteristic parameter. The circumcircle of $\mathcal{B} \in \mathcal{A}_2$ is the smallest circle containing $\mathcal{B}$. It is spanned either by two diametral opposite points of $\partial\mathcal{B}$ or by three points of $\partial\mathcal{B}$ forming an acute angled triangle ($\partial\mathcal{B}$ denotes the boundary of $\mathcal{B}$). This implies the inequality

$$\frac{D}{2} \leq R \leq \frac{D}{\sqrt{3}} \tag{2.1}$$

for the circumradius $\mathcal{R}(\mathcal{B}) = R$ of $\mathcal{B} \in \mathcal{A}_2$ with $\mathcal{D}(\mathcal{B}) = D$. The functional $\mathcal{R}$ is continuous and convex on $\mathcal{A}_2$. For the pairs (L, D) and (L, R) the following inequalities hold:

$$2D \leq L \leq \pi D, \tag{2.2}$$

$$4R \leq L \leq 2\pi R. \tag{2.3}$$

But these three inequalities do not form a complete system because there is no convex figure with $D = 1,\ R = 1/\sqrt{3}$ and $L = 4/\sqrt{3}$. To sharpen this system of inequalities we consider the problem

$$(\mathrm{P}_2): \qquad \begin{array}{l} \mathcal{L}(\mathcal{B}) \to \text{Min!} \\ \qquad \mathcal{B} \in \mathcal{A}_2(R, D) \end{array}$$

with the linear and continuous objective functional $\mathcal{L}$ over the compact feasible domain $\mathcal{A}_2(R, D) := \{\mathcal{B} \in \mathcal{A}_2^d \ / \ \mathcal{D}(\mathcal{B}) = D,\ \mathcal{R}(\mathcal{B}) = R\}$. Solutions of this problem exist.

Lemma 1. *The minimal perimeter length*

$$L = D\left(2 + \sqrt{4 - \left(\frac{D}{R}\right)^2}\right)$$

in the class $\mathcal{A}_2(R, D)$ is attained for an isosceles triangle.

Proof. Let $\bar{\mathcal{B}}$ denote an isosceles triangle of leg length D and circumradius R. Then the optimal figure $\hat{\mathcal{B}}$ of (P_2) satisfies the condition

$$\mathcal{L}(\hat{\mathcal{B}}) \leq 2D + S = \mathcal{L}(\bar{\mathcal{B}})$$

where $S = \frac{D}{R}\sqrt{4R^2 - D^2}$. If $2R = D$ then $\mathcal{L}(\hat{\mathcal{B}}) = 2D$. Otherwise, the circumcircle is spanned by an acute angled triangle Δ_R. From $2D + S \leq \mathcal{L}(\Delta_R) \leq \mathcal{L}(\mathcal{B})$ for all $\mathcal{B} \in \mathcal{A}_2(R, D)$ it follows the assumption. □

Adding the inequality

$$L \geq D\left(2 + \sqrt{4 - \left(\frac{D}{R}\right)^2}\right) \tag{2.4}$$

to (2.1), (2.2) and (2.3) this system becomes a complete one. The feasible points are illustrated in Fig. 1 for $D = 1$.

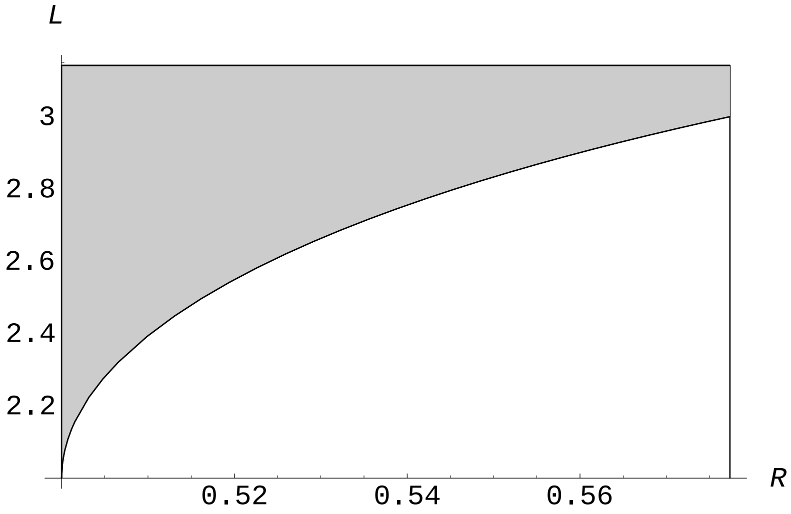

Figure 1

It was an idea of Blaschke (see [2]) to represent relations between three fundamental parameters of a convex body in a plane diagram. Santalo [20] has summarized such Blaschke diagrams for plane convex figures. If we put

$$x = \frac{\sqrt{3}R}{D}, \quad y = \frac{L}{\pi D}$$

the inequalities (2.1) up to (2.4) lead to a plane diagram which is contained in the unit square.

For $L \in [2D, 3D]$ and $L = \pi D$ the area-minimal figures of $\mathcal{A}_2(L, D)$ have maximal circumradius. For $L \in (3D, \pi D)$ we conjecture that the optimal figure $\mathcal{B}^*$ of problem (P$_1$) has maximal circumradius, too. In the following we determine area-minimal figures of $\mathcal{A}_2(L, D)$ for $L \in (3D, \pi D)$ and $R = D/\sqrt{3}$.

3. Area-minimal inpolyeders of the Reuleaux triangle

Figures of $\mathcal{A}_2(L, D)$ with circumradius $R = D/\sqrt{3}$ are contained in the corresponding Reuleaux triangle. The circumcircle is spanned by an equilateral triangle. Hence, we can decompose the figure into three sector indomains.

3.1. Area-minimal sector indomains

In 1929 Favard [6] has solved the problem of area-minimal figures with given perimeter length and circumradius. We have generalized this problem (see [10]) substituting the circumcircle by a circular sector S_ω with radius 1 (w.l.o.g.) and with central angle $\omega \in [0, 2\pi]$. Let Δ_ω be the isosceles triangle corresponding to S_ω. Then the plane set $\mathcal{B}_\omega$ denotes an ω-sector indomain if $S_\omega \cap \Delta_\omega \subseteq \mathcal{B}_\omega \subseteq S_\omega$ and $\Delta_\omega \cup \mathcal{B}_\omega$ is convex. The perimeter of $\mathcal{B}_\omega$ is formed by the legs of Δ_ω and a convex curve of length L.

It turns out that the area-minimal figure is polyhedral (see [10]). A polyhedral sector indomain is called *sector inpolyeder* if all vertices except the vertex of the sector belong to the sector arc. It can be described by corresponding central angles. This inpolyeder is called almost regular if all but one central angles are equal, the remaining one is not greater than the others. It is called regular if all central angles are equal. A regular sector inpolyeder with $n \geq 2$ central angles $\frac{\omega}{n}$ is of perimeter length $2 + l_n^\omega$ with

$$l_n^\omega := 2n \cdot \sin \frac{\omega}{2n}. \tag{3.1}$$

The lower bound for the partial perimeter length L is given by

$$l_1^\omega := \begin{cases} 2 \sin \frac{\omega}{2} & \text{for } \omega \in [0, \pi] \\ 2\left(1 - \cos \frac{\omega}{2}\right) & \text{for } \omega \in [\pi, 2\pi]. \end{cases} \tag{3.2}$$

The following theorem describes the structure of an area-minimal sector indomain and the dependence of its area from the parameter L.

Theorem 1. *For $L \in (l_{n-1}^\omega, l_n^\omega]$ the area-minimal sector indomain is a uniquely determined almost regular sector inpolyeder with n central angles. Depending on L the minimal area is piece-wisely given by*

$$f_\omega(L) = \begin{cases} \frac{L^2}{4} \cot \frac{\omega}{4} - \sin \frac{\omega}{2} & \text{for } n = 2 \\ \frac{n-1}{2} \sin \alpha_\omega(L) + \frac{1}{2} \sin(\omega - (n-1)\alpha_\omega(L)) & \text{for } n \geq 3 \end{cases}$$

with $\alpha_\omega(L)$ according

$$L = 2(n-1) \sin \frac{\alpha_\omega(L)}{2} + 2 \sin \frac{\omega - (n-1)\alpha_\omega(L)}{2}.$$

Proof. See [10].

The implicitely defined optimal value function f_ω is called *fonction penetrante*. Its properties are summarized in the following.

Theorem 2. *The function f_ω is a continuous and strictly monotonously increasing function on $[l_1^\omega, \omega]$ with the following properties:*

a) *f_ω is continuously differentiable on each interval $(l_{n-1}^\omega, l_n^\omega)$ for $n \geq 2$, with negative third and fourth derivatives for $n \geq 3$.*

b) *f_ω is strictly convex on $[l_1^\omega, l_2^\omega]$, it has a unique turning point ${}^1\lambda_n^\omega$ on each interval $(l_{n-1}^\omega, l_n^\omega)$, $n \geq 3$, such that it is strictly convex on $[l_{n-1}^\omega, {}^1\lambda_n^\omega]$ and strictly concave on $[{}^1\lambda_n^\omega, l_n^\omega]$.*

c) *The function $\tilde{f}_\omega$ implicitely defined on $[l_2^\omega, \omega]$ by*

$$\tilde{f}_\omega(L) = \frac{x(L)}{2} \sin \frac{\omega}{x(L)},$$

$$L = 2x(L) \cdot \sin \frac{\omega}{2x(L)}, \quad x(L) \geq 2,$$

is a lower convex envelope of f_ω *with*

$$f_\omega(l_n^\omega) = \tilde{f}_\omega(l_n^\omega) \quad \text{for all } n \geq 2.$$

Proof. See [11].

We call the function $\tilde{f}_\omega$ *smoothed 'fonction penetrante'.*

3.2. Area-minimal inpolyeders of regular Reuleaux polygons

A convex figure is a set of constant width D if each boundary point of the figure is incident with a diameter. Reuleaux-m-gons $\mathcal{R}_m$ are special sets of constant width. Their boundary is formed by a finite and odd number m of circular arcs with radius D (see [1], [5]). The centers of these arcs are the vertices of the Reuleaux polygon. If these vertices form a regular polygon then the Reuleaux polygon is called regular, too. Another definition of Reuleaux polygons is given in [16]. Blaschke [1] and later Firey [7] have found that the area of regular Reuleaux polygons increases with m. Moreover they have shown that for fixed m the infimum of areas of all m-sided Reuleaux polygons equals the area of the Reuleaux triangle (see also Sallee [19]).

An inpolyeder ${}^m\mathcal{P}_k$ of a Reuleaux polygon $\mathcal{R}_m$ is a convex polyeder inscriped in $\mathcal{R}_m$ such that the vertices of the Reuleaux polygon are vertices of the inpolyeder. All $k \geq m$ vertices of ${}^m\mathcal{P}_k$ belong to the boundary of $\mathcal{R}_m$. We ask for area-minimal inpolyeders of regular Reuleaux polygons where the perimeter length is fixed.
In order to achieve this we decompose the inpolyeder ${}^m\mathcal{P}_k$ of the Reuleaux polygon $\mathcal{R}_m$ into m sector inpolyeders. Each vertex of $\mathcal{R}_m$ corresponds to the vertex of a sector S_ω with angle $\omega = \frac{\pi}{m}$. Now, we determine area-minimal m-tuples of sector inpolyeders while fixing the total perimeter length. Because of Theorem 1 only m-tuples of almost regular inpolyeders come into question as optimal solutions. Using the optimal value function f_ω the problem can be formulated as follows:

$$({}^{\mathrm{m}}\mathrm{P}^{\omega,\mathrm{L}}): \qquad \sum_{i=1}^{m} f_\omega(L_i) \to \text{Min!}$$

$$\sum_{i=1}^{m} L_i = L$$

$$\underline{L}_\omega \leq L_i \leq \omega.$$

We have to find the optimal partition $\mathbb{L}^* = (L_1^*, ..., L_m^*)$ of the total perimeter length L and the corresponding m-tuple of almost regular sector inpolyeders. It turns out that the convexity defect of the 'fonction penetrante' influences the solution behavior, essentially. The unique zeros ${}^m\lambda_n^\omega$ of the defect functions

$$
{}^mD_\omega^n(\Lambda) := \begin{cases} (m-1)\cdot f_\omega(l_{n-1}^\omega) + f_\omega(m\Lambda - (m-1)l_{n-1}^\omega) - m\cdot f_\omega(\Lambda) \\ \qquad \text{for } \Lambda \in \left[l_{n-1}^\omega, \frac{1}{m}\left((m-1)l_{n-1}^\omega + l_n^\omega\right)\right] \\ (m-1) f_\omega\left(\frac{m\Lambda - l_n^\omega}{m-1}\right) + f_\omega(l_n^\omega) - m\cdot f_\omega(\Lambda) \\ \qquad \text{for } \Lambda \in \left[\frac{1}{m}\left((m-1)l_{n-1}^\omega + l_n^\omega\right), l_n^\omega\right] \end{cases} \tag{3.3}
$$

defined on $[l_{n-1}^\omega, l_n^\omega]$ describe branch points of the solution structure.

Theorem 3. *Let n be determined according $\frac{L}{m} \in (l_{n-1}^\omega, l_n^\omega]$. Then $L_i^* \in [l_{n-1}^\omega, l_n^\omega]$ holds for $i = 1, ..., m$.*
If $n = 2$ then the symmetric solution $\mathbb{L}^* = (L/m, ..., L/m)$ is optimal.
If $n \geq 3$ then the zeros ${}^k\lambda_n^\omega$ of the defect functions ${}^kD_\omega^n$, $k = 2, ..., m$, satisfy the recursive conditions

$$
{}^k\lambda_n^\omega \in \left(\frac{1}{k}\left(l_{n-1}^\omega + (k-1)\,{}^{k-1}\lambda_n^\omega\right), {}^{k-1}\lambda_n^\omega\right)
$$

and define branch points of the solution structure of problem $({}^{\mathrm{m}}\mathrm{P}^{\omega,\mathrm{L}})$ in the following manner:

1.) $\mathbb{L}^* = \left(\frac{L}{m}, ..., \frac{L}{m}\right)$ *for* $\frac{L}{m} \in \left(l_{n-1}^\omega, {}^m\lambda_n^\omega\right]$

2.) $\mathbb{L}^* = \left(\underbrace{l_{n-1}^\omega, ..., l_{n-1}^\omega}_{m-(j+1)}, L - j\cdot l_n^\omega - (m-j-1)l_{n-1}^\omega, \underbrace{l_n^\omega, ..., l_n^\omega}_{j}\right)$

for $L \in \left[(m-j)\cdot {}^{m-j}\lambda_n^\omega + j\cdot l_n^\omega,\ (m-(j+1))l_{n-1}^\omega + (j+1)l_n^\omega\right]$
with $j \in \{0, ..., m-1\}$

3.) $\mathbb{L}^* = \left(\frac{L-(j+1)l_n^\omega}{m-(j+1)}, ..., \frac{L-(j+1)l_n^\omega}{m-(j+1)}, \underbrace{l_n^\omega, ..., l_n^\omega}_{j+1}\right)$

for $L \in \Big[Max\left\{(m-j)\cdot {}^{m-j}\lambda_n^\omega + j\cdot l_n^\omega, (m-(j+1)l_{n-1}^\omega + (j+1)l_n^\omega\right\},$
$\qquad (j+1)l_n^\omega + (m-(j+1))\cdot {}^{m-(j+1)}\lambda_n^\omega\Big]$
with $j \in \{0, ..., m-2\}$.

All intervals are nonempty for n large enough.

Proof. See [12], [13].

Remark. For $\frac{L}{m} \in (l^{\omega}_{n-1}, {}^{1}\lambda^{\omega}_{n})$, $n \geq 3$, the symmetric partition $\mathbb{L}^* = (\frac{L}{m}, ..., \frac{L}{m})$ is a local minimal solution of problem $({}^{\mathrm{m}}\mathrm{P}^{\omega,\mathrm{L}})$. This property follows from the sufficient optimality conditions for a local minimum.

The results show that regular and almost regular sector inpolyeders can participate in the solution. The number of vertices of two such sector inpolyeders may differ by one at most. Symmetric solution structures (first case) and nonsymmetric ones (second, third case) alternate with each other. An area-minimal inpolyeder ${}^{m}\mathcal{P}^{*}_{k}$ of the regular Reuleaux-m-gon consists of the optimal m-tuple of sector inpolyeders with central angle $\omega = \frac{\pi}{m}$. For given perimeter length $L = m \cdot l^{\omega}_{n}$, $n \geq 1$, this inpolyeder is regular. An area-minimal inpolyeder ${}^{3}\mathcal{P}^{*}_{k}$ of the Reuleaux triangle ($m = 3$, $\omega = \frac{\pi}{3}$) is illustrated in Figure 2 for $D = 1$, $L = 3,1176$ and $k = 9$.

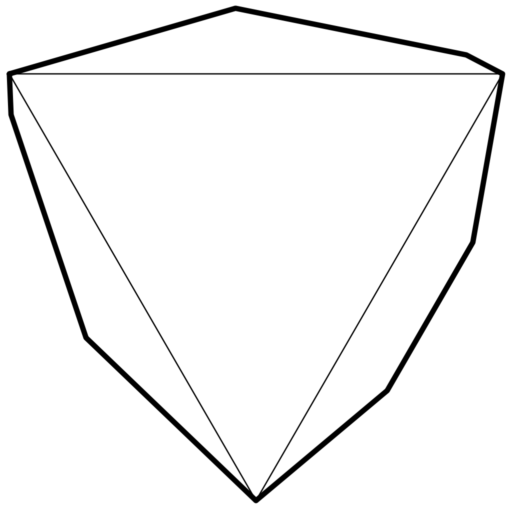

Figure 2

A Reuleaux-m-gon $\mathcal{R}_m$ is associated with its diagonal-polygon $\mathcal{D}_m$ (see [1]). This diagonal polygon is a star polygon which joins the vertices of $\mathcal{R}_m$ with each other via diameter diagonals (straight lines of length D). $\mathcal{D}_3$ is the equilateral triangle of diameter D. According to Blaschke it holds

$$\frac{\pi}{2} \cdot D^2 = \mathcal{F}(\mathcal{R}_m) + 2\mathcal{F}(\mathcal{D}_m)$$

for the corresponding areas. Analogously we get the relation

$$\mathcal{F}(\mathcal{P}^{\pi}_{k}) = \mathcal{F}({}^{m}\mathcal{P}_k) + 2\mathcal{F}(\mathcal{D}_m) \tag{3.4}$$

for the area of inpolyeders ${}^{m}\mathcal{P}_k$ of regular Reuleaux-m-gons. In this, $\mathcal{P}^{\pi}_{k}$ denotes the inpolyeder of the semicircle which is composed of the m sector inpolyeders corresponding to ${}^{m}\mathcal{P}_k$. The k central angles of $\mathcal{P}^{\pi}_{k}$ are those of the sector inpolyeders.

Lemma 2. *For given perimeter length $L = 3 \cdot l^{\frac{\pi}{3}}_{n}$, $n \geq 1$, the area of any inpolyeder of a regular Reuleaux-m-gon is not smaller than that of the area-minimal inpolyeder of the Reuleaux triangle.*

Proof. Let ${}^m\mathcal{P}_k$ be an inpolyeder of the regular Reuleaux-m-gon with $\mathcal{L}({}^m\mathcal{P}_k) = L$. We consider the corresponding inpolyeder $\mathcal{P}_k^\pi$ of the semicircle. Because of $3 \cdot l_n^{\frac{\pi}{3}} = l_{3n}^\pi$ and Theorem 1 the unique area-minimizing inpolyeder of the semicircle is regular with central angles $\alpha_\pi(L) = \frac{\pi}{3n}$ and area $\frac{3n}{2}\sin\frac{\pi}{3n}$. This inpolyeder of the semicircle can be composed to an area-minimal inpolyeder of the Reuleaux triangle. Otherwise, it holds $\mathcal{F}(\mathcal{D}_m) \leq \mathcal{F}(\mathcal{D}_3)$ for all corresponding diagonal-polygons (see [1]). Hence, (3.4) implies the inequality

$$\mathcal{F}({}^m\mathcal{P}_k) \geq \frac{3n}{2}\sin\frac{\pi}{3n} - 2\mathcal{F}(\mathcal{D}_3) = \mathcal{F}({}^3\mathcal{P}^*_{3n})$$

and consequently the assertion. □

This lemma shows that in the considered class of figures the area-minimal one has maximal circumradius.

Lemma 3. *The circumradius of an area-minimal figure $\mathcal{B}^* \in \mathcal{A}_2(L, D)$ is greater than $\frac{D}{2}$.*

Proof. For $L \in [2D, 3D] \cup \{\pi D\}$ the assertion follows from the known structure of $\mathcal{B}^*$.
For $L \in (3D, \pi D)$ we consider a figure $\mathcal{B}^o \in \mathcal{A}_2(L, D)$ with $\mathcal{R}(\mathcal{B}^o) = \frac{D}{2}$. Assuming that $\mathcal{B}^o$ is area-minimal. Then the Theorems 1 and 3 imply that $\mathcal{B}^o$ is composed of two almost regular inpolyeders of the circumcircle. Furthermore, each vertex of $\mathcal{B}^o$ has to be an endpoint of a diameter of $\mathcal{B}^o$. If the central angles of $\mathcal{B}^o$ are arbitrarily ordered then it follows that $\mathcal{B}^o$ has to be a regular inpolyeder of the circumcircle.
We consider two diametral opposite vertices A and B of $\mathcal{B}^o$ together with their neighbouring vertices A_1, A_2 and B_1, B_2. They form two isosceles triangles. Variing the vertices A, B on an ellipse with focuses A_1, A_2 and B_1, B_2, resp., we get a figure $\tilde{\mathcal{B}} \in \mathcal{A}_2(L, D)$ of smaller area which contradicts the assumption. □

These results support our conjecture about the maximality of the circumradius of $\mathcal{B}^*$.

4. Lower bounds for the area

We consider figures $\mathcal{B} \in \mathcal{A}_2(L, D)$ with area F and maximal circumradius $R = D/\sqrt{3}$ for $L \in [3D, \pi D)$. The area-minimal one is an inpolyeder of the Reuleaux triangle composed of a triple of almost regular sector inpolyeders with central angle $\omega = \frac{\pi}{\sqrt{3}}$.

Theorem 4. *Let $\mathcal{B} \in \mathcal{A}_2(L, D)$ be a convex figure with $\mathcal{R}(\mathcal{B}) = D/\sqrt{3}$.*
If $\frac{L}{D} \in [3, 12\sin\frac{\pi}{12}]$ then

$$F \geq \frac{\sqrt{3}+2}{12}L^2 - \frac{\sqrt{3}+3}{2}D^2.$$

If $\frac{L}{D} \geq 12\sin\frac{\pi}{12}$ then

$$F \geq \frac{LD}{4}\sqrt{4 - \left(\frac{L}{3Dx(\frac{L}{D})}\right)^2} - \frac{\sqrt{3}}{2}D^2$$

with $x(\frac{L}{D}) \geq 2$ *according*

$$\frac{L}{D} = 6x \cdot \sin \frac{\pi}{6x}.$$

Proof. For $m = 3$ and $R = D/\sqrt{3}$, (3.4) implies

$$F \geq \sum_{i=1}^{3} f_{\frac{\pi}{3}} \left(\frac{L_i^*}{D} \right) \cdot D^2 - \frac{1}{2}\sqrt{3}\, D^2 \tag{4.1}$$

where $\frac{1}{D}(L_1^*, L_2^*, L_3^*)$ denotes the optimal solution of problem $({}^3\mathrm{P}^{\omega,\mathrm{L/D}})$ for $\omega = \frac{\pi}{3}$.
If $\frac{L}{3D} \in [1, 4\sin\frac{\pi}{12}]$ then $L_1^* = L_2^* = L_3^* = \frac{L}{3}$ and

$$f_{\frac{\pi}{3}} \left(\frac{L}{3D} \right) = \frac{1}{36} \frac{L^2}{D^2} \left(\sqrt{3} + 2 \right) - \frac{1}{2}$$

follows from Theorem 3. Hence we get the first estimation which is sharp for the corresponding area-minimal figure.
If $\frac{L}{3D} \geq 4\sin\frac{\pi}{12}$ then we substitute f_ω by its lower convex envelope $\tilde{f}_\omega$. The convexity of $\tilde{f}_\omega$ and Theorem 2 imply

$$F \geq 3 \cdot \tilde{f}_{\frac{\pi}{3}} \left(\frac{L^*}{3D} \right) \cdot D^2 - \frac{1}{2}\sqrt{3}\, D^2$$

from (4.1). It follows

$$F \geq \frac{3}{2} D^2 \cdot x \left(\frac{L}{D} \right) \cdot \sin \frac{\pi}{3x\left(\frac{L}{D}\right)} - \frac{1}{2}\sqrt{3}\, D^2 \tag{4.2}$$

with $x(\frac{L}{D}) \geq 2$ from

$$\frac{L}{3D} = 2x \cdot \sin \frac{\pi}{6x}. \tag{4.3}$$

Substituting

$$\sin \frac{\pi}{3x} = 2 \cdot \frac{L}{6Dx} \sqrt{1 - \frac{L^2}{36D^2x^2}}$$

in the right hand side of (4.2) we get the second inequality. □

A comparison with the inequality (1.4) of Kubota follows in the first case from

$$\frac{\sqrt{3}+2}{12} L^2 - \frac{3}{2} D^2 \geq \frac{\sqrt{3}}{4} LD$$

and in the second case from

$$\frac{L}{3D} \leq x \left(\frac{L}{D} \right)$$

because of (4.3) and $x(\frac{L}{D}) \geq 2$. In the limiting case $L \to \pi D$ and $x \to +\infty$ the lower bound of F tends to the area of the Reuleaux triangle. It remains to show that the circumradius of an isoperimetric and isodiametric area-minimal figure is necessarily maximal.

References

[1] Blaschke, W.: *Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts.* Math. Ann. **76** (1915), 504–513.

[2] Blaschke, W.: *Eine Frage über konvexe Körper.* Jahresber. Deutsche Math.-Ver. **25** (1916), 121–125.

[3] Bonnesen, T., Fenchel, W.: *Theorie der konvexen Körper.* Springer-Verlag, Berlin 1936.

[4] Burago, Yu. D., Zalgaller, V. A.: *Geometric Inequalities.* Springer-Verlag, Berlin 1988.

[5] Eggleston, H. G.: *Convexity.* Cambridge University Press 1958.

[6] Favard, J. M.: *Problèmes d' Extremums Relatifs aux Courbes Convexes (I).* Ann. Sci. École Norm. Sup. **46** (1929), 345–369.

[7] Firey, W. J.: *Isoperimetric Ratios of Reuleaux Polygons.* Pacific J. Math. **10** (1960), 823–829.

[8] Jaglom, I. M., Boltjanski, W. G.: *Konvexe Figuren.* Dt. Verl. Wiss., Berlin 1956.

[9] Klötzler, R.: *Beweis einer Vermutung über n-Orbiformen kleinsten Inhalts.* ZAMM **55** (1975), 557–570.

[10] Kripfganz, A.: *The Generalized Favard Problem.* Beiträge Algebra Geom. **36** (1995), 185–202.

[11] Kripfganz, A.: *Favard's 'Fonction Penetrante' – A Roughly Convex Function.* Optimization **38** (1996), 329–342.

[12] Kripfganz, A.: *An Isoperimetric Partition Problem.* Beiträge Algebra Geom. **38** (1997), 161–181.

[13] Kripfganz, A.: *Solution Branching of a General Perimeter Partition Problem.* Beiträge Algebra Geom. **39** (1998), 447–459.

[14] Kubota, T.: *Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen.* Sci. Rep. Tohoku Imp. Univ. **12** (1923), 45–65.

[15] Kubota, T.: *Eine Ungleichheit für Eilinien.* Math. Z. **20** (1924), 264–266.

[16] Kupitz, Y., Martini, H., Wegner, B.: *A Linear-Time Construction of Reuleaux Polygons.* Beiträge Algebra Geom. **37** (1996), 415–427.

[17] Lebesgue, H.: *Sur le problème des isopérimètres et sur les domaines de largeur constante.* Bull. Soc. Math. France C. R. (1914), 72–76.

[18] Mitrinovič, D. S., Pečarić, J. E., Volonec, V.: *Recent Advances in Geometric Inequalities.* Kluwer Akademic Publishers 1989.

[19] Sallee, G. T.: *Maximal Areas of Reuleaux Polygons.* Canad. Math. Bull. **13** (1970), 175–179.

[20] Santalo, L. A.: *Sobre los sistemas completos de desigualdades entre tres elementos de una figura convexa plana.* Math. Notae **17** (1959/61), 82–104.

[21] Schneider, R.: *Convex Bodies: The Brunn-Minkowski Theory.* Cambridge University Press 1993.

[22] Sholander, M.: *On Certain Minimum Problems in the Theory of Convex Curves.* Trans. Amer. Math. Soc. **73** (1952), 139–173.

Received November 27, 1997