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What is the Minimum Length of a Non-Extendable Lace?

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Abstract. A family $\{C_1, \dots, C_n\}$ of pairwise distinct, non-overlapping, congruent circles in the plane form a *lace* provided C_i touches C_{i+1} for all $i = 1, \dots, n-1$. If, additionally, C_n touches C_1, the lace is named a *loop.* A lace (loop) $\{C_1, \dots, C_n\}$ is called *extendable* if it is properly contained in another lace (respectively, loop). In the paper various problems and results on minimum lengths of non-extendable laces and loops are discussed.

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1. Main Problems and Results

According to [1], a family of n non-overlapping congruent circles $C_1, \dots, C_n$ form a *snake* of length n provided C_i touches C_j if and only if $|i - j| = 1$. We will say that a snake $\{C_1, \dots, C_n\}$ is *extendable* provided it is a proper subset of another snake. (Note that in [1] and [2] non-extendable snakes are called *limited* and *maximal*, respectively.)

It is proved in [1] that the minimum length of a non-extendable snake formed by congruent circles is 10. This assertion is generalized in [2] for the case of an arbitrary planar convex body. Namely, it is proved that the minimum length of a non-extendable snake formed by translates of a given planar convex body K is 11 if K is a parallelogram and is 10 otherwise.

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For the sake of completeness, we mention here one more related result: it was proved in [4], confirming a conjecture from [3], that the minimum length of a non-extendable snake formed by edge-to-edge touching squares is 19.

Here we modify the definition of snake as follows. We say that a family $\{C_1, \dots, C_n\}$ of pairwise distinct, non-overlapping, congruent circles form a *lace* of length n if C_i touches C_{i+1} for all $i = 1, \dots, n-1$. If, additionally, C_n touches C_1, the lace is called a *loop* of length n. Following the graph-theoretic terminology, $C_1, \dots, C_n$ form a lace (loop) if and only if there is a Hamiltonian path (respectively, a Hamiltonian cycle) through the skeleton of the family.

Further, a lace (loop) $\{C_1, \dots, C_n\}$ is called *extendable* provided it is a proper subfamily of another lace (respectively, loop). A lace (loop) of length n is *weakly extendable* if it is a subfamily of a lace (respectively, loop) of length $n+1$. It is easily seen that any loop is extendable to a lace.

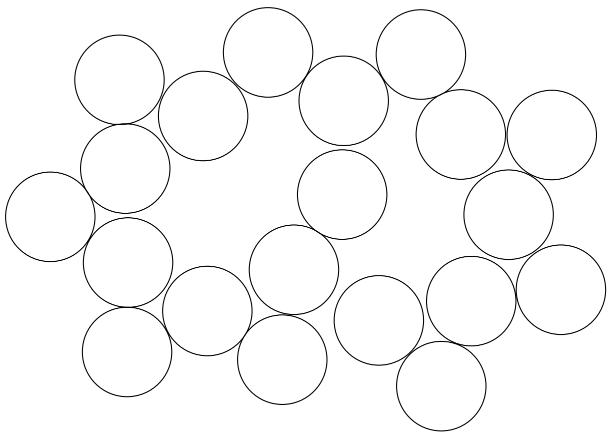

Figure 1: A weakly non-extendable lace of length 20 formed by congruent circles.

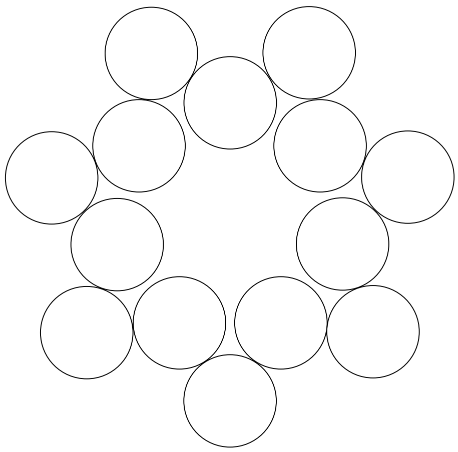

Figure 2: A weakly non-extendable loop of length 14 formed by congruent circles.

Conjecture 1. *The minimum length of a weakly non-extendable lace formed by congruent circles is 20.*

Theorem 1. *The minimum length of a weakly non-extendable loop formed by congruent circles is 14.*

Conjecture 2. *The minimum length of a non-extendable lace or loop formed by congruent circles is 21.*

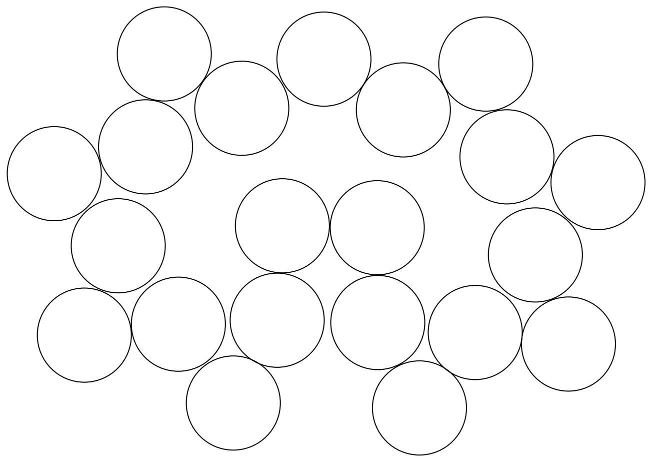

Figure 3: A non-extendable loop of length 21 formed by congruent circles.

Remark. An example of a non-extendable lace of length 21 formed by congruent circles can be obtained from the loop in Figure 3 by making two central circles disjoint.

The following two assertions give initial steps towards proofs of Conjectures 1 and 2.

Theorem 2. *The length of a weakly non-extendable lace formed by congruent circles is at least 15.*

Theorem 3. *The length of a non-extendable loop formed by congruent circles is at least 15.*
The above considerations can be obviously generalized for the case of laces or loops formed by translates of a given planar convex body. The following result shows that in this case even the existence of a finite weakly non-extendable lace or loop is not guaranteed.

Theorem 4. *Any finite lace or loop formed by translates of a given square is weakly extendable.*

Clearly, Theorem 4, as well as its proof, also hold for the case of a parallelogram.

We enclose this section with some open problems.

Problem 1. Do there exist finite non-extendable, even weakly non-extendable, laces or loops formed by *translates* of any given planar convex body distinct from the parallelogram and circles?

Problem 2. What is the minimum length of a non-extendable snake formed by *congruent copies* of a given planar convex body?

Problem 3. What is the minimum length of a non-extendable (weakly non-extendable) lace or loop formed by *congruent copies* of a given planar convex body?

2. Proofs

Proof of Theorem 1

Since Figure 2 gives an example of a weakly non-extendable loop of length 14 formed by congruent circles, it remains to show that the length of any weakly non-extendable loop of congruent circles is at least 14.

Let $\mathcal{L} = \{C_1, \dots, C_n\}$ be a weakly non-extendable loop formed by congruent circles of radii 1, say, and let b_i be the center of C_i, $i = 1, \dots, n$. Clearly, $b_1, \dots, b_n$ are the vertices of a simple closed polygonal line, L, formed by n closed line segments $[b_i, b_{i+1}]$, $i = 1, \dots, n$, of length two each (here we put $b_{n+1} = b_1$). Denote by P the convex hull of L, and let $b_{i_1}, \dots, b_{i_m}$, $1 \le i_1 < \dots < i_m \le n$, be the vertices of P. Since $b_1, \dots, b_n$ are not on a line, P has nonempty interior and therefore $m \ge 3$.

Lemma 1. *One has* $1 < i_l - i_k < n - 1$ *for any* $1 \le k < l \le m$. *In particular,* $n \ge 2m$.

Proof. Assume for a moment that $i_l - i_k = 1$ for a pair of integers $1 \le k < l \le m$ (the case $i_l - i_k = n - 1$ is similar). This means that $l = k + 1$ and hence the segment $[b_{i_k}, b_{i_{k+1}}]$ is a side of P. Since P is convex, it lies entirely in a closed half-plane, H, determined by the line through b_{i_k} and $b_{i_{k+1}}$. If C is the circle of radius 1 touching both $C_{i_k}, C_{i_{k+1}}$ and lying outside H, then $\{C_1, \dots, C_{i_k}, C, C_{i_{k+1}}, \dots, C_n\}$ is a loop properly containing $\mathcal{L}$, a contradiction. □

Lemma 2. *Every interior angle of* P *is of more than 120°.*

Proof. Let b_r be a vertex of P. After a certain renumbering (if necessary) we may assume that $1 < r < m$. In this case the angle $\angle b_{r-1} b_r b_{r+1}$ is less than $180°$. Due to Lemma 1, neither b_{r-1} nor b_{r+1} is a vertex of P. Since both circles C_{r-1} and C_{r+1} touch C_r and do not overlap each other, one has $\angle b_{r-1} b_r b_{r+1} \ge 60°$. Denote by C' the circle of radius 1 which touches both C_r and C_{r-1} and whose center lies outside the convex angular region R formed by the half-lines $[b_r, b_{r-1})$ and $[b_r, b_{r+1})$, as shown in Figure 4. By the hypothesis, $\{C_1, \dots, C_{r-1}, C', C_r, \dots, C_n\}$ is not a loop. Hence C' overlaps a circle $C_k \in \mathcal{L}$. It is easily seen that the last is possible only if b_k lies outside the region R and $\angle b_k b_r b_{r-1} > 30°$. By a similar argument, there is a circle $C_j \in \mathcal{L}$, such that its center b_j lies outside R and $\angle b_{r+1} b_r b_j > 30°$.

Summing up, we have that the interior angle of P with the apex b_r is at least

$$\angle b_k b_r b_j = \angle b_k b_r b_{r-1} + \angle b_{r-1} b_r b_{r+1} + \angle b_{r+1} b_r b_j > 120°. \qquad \square$$

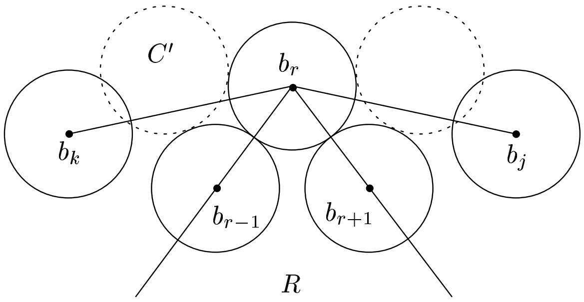

Figure 4: A step in the proof of Lemma 2.

Now we are able to show that $n \geq 14$. Indeed, the sum of all interior angles of P equals $180°(m-2)$, where m is the number of vertices of P. According to Lemma 2, we have $180°(m-2) > 120°m$, i.e., $m > 6$. From Lemma 1 it follows that $n \geq 2m \geq 14$. □

Proof of Theorem 2

Proof of Theorem 2 is similar to that of Theorem 1. Let $\mathcal{L} = \{C_1, \dots, C_n\}$ be a weakly non-extendable lace formed by congruent circles of radii 1, say, and let b_i be the center of C_i, $i = 1, \dots, n$. The points $b_1, \dots, b_n$ are the vertices of a simple self-nonintersecting polygonal line, L, formed by n line segments $[b_1, b_2], \dots, [b_{n-1}, b_n]$ such that $\|b_i - b_{i+1}\| = 2$ for all $i = 1, \dots, n-1$ and $\|b_1 - b_n\| > 2$. Denote by P the convex hull of L, and let $b_{i_1}, \dots, b_{i_m}$, $1 \leq i_1 < \cdots < i_m \leq n$, be the vertices of P. Since $b_1, \dots, b_n$ are not on a line, P has nonempty interior and whence $m \geq 3$.

Lemma 3. *One has* $1 < i_1$ *and* $i_m < n$.

Proof. Assume, for contradiction, that $i_1 = 1$ (the case $i_m = n$ is similar). This means that b_1 is a vertex of P, and one can easily draw a circle C' touching C_1 such that C' is outside P and its center lies on the bisector line of the angle $\angle b_{i_m} b_1 b_{i_2}$. Clearly, C' touches only one of the circles $C_i \in \mathcal{L}$, namely C_1. This implies that $\{C', C_1, \dots, C_n\}$ is a lace properly containing $\mathcal{L}$, a contradiction. □

By repeating the proof of Lemma 1, which holds also for the case when $\mathcal{L}$ is a lace, we obtain that $1 < i_l - i_k$ for any integers $1 \leq k < l \leq m$. This and Lemma 3 imply the inequality $n \geq 2m + 1$.

Since Lemma 2 also holds for the case of laces, we get that every interior angle of P is of more than $120°$. As in Theorem 1, the last implies $m \geq 7$, and hence $n \geq 2m + 1 \geq 15$. □

Proof of Theorem 3

Let $\mathcal{L} = \{C_1, \dots, C_{14}\}$ be a loop of length 14 formed by congruent circles of radii 1, say. If $\mathcal{L}$ is weakly extendable, it is also extendable. Hence it remains to consider the case when $\mathcal{L}$ is weakly non-extendable.

Denote by b_i the center of C_i, $i = 1, \dots, 14$. As it is shown in the proof of Theorem 1, the convex hull of $\{b_1, \dots, b_{14}\}$ is a convex polygon, P, with at least seven vertices such that no two points of the form b_i, b_{i+1}, $i = 1, \ldots, 14$, are simultaneously vertices of P (here we put $b_{15} = b_1$). Hence P is a convex heptagon with the vertices $b_1, b_3, b_5, b_7, b_9, b_{11}, b_{13}$, say.

Let Q be the simple polygon with the vertices $b_2, b_4, b_6, b_8, b_{10}, b_{12}, b_{14}$ (see Figure 5).

Lemma 4. *The polygon Q is a convex heptagon.*

Proof. To prove this, consider the isosceles triangles $\Delta(b_1 b_2 b_3)$ and $\Delta(b_2 b_3 b_4)$, where

$$\|b_1 - b_2\| = \|b_2 - b_3\| = \|b_3 - b_4\| = 2.$$

As it was shown in the proof of Lemma 2, $\angle b_1 b_3 b_2 > 30°$ and $\angle b_2 b_3 b_4 \geq 60°$. Hence $\angle b_1 b_2 b_3 < 120°$ and $\angle b_3 b_2 b_4 \leq 60°$. Similarly, $\angle b_1 b_2 b_{14} \leq 60°$.

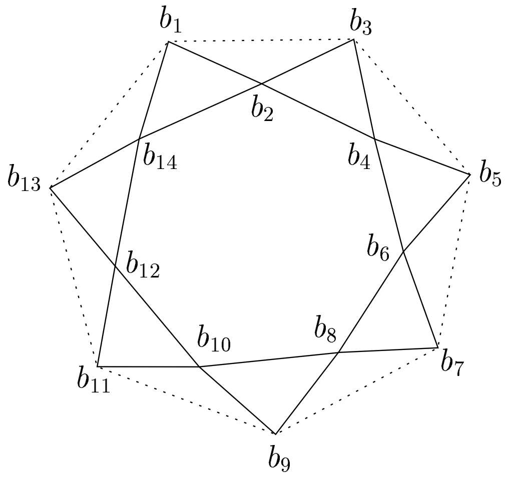

Figure 5: Polygon Q with vertices $b_2, b_4, b_6, b_8, b_{10}, b_{12}, b_{14}$.

As a result, the interior angle $\angle b_4 b_2 b_{14}$ of the polygon Q is more than $120°$. By the same argument, any interior angle of Q is more than $120°$.

The sum of interior angles of the heptagon Q equals $900°$, while the sum of any six of them is more than $720°$. Hence each of the interior angles of Q is less than $180°$, i.e., Q is a convex heptagon. □

Lemma 5. *There is a circle of radius 1, lying inside the heptagon Q such that it touches C_2 and is disjoint to any of the circles C_i, $i = 4, 6, 8, 10, 12, 14$.*

Proof. Denote by C the circle of radius 1 lying inside the polygon Q and touching the circle C_2 such that the center of C lies on the bisector line of the interior angle $\angle b_4 b_2 b_{14}$ of Q (see Figure 6).

We claim that C does not intersect any of the circles $C_4, C_6, C_8, C_{10}, C_{12}, C_{14}$. Due to the symmetry argument it is sufficient to show that C does not intersect any of C_4, C_6, C_8.

Denote by b the center of C and consider the triangle $\Delta(b, b_2, b_4)$. Since $120° < \angle b_4 b_2 b_{14} < 180°$ (see Lemmas 2 and 4) and since b lies on the bisector line of the angle $\angle b_4 b_2 b_{14}$, one has $60° < \angle b b_2 b_4 < 90°$. Having in mind that $\|b_2 - b_4\| \geq \|b - b_2\| = 2$, we get $\|b - b_4\| > 2$. This implies that C is disjoint to C_4.

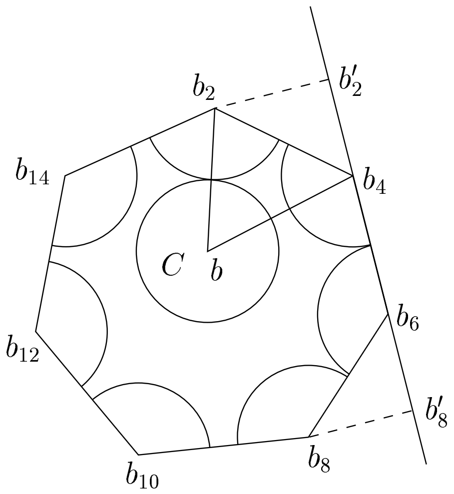

Figure 6: Unit circle C inside polygon Q.

Clearly, $[b, b_2]$ is the shortest side of the triangle $\Delta(bb_2b_4)$. Then $\angle b_2b_4b < 60°$, and hence

$$\angle bb_4b_6 = \angle b_2b_4b_6 - \angle b_2b_4b > 120° - 60° = 60°.$$

Considering the triangle $\Delta(bb_4b_6)$, we have, as above, $\|b - b_6\| > 2$, i.e., C is disjoint to C_6. It remains to show that C is disjoint to C_8. Since

$$\|b - b_8\| \geq \|b_2 - b_8\| - \|b - b_2\| = \|b_2 - b_8\| - 2,$$

it is sufficient to prove that $\|b_2 - b_8\| > 4$. Indeed, let b_2' and b_8' be the projections of the points b_2 and b_8, respectively, on the line l through the points b_4 and b_6 (see Figure 6). Since $\angle b_2b_4b_2' = 180° - \angle b_2b_4b_6 < 60°$, one has $\|b_2' - b_4\| > \frac{1}{2}\|b_2 - b_4\| \geq 1$. Similarly, $\|b_6 - b_8'\| > 1$, and whence

$$\|b_2 - b_8\| \geq \|b_2' - b_8'\| = \|b_2' - b_4\| + \|b_4 - b_6\| + \|b_6 - b_8'\| > 4. \qquad \square$$

Lemma 6. *There is a circle of radius 1, lying inside the heptagon Q such that it touches both C_2 and C_4 and is disjoint to any of the circles C_i, $i = 6, 8, 10, 12, 14$.*

Proof. Let C be a circle of radius 1 satisfying the condition of Lemma 5. We rotate continuously C around C_2 in the direction of C_4 until it will touch a circle C_j, $j \in \{4, 6, 8, 10, 12, 14\}$. Denote by C' this new position of C and let b' be the center of C'.

We are going to prove that $j = 4$. Due to the orientation of the rotation of C' around C_2, we have $12 \neq j \neq 14$. Since the inequality $\|b - b_8\| > 2$ follows only from the relations $\|b_2 - b_8\| > 4$ and $\|b - b_2\| = 2$ (see the proof of Lemma 5), we have, as above, that $\|b' - b_8\| > 2$. Similarly, $\|b' - b_{10}\| > 2$. The last means that $10 \neq j \neq 12$.

It remains to show that $j \neq 6$. Assume, for contradiction, that C' touches C_6 and consider the triangles $\Delta(b'b_2b_4)$ and $\Delta(b'b_4b_6)$. Since the line segments $[b', b_2]$ and $[b', b_6]$ are

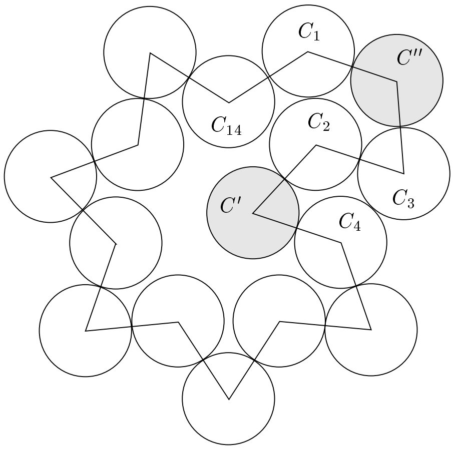

Figure 7: Unit circles C' and C'' in the last step of the proof of Theorem 3.

the shortest sides in these triangles, we have $\angle b_2b_4b' \leq 60°$ and $\angle b'b_4b_6 \leq 60°$. The last is in contradiction with $\angle b_2b_4b_6 > 120°$ (see Lemma 2). Hence $j \neq 6$. □

Now we are ready to finish the proof of Theorem 3. Indeed, let C' be the circle satisfying the condition of Lemma 6, and denote by C'' the circle lying outside the heptagon P and touching C_1 such that the center of C'' lies on the bisector line of the angle $\angle b_2b_1b_{14}$. We rotate continuously C'' around C_1 until it will touch C_3. (The positions of C' and C'' are shown by the shaded circles in Figure 7). The circle C'' is disjoint to C_2 since the loop $\mathcal{L}$ is assumed to be weakly non-extendable. Clearly, $\{C_1, C'', C_3, C_2, C', C_4, C_5, \ldots, C_{14}\}$ is a loop properly containing $\mathcal{L}$. □

Proof of Theorem 4

We will organize the proof of the theorem for both cases (i.e., for laces and loops) simultaneously. Let K be a given square, whose sides are parallel to the coordinate axes in the plane. Assume, for contradiction, the existence of a finite weakly non-extendable lace or loop $\mathcal{L} = \{K_1, \ldots, K_n\}$ formed by translates of K. Denote by R be the least (under inclusion) rectangle containing $\mathcal{L}$ such that its sides are parallel to those of K. In what follows, a, b, c, d denote the vertices of R in the clockwise order of the boundary of R, where $[a, b]$ is the upper horizontal side of R. Clearly, each side of R is touched from inside by a certain translate $K_i \in \mathcal{L}$.

Claims 1–4 below deal with some preliminary properties of $\mathcal{L}$.

Claim 1. *No two squares K_i and K_{i+1} simultaneously touch the same side of R.*

Assume that for a certain $i = 1, \ldots, n$ both squares K_i and K_{i+1} touch the same side of R. In this case we can extend $\mathcal{L}$ to $\mathcal{L}' = \{K_1, \ldots, K_i, K', K_{i+1}, \ldots, K_n\}$ by adding a new translate K' of K placed outside R and touching both K_i, K_{i+1}. Clearly, $\mathcal{L}'$ is a lace or a loop if and only if so is $\mathcal{L}$. □

Claim 2. *None of the squares K_i, $i = 1, \dots, n$, contains the vertex b of R.*

Indeed, assume for a moment that a square K_i contains b. If $\mathcal{L}$ is a lace and i equals one of the indices $1, n$, then one can place outside R a new translate of K having a common side with K_i. In this case we get an extension of the lace $\mathcal{L}$.

Let $\mathcal{L}$ be a loop. Then K_i has in $\mathcal{L}$ two neighbours, K_{i-1} and K_{i+1} (we can renumber, if necessary, the members of $\mathcal{L}$ to have $1 < i < n$). It is easily seen that at least one of the squares K_{i-1}, K_{i+1} has a common side with K_i and whence touches a side of R. The last is impossible by Claim 1. □

Claim 3. *If $\mathcal{L}$ is a lace, then none of the squares K_1, K_n touches a side of R.*

Indeed, if one of the squares K_1, K_n touched a side of R, then as in the proof of Claim 1, one could extend $\mathcal{L}$ to a larger lace by adding a new translate of K exterior to R. □

From Claim 3 it follows that if a square K_i touches a side of R, then it has in $\mathcal{L}$ two neighbours, K_{i-1} and K_{i+1} (if $\mathcal{L}$ is a loop and occasionally $i = 1$ or $i = n$, we put $i - 1 = n$ or $i + 1 = 1$, respectively).

Claim 4. *If a square K_i touches a side of R along its side $[e, f]$ and $[e', f']$ is the second side of K_i parallel to $[e, f]$, then both squares K_{i-1}, K_{i+1} touch K_i along $[e', f']$.*

Assume for a moment that K_{i+1} touches K_i along a side distinct from $[e', f']$. In this case we can extend $\mathcal{L}$ to $\{K_1, \dots, K_i, K', K_{i+1}, \dots, K_n\}$ by adding a new translate K' of K, such that K' touches both K_i, K_{i+1}, has a common side with K_{i+1}, and partly lies in R. □

The contradiction with the finiteness of $\mathcal{L}$ will be obtained by constructing an infinite sequence $S_1, S_2, \dots$ of pairwise distinct members in $\mathcal{L}$.

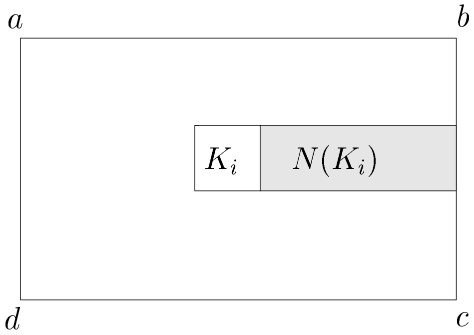

Figure 8: Square K_i with associated rectangle $N(K_i)$.

Before doing this, we need one more notion. In what follows, we associated to any square $K_i \in \mathcal{L}$ disjoint to the side $[b, c]$ of R a rectangle $N(K_i)$ whose sides are parallel to those of K, and such that:

(i) the left vertical side of $N(K_i)$ conicides with the right vertical side of K_i,

(ii) the right vertical side of $N(K_i)$ lies on the side $[b, c]$ of R (see Figure 8).

Let by K_{i_1} the square in $\mathcal{L}$ that touches the side $[a, b]$ of R and is the closest to the vertex b. Put $S_1 = K_{i_1}$. By Claim 1, S_1 does not contain the vertex b. Clearly, no other square $K_i \in \mathcal{L}$ lies in $N(S_1)$. Due to Claim 4, neither of K_{i_1-1}, K_{i_1+1} overlaps $N(S_1)$.

Claim 5. *There is a square $K_i \in \mathcal{L}$ overlapping $N(S_1)$.*

Indeed, suppose that none of the squares in $\mathcal{L}$ overlaps $N(S_1)$ and consider the translate K' of K, lying in $N(S_1)$ and having with S_1 a common side. Obviously, the addition of K' gives an extension of $\mathcal{L}$, since from Claim 4 it follows that K' touches one of K_{i_1-1}, K_{i_1+1}. The obtained contradiction shows that at least one square K_i overlaps $N(S_1)$. □

Now we choose in $\mathcal{L}$ a square, which overlaps the rectangle $N(S_1)$ and is the closest to the side $[a, b]$ of R. If there are more than one such squares, we choose among them the closest to the side $[b, c]$ of R and denote this square by S_2. Note that the existence of S_2 is guaranteed by Claim 5 and finiteness of $\mathcal{L}$.

Claim 6. *The square S_2 is disjoint to the side $[b, c]$ of R.*

Assume, for contradiction, that the square S_2 touches the side $[b, c]$. If $[v, z]$ is the upper horizontal side of S_2, then consider the translate K' of K having $[v, z]$ as the lower horizontal side. Due to the choice of S_1 and to Claim 4, K' does not overlap any square $K_i \in \mathcal{L}$ and touches one of the squares K_{j-1}, K_{j+1}, where j is the ordinal number of S_2 in $\mathcal{L}$. This means that $\mathcal{L}$ can be extended by the addition of K', a contradiction. □

Assume that we have constructed a sequence of pairwise distinct squares $S_1, \ldots, S_m \in \mathcal{L}$, $m \geq 2$, all disjoint to the side $[b, c]$ of R, and such that for any $i = 2, \ldots, m$,

(i) S_i overlaps the rectangle $N(S_{i-1})$,

(ii) among all squares overlapping $N(S_{i-1})$, S_i is the closest to the side $[a, b]$, and among all such squares it is the closest to the side $[b, c]$ of R.

We are going to show the existence of the next such square, S_{m+1}.

Claim 7. *There is a square S_{m+1} overlapping $N(S_m)$ and disjoint to the side $[b, c]$ such that among all such squares S_{m+1} is the closest, firstly, to the side $[a, b]$, and secondly, to the side $[b, c]$ of R.*

The proof of Claim 7 follows from that of Claims 5 and 6, with the respective modifications. From the choice of $S_m \,(= K_j)$ it follows that none of the neighbours K_{j-1}, K_{j+1} of S_m overlaps $N(S_m)$, since otherwise we can add to $\mathcal{L}$ a new translate of K, as it is described in Claim 4. Then the translate K' of K, lying in $N(S_m)$ and having a common vertical side with S_m, touches one of the squares K_{j-1}, K_{j+1}. Since $\mathcal{L}$ cannot be extended by the addition of K', the last overlaps a new square $K_i \in \mathcal{L}$. Hence there is a square K_i overlapping $N(S_m)$. Among all such squares we choose in $\mathcal{L}$ the closest to the side $[a, b]$ of R, and if there are more than one such squares, we choose among them the closest to the side $[b, c]$ of R. Denote this square by S_{m+1}.

In a similar way, by repeating the proof of Claim 6, we obtain that S_{m+1} is disjoint to the side $[b, c]$ of R. □

Summing up, we get that any finite sequence of squares $S_1, S_2, \ldots \in \mathcal{L}$ can be extended to a larger one. The last is in contradiction with the finiteness of $\mathcal{L}$. Hence any finite lace or loop formed by translates of a square is weakly extendable. □

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