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Pseudo-Ovals, Elation Laguerre Planes, and Translation Generalized Quadrangles

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Abstract. We extend known results on Laguerre planes and generalized quadrangles.

1. Introduction

Almost since the very beginning of the theory of generalized quadrangles connections to circle geometries were observed and investigated. Strong results are known in the finite as well as in the topological case. In the present paper we focus on the interplay between a restricted class of Laguerre planes and certain translation generalized quadrangles.

We start with an investigation of elation Laguerre planes. It turns out that they are close relatives of the translation planes, cf. Proposition 2.8 and Theorem 2.10. This yields improvements of results due to Steinke [12] and Löwen [7].

Payne and Thas [9] proved that a finite Laguerre plane of (odd) order s can be constructed from an anti-regular point in a finite generalized quadrangle of order s, and conversely. In [11, 3.1] Schroth constructs a Laguerre plane from a not necessarily finite generalized quadrangle, provided all points are anti-regular. From his proof it is clear that the assumption can be weakened slightly. This led us to introduce the notion of *strong anti-regularity*, which is shown to characterize the constructibility of a Laguerre plane from a generalized quadrangle, cf. Theorem 3.1.

The paper is mainly inspired by Löwen's paper [7], where the notion of (dual) pseudo-ovals is introduced. These structures help to understand planar translation generalized quadrangle and elation Laguerre planes as concepts which are essentially dual to each other,

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cf. Lemma 5.1 and Theorem 5.3. We give a characterization of the situation in which the tangents of a pseudo-oval form a dual pseudo-oval, cf. Theorem 5.4.

In contrast to previous work by Payne and Thas [10], Thas [15], Casse, Thas, and Wild [2], Schroth [11], Steinke [12] and [14], Löwen [7], and others (see the references in the cited papers), we here consider (possibly) infinite incidence structures without any topological assumptions.

I am indebted to Rainer Löwen for his comments on an earlier version of this paper. In particular, he required a correction in the proof of Theorem 4.9.

2. Elation Laguerre planes

An incidence structure $\mathfrak{L} = (P, \mathcal{K})$ is called a *Laguerre plane* with point set P and set of circles $\mathcal{K}$ if there is an equivalence relation $\|$, called *parallelity*, on P such that the conditions below are satisfied. Two circles *touch* each other if they have a unique common point.

LP1. For every three pairwise non-parallel points there is a unique circle through them.

LP2. For each point p on a circle K and each point $q \notin K$ which is not parallel to p there is a unique circle through q that touches K in p.

LP3. For every pair $(p, K) \in P \times \mathcal{K}$ there is a unique point on K which is parallel to p.

LP4. Every circle contains at least three points; there is at least one circle and a point outside.

An equivalence class $[p]$ with respect to $\|$ is called a *generator*. Automorphisms of $\mathfrak{L}$ are required to preserve incidence; it follows that parallelity is invariant as well.

Classical examples of Laguerre planes arise from conics or, more generally, from ovals in desarguesian projective planes. We give an outline of the construction. Let $\mathcal{O}$ be an oval in an affine plane $\mathfrak{A}$ which is embedded in affine 3-space over some skew field F. Fix a class of parallel lines not parallel to $\mathfrak{A}$. Let G be the set of those lines from this parallel class which contain a point of $\mathcal{O}$. As the point set P of the Laguerre plane take all points on lines in G. Each affine plane not parallel to the distinguished parallel class intersects P in a *circle*. The set G partitions the point set and thus induces a parallelity relation on P. Laguerre planes which are arising in this way are called *ovoidal*. An ovoidal Laguerre plane constructed from a conic is *classical*. The distinguished parallel class corresponds to a point in the projective closure of the affine 3-space. Assume that the generators pass through the far point $(1, 0, 0, 0)$. Then the stabilizer of all lines through this point contains a group of collineations induced by the matrix group

$$\mathrm{E} = \left\{ \begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & & 1 & \\ * & & & 1 \end{pmatrix} \right\}.$$

The group E acts faithfully on the projective space and on the Laguerre plane. It fixes each generator and it acts regularly on the set of circles. Obviously, E is isomorphic to the additive group of the vector space F^3. In fact, E is a group of translations in the dual space.

Let $\mathfrak{L} = (P, \mathcal{K})$ be a Laguerre plane and $p \in P$ a point. The set of circles through p is denoted by $\mathcal{K}_p$. We define the derivation at p, which is the incidence structure

$$\mathfrak{D}_p = (P \setminus [p], \mathcal{K}_p \cup \{[q] \mid p \nparallel q\})$$

with the induced incidence relation.

The next two lemmas are well-known.

Lemma 2.1. *The derivation at an arbitrary point of a Laguerre plane is an affine plane.*

An automorphism of $\mathfrak{L}$ which acts trivially on the set of generators and which fixes each point on the two generators g and h is called an *axial automorphism* with *axes* g and h. The axial automorphisms with respect to a fixed pair of generators form a group.

There is an easy observation on groups of axial automorphisms of an arbitrary Laguerre plane, which can be derived from Lemma 2.1.

Lemma 2.2. *Let p and q be non-parallel points, and let Σ denote the group of axial automorphisms with axes $[p]$ and $[q]$. Then Σ acts semi-regularly on the set of circles through p and q.*

The projective closure of $\mathfrak{D}_p$ is denoted by $\overline{\mathfrak{D}_p}$. Note that the set of generators forms a parallel class in $\mathfrak{D}_p$ and thus corresponds to a pencil of lines in $\overline{\mathfrak{D}_p}$. We denote the corresponding point in the projective closure by ∞.

The Laguerre plane $\mathfrak{L}$ is an *elation Laguerre plane* if there is a group E of automorphisms of $\mathfrak{L}$ which acts trivially on the set of generators and regularly on the set of circles. Above we already saw that an ovoidal Laguerre plane is an elation Laguerre plane. It is not clear how restrictive this definition is in the finite case, as all known finite Laguerre planes are ovoidal. In the infinite case, free constructions lead to many non-elation Laguerre planes; see Funk and Strambach [3, p.751].

Let $\mathfrak{L}$ be an elation Laguerre plane with elation group E. Clearly, the stabilizer E_p of a point p induces a group of collineations on $\overline{\mathfrak{D}_p}$, which acts regularly on the set of lines not passing through ∞ and which fixes all lines through ∞.

Lemma 2.3. *The projective plane $\overline{\mathfrak{D}_p}$ is a dual translation plane.*

The geometric concept of a translation (affine) plane is well-known to be equivalent to the algebraic concept of a (planar) spread. Let Γ be a group. A *spread* on Γ is a partition π of Γ into subgroups such that Γ is the product of any two elements of π. A group admitting a spread, or equivalently the translation group of a translation plane, is isomorphic to the additive group of some vector space.

Let $\mathrm{E}_{p,q}$ denote the group of elations fixing both p and q; and let $\mathrm{E}_{p,p}$ denote the set of those elations fixing p but acting non-trivially on each generator other than $[p]$.

Applying standard results from the theory of translation planes and planar spreads to the plane $\overline{\mathfrak{D}_p}$ give the following lemma, cf. Lüneburg [8] or Knarr [6].

Lemma 2.4. *The set $\delta_p = \{\mathrm{E}_{p,p}\} \cup \{\mathrm{E}_{p,q} \mid p \nparallel q\}$ is a spread for the group E_p. In particular $\mathrm{E}_p = \mathrm{E}_{p,q} \oplus \mathrm{E}_{p,r}$ where p, q, r are pair-wise non-parallel points.*

A key observation is that the group $\mathrm{E}_{p,q}$ is a group of collineations of $\overline{\mathfrak{D}_p}$ with center ∞ and axis $[q]$. Thus, as a collineation of $\mathfrak{L}$, it fixes the generator $[q]$ point-wise. But, by symmetry of arguments, it also fixes all points parallel to p. This property extends to E_p in view of Lemma 2.4.

Lemma 2.5. *The group* E_p *fixes all points parallel to* p*. Moreover,* $\mathrm{E}_p = \mathrm{E}_{p'}$ *for* $p' \parallel p$*.*

Lemma 2.6. *The group* $\mathrm{E}_{p,q}$ *is the full group of axial automorphisms with axes* $[p]$ *and* $[q]$*.*

Proof. Let k, k' be circles through p and q. Then there is a collineation $\epsilon \in \mathrm{E}$ which maps k to k'. By axiom LP3, $p^\epsilon = p$ and $q^\epsilon = q$. From Lemma 2.5 it follows that ϵ is an axial automorphism with axes $[p]$ and $[q]$. We infer that $\mathrm{E}_{p,q}$ contains a group of axial automorphisms which acts transitively on the set of circles through p and q. The claim follows from Lemma 2.2. □

The elation group can be generated by any two point stabilizers.

Lemma 2.7. $\mathrm{E} = \mathrm{E}_x \mathrm{E}_y$ *for arbitrary* $x \nparallel y$*.*

Proof. Let $\epsilon \in \mathrm{E}$ and let k be an arbitrary circle. There are unique points $x' \in k$ and $y' \in k^\epsilon$ with $x' \parallel x$ and $y' \parallel y$, respectively. Choose a circle l through x' and y'. Then there is an element $\xi \in \mathrm{E}_x = \mathrm{E}_{x'}$ mapping k to l and an element $\eta \in \mathrm{E}_y = \mathrm{E}_{y'}$ mapping l to k^ϵ. Hence $k^{\xi\eta} = k^\epsilon$ and $\xi\eta = \epsilon$ due to the regularity of the action of E on $\mathcal{K}$. □

Proposition 2.8. *Let* x, y, z *be three pairwise non-parallel points. Then* $\mathrm{E} = \mathrm{E}_{y,z} \oplus \mathrm{E}_{x,z} \oplus \mathrm{E}_{x,y}$*. In particular,* E *is isomorphic to the additive group of a vector space.*

Proof. The group $\mathrm{E}_{x,y}$ is a central subgroup of both E_x and E_y. From Lemma 2.4 and Lemma 2.7 we infer that E is generated by the mutually trivially intersecting abelian groups $\mathrm{E}_{y,z}$, $\mathrm{E}_{x,z}$, $\mathrm{E}_{x,y}$ which centralize each other. □

Löwen proved that the elation group of a locally compact connected elation Laguerre plane is uniquely determined, [7, 5.8], thereby generalizing Steinke's result in the 4-dimensional case, [13, 3.2]; see also Steinke [12].

Corollary 2.9. *The elation group of an elation Laguerre plane is generated by all axial automorphisms with two axes. In particular, it is uniquely determined.*

Let $K(\mathfrak{L})$ be the ring of all endomorphisms of the group E which leaves each group E_x invariant. Clearly, $K(\mathfrak{L})$ also maps each group $\mathrm{E}_{x,y}$ into itself; this is obvious from the definition for $x \neq y$, and it thus follows from Lemma 2.4 for $x = y$. Therefore, each element of $K(\mathfrak{L})$ induces an element of the kernel of the spread δ_p for arbitrary p.

The ring $K(\mathfrak{L})$ is called the *kernel* of the elation Laguerre plane $\mathfrak{L}$. Note that in contrast to this definition, Steinke and Löwen use the term 'kernel' for the set of automorphisms of $\mathfrak{L}$ fixing all generators.

Theorem 2.10. *The kernel of an elation Laguerre plane is a skew field.*

Proof. Let α be a non-zero element of the kernel. As the elation group E is generated by the groups $\mathrm{E}_{x,y}$, there are non-parallel points p, q such that $\mathrm{E}_{p,q}^{\alpha} \neq 0$. Therefore also $\mathrm{E}_p^{\alpha} \neq 0$ and the restriction of ϵ to E_p is a non-trivial element of the kernel of E_p. Thus it is bijective. Similarly for E_q. We infer that the restriction of α to $\mathrm{E}_{p,r}$ and $\mathrm{E}_{q,r}$ is bijective for all points r, which are neither parallel to p nor parallel to q. By Proposition 2.8, α itself is bijective.

The inverse of ϵ is an endomorphism of E which leaves invariant the subgroups E_x for all points x. □

Corollary 2.11. *The kernel of an elation Laguerre plane is a subfield of the kernel of the spread of any derivation.*

It is obvious that, similarly to the situation in translation (affine) planes, elation Laguerre planes can be described in algebraic terms. This amounts to a suitable dualization of Theorem 4.4, see also Theorem 5.3. We omit the details.

3. Anti-regularity in generalized quadrangles

An incidence structure $\mathfrak{Q} = (\mathcal{P}, \mathcal{L})$ is called a (thick) *generalized quadrangle* if the following conditions hold.

GQ1. For each anti-flag $(p, L) \in \mathcal{P} \times \mathcal{L}$, $p \notin L$, there is a unique point on L which is joined to p and a unique line through p which meets L.

GQ2. Each line contains at least three points, and at least three lines pass through each point.

Let $\mathfrak{Q}$ be a generalized quadrangle. Let $\mathfrak{Q}_d(v)$ denote the set of points or lines which are at distance d from the point or line v in the incidence graph of $\mathfrak{Q}$. Often we write $\mathfrak{Q}(v)$ instead of $\mathfrak{Q}_1(v)$ and $\mathrm{Opp}(v)$ instead of $\mathfrak{Q}_4(v)$. If $v \in \mathrm{Opp}(w)$, then v and w are called *opposite.*

A set of three mutually opposite points $x, y, z \in \mathcal{P}$ is called a *triad.* An *anti-regular* triad (x, y, z) satisfies the condition $|\mathfrak{Q}_2(x) \cap \mathfrak{Q}_2(y) \cap \mathfrak{Q}_2(z)| \in \{0, 2\}$. A point p is called *anti-regular* if each triad containing p is anti-regular. An anti-regular point p is called *strongly anti-regular* if additionally each triad contained in $\mathfrak{Q}_2(p)$ is anti-regular.

Fix a point ∞ and define an incidence structure $\mathfrak{L}(\infty) = (\mathfrak{Q}_2(\infty), \mathrm{Opp}(\infty))$ and an equivalence relation $\parallel$ on the point set by putting $x \parallel y$ if x is joined to y in $\mathfrak{Q}$. The *point* $x \in \mathfrak{Q}_2(\infty)$ is incident with the *circle* $u \in \mathrm{Opp}(\infty)$ if and only if x is joined to u in $\mathfrak{Q}$. There are no other incidences.

There is an important geometric characterization of strong anti-regularity.

Theorem 3.1. *The incidence structure $\mathfrak{L}(\infty)$ is a Laguerre plane if and only if the point ∞ is strongly anti-regular.*

Proof. Assume that $\mathfrak{L}(\infty)$ is a Laguerre plane. The axiom LP1 implies that each triad in $\mathfrak{Q}_2(\infty)$ is anti-regular. Now let (∞, k, m) be a triad such that there is a point p joined to ∞, k, m. We have to show that there is a unique other point which also joined to ∞, k, m. It is clear that there is at most one such point because of LP1. Assume that p is the unique point of the intersection; that is the circles k and m touch in p. Choose an arbitrary line $L \in \mathfrak{Q}(\infty)$ not containing p. There is unique point q on L which is joined to k. Let m' be the unique point on the line joining p and m which is joined to q. By assumption, $m' \neq m$; and, obviously, the circles m and m' touch in p. So k and m' both are circles containing q which touch m in p. This contradicts LP2. We conclude that the triad (∞, k, m) is anti-regular.

The construction of a Laguerre plane from a generalized quadrangle with a strongly anti-regular point is due to Schroth [11, 3.1]. In his claim Schroth required the anti-regularity of each point, but a careful examination of his proof shows that the strong anti-regularity of one point suffices. □

The following is obvious from the construction.

Corollary 3.2. *Let ∞ be a strongly anti-regular point. Then the collineations of $\mathfrak{Q}$ fixing all lines through ∞ bijectively correspond to the automorphisms of $\mathfrak{L}(\infty)$ which fix all generators.*

We want to consider the finite case for a moment. Given any two lines L, M of an arbitrary generalized quadrangle, there is a bijection between the point set of L and the point set of M. Dually, there are bijections between any two pencils of lines. Assuming that $\mathfrak{Q}$ is finite, each line is incident with, say, $1 + s$ points, and each point is incident with $1 + t$ lines. Then (s, t) are the *parameters* of the generalized quadrangle.

Theorem 3.3. (Payne and Thas [9], [10, 1.3.2]) *Let $\mathfrak{Q}$ be a finite generalized quadrangle with parameters (s, s) and an anti-regular point p. Then $\mathfrak{L}(p)$ is a Laguerre plane of order s, and s is odd.*

Corollary 3.4. *Let $\mathfrak{Q}$ be a finite generalized quadrangle with parameters (s, s). Then a point is anti-regular if and only if it is strongly anti-regular.*

As already mentioned, all known finite Laguerre planes are ovoidal. Moreover, an oval in a projective plane of odd order is a conic by Segre's celebrated theorem, cf. Hughes and Piper [4, XII.6]; all known finite Laguerre planes of odd order are classical. All known finite generalized quadrangles with parameters (s, s), which have an anti-regular point, are dual to the symplectic quadrangle over $\mathrm{GF}(s)$.

4. Planar translation generalized quadrangles

The quadrangle $\mathfrak{Q}$ is called a *translation generalized quadrangle* (TGQ) with *translation center* ∞ if it admits an abelian group T of collineations which fixes $\infty \in \mathcal{P}$ line-wise and acts transitively (and therefore regularly) on the set $\mathrm{Opp}(\infty)$. The group T is called the *translation group* with respect to ∞; it is uniquely determined, cf. [5, 2.5].

Assume that $\mathfrak{Q}$ is a TGQ, and let T be the translation group with respect to the center ∞. For $L \in \mathfrak{Q}(\infty)$ let

$$\mathrm{T}_L = \{\tau \in \mathrm{T} \mid \tau \text{ fixes all lines intersecting } L\}$$

and

$$\mathrm{T}_L^* = \{\tau \in \mathrm{T} \mid \tau \text{ fixes } L \text{ point-wise}\}.$$

Obviously $\mathrm{T}_L \leq \mathrm{T}_L^*$. The group T_L acts regularly on the set $X \setminus \{\pi(\infty, X)\}$ for any line X intersecting L. Analogously, $\mathrm{T}_L^*/\mathrm{T}_L$ acts regularly on the set $\mathfrak{Q}(p) \setminus \{L\}$ for any point

$p \in L \setminus \{\infty\}$. The group T is generated by the union of all these subgroups, [5, 2.4]. We define

$$K(\infty) = \{a \in \mathrm{End}(\mathrm{T}) \mid a(\mathrm{T}_L) \subseteq \mathrm{T}_L \text{ for all } L \in \mathfrak{Q}(\infty)\},$$

the *kernel* of $\mathfrak{Q}$ with respect to ∞.

In the finite case the situation is known to be similar to translation projective planes.

Theorem 4.1. (Payne and Thas [10, 8.5]) *The kernel K of a finite* TGQ *is a field.*

In [5, 3.12], it was claimed that a similar result holds without the assumption of finiteness. However, the Proposition 3.10 in [5], which is used in the proof of [5, 3.12], is false.

Due to the lack of counterexamples one might be tempted to believe in the following.

Conjecture 4.2. *The kernel of an arbitrary TGQ is a skew field.*

We will show that the conjecture holds for special classes of TGQs. A TGQ whose kernel is a skew field is called *linear*. Note that in spite of the erroneous Proposition 3.10 in [5] the results from Sections 4 and 5 of [5] still hold for linear TGQs.

We start with a simple observation that follows from the fact that each unit in the ring $K(\infty)$ is a bijection on T.

Lemma 4.3. *If an element of the kernel $K(\infty)$ is invertible, then its inverse is also contained in $K(\infty)$.*

There is an entirely algebraic description of TGQs. Let ω be an arbitrary set. Take an abelian group A with families of subgroups $\mathcal{T} = \{\mathrm{A}_L \mid L \in \omega\}$, $\mathcal{T}^* = \{\mathrm{A}_L^* \mid L \in \omega\}$. Define an incidence structure

$$\mathfrak{T}(\mathrm{A}, \mathcal{T}, \mathcal{T}^*) = (\mathcal{P}, \mathcal{L}, \mathcal{F})$$

as follows. Set

$$\mathcal{P} = \{\omega\} \cup \{\mathrm{A}_L^* + a \mid L \in \omega,\ a \in \mathrm{A}\} \cup \mathrm{A}$$

and

$$\mathcal{L} = \omega \cup \{\mathrm{A}_L + a \mid L \in \omega,\ a \in \mathrm{A}\}.$$

Now let $(p, G) \in \mathcal{P} \times \mathcal{L}$ be a flag if and only if $p \in G$ or $p \ni G$ or $p \subset G$ or $p \supset G$.

Theorem 4.4. ([5, 3.5]; see also Bader and Payne [1]) *If*

A. $0 < \mathrm{A}_L < \mathrm{A}_L^*$ *and* $|\omega| \geq 3$ *and*

B. $\mathrm{A} = \mathrm{A}_L^* \oplus \mathrm{A}_M$ *and*

C. *the set* $\pi_L = \{\mathrm{A}_L^*/\mathrm{A}_L\} \cup \{(\mathrm{A}_L + \mathrm{A}_N)/\mathrm{A}_L \mid N \in \omega \setminus \{L\}\}$ *is a partition of the group* A/A_L,

for all $L, M \in \omega$ with $L \neq M$, then the incidence structure $\mathfrak{T}(\mathrm{A}, \mathcal{T}, \mathcal{T}^*)$ is a TGQ *with translation center ω and translation group* A.

Each TGQ arises in this way. Let $\mathfrak{Q} = (\mathcal{P}, \mathcal{L})$ be a TGQ with translation center ∞ and translation group T. Define groups T_L and T_L^* for $L \in \mathfrak{Q}(\infty)$ as above.

Proposition 4.5. ([5, 3.6]) $\mathfrak{Q} \cong \mathfrak{T}(\mathrm{T}, \{\mathrm{T}_L \mid L \in \mathfrak{Q}(\infty)\}, \{\mathrm{T}_L^* \mid L \in \mathfrak{Q}(\infty)\})$.

The TGQ $\mathfrak{Q}$ is called *planar* if the partition π_L is a planar spread for all $L \in \mathfrak{Q}(\infty)$. This property can easily be characterized in terms of the translation group.

Lemma 4.6. *The* TGQ *is planar if and only if* $\mathrm{T} = \mathrm{T}_L \oplus \mathrm{T}_M \oplus \mathrm{T}_N$ *for any three distinct lines* $L, M, N \in \mathfrak{Q}(\infty)$.

Theorem 4.7. *A planar* TGQ *is linear.*

See also [5, proof of 3.11].

Proof. Let a be a non-zero element of the kernel. By assumption a induces an non-trivial element of the kernel of the planar spread on each factor T/T_L, where L passes through ∞. In particular, for $\mu \in \mathrm{T}_M$ with $\mu \neq 0$ there is an element $\tau \in \mathrm{T}$ such that $a(\tau) \in \mathrm{T}_L + \mu$. Choose $N \in \mathfrak{Q}(\infty)$ with $N \neq L, M$. We have $\mathrm{T} = \mathrm{T}_L \oplus \mathrm{T}_M \oplus \mathrm{T}_N$. As T_L, T_M, T_N are a-invariant we infer that $\tau \in \mathrm{T}_M$. Hence $a(\tau) = \mu$, and a is onto on T_M. Thus a is also onto on T. The claim follows from [5, 3.7]. □

Assume now that the translation center ∞ is a strongly anti-regular point.

Lemma 4.8. *If* ∞ *is strongly anti-regular, then the Laguerre plane* $\mathfrak{L}(\infty)$ *is an elation Laguerre plane with elation group* T. *Moreover, the kernel of the* TGQ $\mathfrak{Q}$ *is contained in the kernel of the elation Laguerre plane* $\mathfrak{Q}(\infty)$.

Proof. By Theorem 3.1 and Corollary 3.2 the incidence structure $\mathfrak{L}(\infty)$ is an elation Laguerre plane. Its elation group is canonically isomorphic to T. For any line L passing through the translation center ∞ the group T_L^* is the stabilizer of some point of the Laguerre plane $\mathfrak{L}(\infty)$ in T.

Let α be an element of the kernel of $\mathfrak{Q}$. Then $\alpha(\mathrm{T}_L^*) \subseteq \mathrm{T}_L^*$ by [5, 3.2], and α is contained in the kernel of $\mathfrak{L}(\infty)$. □

Theorem 4.9. *A* TGQ *with a strongly anti-regular translation center is linear.*

Proof. From Lemma 4.8 and Theorem 2.10 we infer that each non-zero element of $K(\infty)$ is invertible. So the result follows from Lemma 4.3. □

For any linear TGQ it makes sense to introduce the pair of *F-dimension parameters*

$$(s, t) = (\dim_F \mathrm{T}_L, \dim_F(\mathrm{T}_L^*/\mathrm{T}_L)),$$

where F is a subfield of the kernel; the definition does not depend on the choice of $L \in \mathfrak{Q}(\infty)$ because of [5, 3.3]. We always have $s \leq t$ as an inequality of arbitrary cardinal numbers, by [5, 3.13]. Note that these dimension parameters are the logarithmic version of the usual parameters of a finite generalized quadrangle. A linear TGQ is called *finite-dimensional* if its $K(\infty)$-parameters are finite. This is equivalent to the translation group being a finite-dimensional $K(\infty)$-vector space.

The following lemma is obvious.

Lemma 4.10. *Let* $\mathfrak{Q}$ *be a linear* TGQ *with finite F-dimension parameters* (s, t). *Then* $\mathfrak{Q}$ *is planar if and only if* $s = t$ *if and only if* $\dim_F \mathrm{T} = 3s$ *if and only if* $\dim_F \mathrm{T} = 3t$.

With finite-dimensionality strong anti-regularity implies planarity.

Lemma 4.11. *A finite-dimensional* TGQ *with a strongly anti-regular translation center is planar.*

Proof. Let ∞ be a strongly anti-regular point. Then we obtain an elation Laguerre plane $\mathfrak{L}(\infty)$ due to Lemma 4.8. Observe that $\dim_F \mathrm{T}_L^* \cap \mathrm{T}_M^* = t$ for distinct $L, M \in \mathfrak{Q}(\infty)$. By Proposition 2.8, $\dim_F \mathrm{T} = 3t$. Apply Lemma 4.10. □

We give a brief account of the finite case.

Theorem 4.12. (Payne and Thas [10, 1.5.2(v)]) *Let $\mathfrak{Q}$ be a finite planar TGQ. Then the translation center of $\mathfrak{Q}$ is (strongly) anti-regular if and only if the characteristic of the kernel is odd.*

There are finite and infinite planar TGQs with a translation center which is not anti-regular. The finite examples necessarily have characteristic 2, cf. Theorem 4.12. Although there is no analogue to Theorem 4.12 known in the infinite case, the same is true for the known infinite examples.

5. Pseudo-ovals and dual pseudo-ovals

So far no assumptions were made concerning the dimensions of the vector spaces that occurred. From now on we restrict our attention to finite-dimensionality. Note that the discussion covers the finite as well as the compact connected topological case without making a reference to either.

Let F be a skew-field, and let $\mathrm{G}_{m,k}$ be the set of k-dimensional subspaces of the left vector space F^m. The projective space $\mathrm{PG}(F^m)$ is the lattice obtained from the union of all sets $\mathrm{G}_{m,k}$ with the obvious operations '$\wedge$' and '$\vee$'. A bijection on $\mathrm{PG}(F^m)$ which reverses inclusion is called a *duality*. If F is a field, then the standard scalar product on F^m induces a duality (even a polarity) on $\mathrm{PG}(F^m)$ by taking the orthogonal complement. If $\perp$ is a duality then the $\perp$-image of $\mathrm{PG}(F^m)$ is isomorphic to the projective space $\mathrm{PG}({F_\circ}^m)$ over the skew-field $F_\circ$ with the opposite multiplication. Clearly, $F \cong F_\circ$ if F is a field.

Following Löwen [7, 2.3], a set $\mathcal{O} \subset \mathrm{G}_{3m,n}$ is called an *n-dimensional F-pseudo-oval* if each point $p \in \mathcal{O}$ has a unique *tangent* $t_p \in \mathrm{G}_{3m,2n}$, with $p < t_p$, such that the set

$$\pi_p = \{t_p\} \cup \{p \vee q \mid q \in \mathcal{O},\ q \neq p\}$$

induces a planar spread on the factor space F^{3n}/p.

From Theorem 4.4 and Theorem 4.7 it is clear that the notion of a pseudo-oval is equivalent to the notion of a finite-dimensional planar TGQ.

Lemma 5.1. *Let $\mathfrak{Q}$ be a planar* TGQ *with translation center ∞, translation group* T, *and kernel K. If $\dim_K \mathrm{T} = 3n$ is finite, then the set $\{\mathrm{T}_L \mid L \in \mathfrak{Q}(\infty)\}$ is an n-dimensional K-pseudo-oval. The tangent through T_L is given by T_L^*. Conversely, each pseudo-oval can be seen as a finite-dimensional planar* TGQ.

It is worth-while to dualize the notion of a pseudo-oval explicitly, cf. Löwen [7, 2.1]. A set $\mathcal{T} \subset \mathrm{G}_{3n,2n}$ is called an n-dimensional dual F-pseudo-oval if for every $t \in \mathcal{T}$ there is a unique *point of tangency* $p_t \in \mathrm{G}_{3m,n}$ such that

$$\sigma_t = \{p_t\} \cup \{s \wedge t \mid s \in \mathcal{T}\}$$

is a planar spread on t.

The following lemma is obvious.

Lemma 5.2. *The image of an n-dimensional F-pseudo-oval under a duality of the projective space* $\mathrm{PG}(F^{3n})$ *is an n-dimensional dual $F_\circ$-pseudo-oval. The image of an n-dimensional dual F-pseudo-oval under a duality is an n-dimensional $F_\circ$-pseudo-oval.*

Löwen proved that each dual pseudo-oval gives rise to an elation Laguerre plane, [7, 6.2a]. For the converse he made strong topological assumptions, [7, 5.7], which are, in fact, not necessary.

Theorem 5.3. *Let $\mathfrak{L} = (P, \mathcal{K})$ be an elation Laguerre plane with elation group* E *and kernel K. If* $\dim_K \mathrm{E} = 3n$ *is finite, then the set* $\{\mathrm{E}_x \mid x \in P\}$ *is an n-dimensional dual K-pseudo-oval. The point of tangency of* E_x *is given by* $\mathrm{E}_{x,x}$*. Conversely, each dual pseudo-oval can be seen as a finite-dimensional elation Laguerre plane.*

Proof. The elation group E of an elation Laguerre plane is a vector space by Proposition 2.8; if its dimension is finite, then it is divisible by 3. The set $\{\mathrm{E}_x \mid x \in P\}$ is an n-dimensional dual F-pseudo-oval by Lemma 2.4.

The converse has been proved by Löwen [7, 6.2a]. □

The following theorem characterizes those dual pseudo-ovals which arise as sets of tangents to pseudo-ovals.

Theorem 5.4. *The set of tangents of an n-dimensional F-pseudo-oval $\mathcal{O}$ is an n-dimensional dual F-pseudo-oval if and only if the planar* TGQ *associated to $\mathcal{O}$ has a strongly anti-regular translation center.*

Proof. Let $\mathcal{O}$ be an n-dimensional F-pseudo-oval such that the set of tangents $\mathcal{T}$ forms a dual pseudo-oval. According to Lemma 5.1 there is a planar TGQ $\mathfrak{Q}$ associated to $\mathcal{O}$. Similarly, $\mathcal{T}$ gives rise to an elation Laguerre plane $\mathfrak{L}$, cf. Theorem 5.3. By construction the quadrangle $\mathfrak{Q}$ is related to the Laguerre plane $\mathfrak{L}$ as in Theorem 3.1 and Lemma 4.8. We infer that the translation center of $\mathfrak{Q}$ is strongly anti-regular. The whole argument can clearly be reversed. □

In view of Lemma 5.2, Theorem 5.4 allows for the following definition. Let $\mathfrak{Q}$ be a finite-dimensional TGQ with a strongly anti-regular translation center. Then the *translation dual* of $\mathfrak{Q}$ is the TGQ (with a strongly anti-regular translation center) obtained by dualizing the set of tangents to the pseudo-oval associated to $\mathfrak{Q}$. This construction is similar to [10, 8.7.2(iv)], where translation duals of finite non-planar TGQs are introduced.

Note that due to Löwen's Duality Theorem [7, 7.1] all compact connected planar TGQs have, in fact, a strongly anti-regular translation center. However, in all cases examined by

Löwen [7, 7.4], the translation dual is isomorphic to the primal TGQ. A similar phenomenon can be observed in the finite case of odd order: The translation center is strongly anti-regular, but the translation dual is isomorphic to the primal TGQ in the classical case. All known finite planar TGQs of odd order are classical.

The Theorem 5.4 can be dualized in terms of Lemma 5.2 in order to characterize the situation where an elation Laguerre plane allows for the construction of a TGQ with a strongly anti-regular translation center.

Let $\mathfrak{L} = (P, \mathcal{K})$ be a Laguerre plane. Define an incidence structure $\mathfrak{Q}(\mathfrak{L})$ as follows. As points take all points and all circles of $\mathfrak{L}$, plus an additional point ∞. A point $p \in P$ is joined (by a line) to a point $q \in P$ if p is parallel to q; a point p is joined to a point $k \in \mathcal{K}$ if p is incident with k in $\mathfrak{L}$; a point $k \in \mathcal{K}$ is joined to a point $l \in \mathcal{K}$ if k and l touch; the point ∞ is joined to all points in P, but it is not joined to any point in $\mathcal{K}$. A line is a subset of the point set such that any two of its points are joined. See Schroth [11, §3.2] for the details.

Corollary 5.5. *The points of tangency of an n-dimensional dual F-pseudo-oval $\mathcal{T}$ form an n-dimensional F-pseudo-oval if and only if the incidence structure $\mathfrak{Q}(\mathfrak{L})$ is a (translation) generalized quadrangle (with a strongly anti-regular translation center), where $\mathfrak{L}$ is the elation Laguerre plane associated to $\mathcal{T}$.*

Proof. Observe that $\mathfrak{Q}(\mathfrak{L})$ is isomorphic to the TGQ associated to the points of tangency of $\mathcal{T}$. □

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