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m-Point Invariants of Real Geometries

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Abstract. Let V be a finite dimensional vector space over a field K with a non-degenerate symmetric, alternating, or hermitian scalar product. We characterize the orbits of m-tuples of vectors of V under the group of isometries of V by the Gram matrix and an additional subspace of K^m. This characterization will then be applied to determine all m-point invariants of real euclidean, spherical, hyperbolic, and de-Sitter geometries.

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1. Introduction

Let P be a set and Γ a group of permutations of P. The pair (P, Γ) is called a *geometry*, the elements of P are called *points*, the elements of Γ are called *motions*. Let m be a natural number. An *m-point invariant* ι of (P, Γ) maps m-tuples of points to an arbitrary non-empty set S and satisfies the functional equation

$$\iota(x_1, \dots, x_m) = \iota(\gamma(x_1), \dots, \gamma(x_m))$$

for all $x_1, \dots, x_m \in P$, $\gamma \in \Gamma$. Let S_0 be the set of orbits of m-tuples of points under Γ. Let $\omega : P^m \to S_0$ map any m-tuple of points to its orbit. Let $\psi : S_0 \to S$ be an arbitrary mapping, then $\iota := \psi \circ \omega : P^m \to S$ is an m-point invariant; on the other hand, for any m-point invariant $\iota : P^m \to S$ there exists a uniquely determined mapping $\psi : S_0 \to S$ such that $\iota = \psi \circ \omega$. We are looking for a concrete characterization of the orbits.

W. Blaschke has determined all invariants of m-tuples of vectors in a real finite dimensional orthogonal space in the following way [2, §10]:

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"Man bestimme zuerst die höchstmögliche Zahl r von linear unabhängigen Vektoren unter den gegebenen. Dann greife man irgendein System von r linear unabhängigen 'Grundvektoren' aus ihnen heraus und kombiniere alle übrigen aus diesen linear. Das vollständige System der unabhängigen Invarianten ist dann gegeben durch die Skalarprodukte der Grundvektoren und die Koeffizienten der gebildeten Linearkombinationen."

A similar result for more general spaces is obtained in Theorem 1. The orbit of m vectors is characterized by their Gram matrix and an additional subspace. Of course, if the vectors are linearly dependent, then the components of their Gram matrix are dependent; in Theorem 2 we determine all possible Gram matrices and related subspaces in an important case.
The 2- and 3-point invariants of some real geometries have been examined in [1]. Theorem 1 and 2 are applied to give all m-point invariants of real euclidean, spherical, hyperbolic, and de-Sitter geometries.

2. Transitivity properties

In this section let V be a finite dimensional left- vector space over a (skew-) field K and let α be an automorphism of K. Let $(\cdot,\cdot) : V \times V \to K$ be an α-scalar product (i.e. for all $a_1, a_2, a_3 \in K,\ v_1, v_2, v_3 \in V$,

$$\begin{aligned}(a_1\, v_1 + a_2\, v_2, v_3) &= a_1\,(v_1, v_3) + a_2\,(v_2, v_3),\\ (v_1, a_2\, v_2 + a_3\, v_3) &= \alpha(a_2)\,(v_1, v_2) + \alpha(a_3)\,(v_1, v_3)\ ),\end{aligned}$$

which is non-degenerate and which is symmetric, alternating, or hermitian; we assume $\operatorname{char} K \neq 2$, if $(\cdot,\cdot)$ is symmetric.

Theorem 1. *Let $m \in \mathbb{N}$ and $x_1, \dots, x_m,\ x'_1, \dots, x'_m \in V$. There is an isometry of $(V, (\cdot,\cdot))$ which takes x_i to x'_i, $i = 1, \dots, m$, iff the Gram matrices of $x_1, \dots, x_m$ and $x'_1, \dots, x'_m$ are equal,*

$$(x_i, x_j) = (x'_i, x'_j), \tag{1}$$

and

$$\sum_{i=1}^{m} \lambda_i x_i = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{m} \lambda_i x'_i = 0 \tag{2}$$

for all $\lambda = (\lambda_1, \dots, \lambda_m) \in K^m$.

In order to prove Theorem 1 we need a theorem of Witt (see e.g. [3]):
Let U_1, U_2 be subspaces of V and let $\varphi : U_1 \to U_2$ be an isometry, i.e. φ is a linear isomorphism which is metric, $(\varphi(x), \varphi(y)) = (x, y)$ for all $x, y \in U_1$. Then φ can be extended to an isometry f of $(V, (\cdot,\cdot))$, $f|_{U_1} = \varphi$.

Proof of Theorem 1. If there is an isometry f which takes x_i to x'_i, $i = 1, \dots, m$, then the conditions (1) and (2) are satisfied. Now let $x_1, \dots, x_m$ and $x'_1, \dots, x'_m$ satisfy (1) and (2). To $x_1, \dots, x_m$ corresponds a linear map $g : K^m \to V$,

$$g(\lambda) = \sum_{i=1}^{m} \lambda_i x_i. \tag{3}$$

To $x_1', \dots, x_m'$ we define $g' : K^m \to V$ analogously. Then

$$(g(\lambda), g(\mu)) = (g'(\lambda), g'(\mu)) \text{ for all } \lambda, \mu \in U, \tag{1'}$$

$$\ker g = \ker g'. \tag{2'}$$

Let $v_1, \dots, v_r$ be a basis of $\operatorname{im} g$. Take $u_1, \dots, u_r \in U$ with $g(u_i) = v_i$, $i = 1, \dots, r$. Then $u_1, \dots, u_r$ are linearly independent and $U = \operatorname{span}\{u_1, \dots, u_r\} \oplus \ker g$. From (2') follows that the images $v_1', \dots, v_r'$ of $u_1, \dots, u_r$ under g' are a basis of $\operatorname{im} g'$. The linear map $\varphi : \operatorname{im} g \to \operatorname{im} g'$ defined by $\varphi(v_i) = v_i'$, $i = 1, \dots, r$, is an isometry because of (1'). From Witt's theorem there is an isometry $f : V \to V$ which satisfies $f|_{\operatorname{im} g} = \varphi$, thus $g' = \varphi \circ g = f \circ g$, and $x_i' = f(x_i)$, $i = 1, \dots, m$. □

3. A characterization of the orbits

Let G be an $m \times m$ matrix and let W be a subspace of K^m. In view of Theorem 1 we may ask: Are there $x_1, \dots, x_m \in V$ such that

$$G \text{ is the Gram matrix of } x_1, \dots, x_m \tag{4}$$

and such that, for all $\lambda \in K^m$,

$$\sum_{i=1}^{m} \lambda_i x_i = 0 \quad \Leftrightarrow \quad (\lambda_1, \dots, \lambda_m) \in W \tag{5}$$

is satisfied? We give necessary and sufficient conditions in an important case:

Theorem 2. *Let K be an ordered commutative field whose positive numbers are squares. Let $m, n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$. Let V be an n-dimensional vector space over K with a non-degenerate, symmetric bilinear form $(\cdot,\cdot)_V$ and signature $(k, n-k)$. Let G be a symmetric $m \times m$-matrix with entries in K and with s positive and t negative eigenvalues, $s + t \le m$. Let W be a subspace of K^m and let $\delta := m - s - t - \dim W$. There are $x_1, \dots, x_m \in V$ with* (4) *and* (5) *iff*

$$s + \delta \le k, \; t + \delta \le n - k, \quad WG := \{wG \mid w \in W\} = 0. \tag{6}$$

Proof. We define a symmetric bilinear form $(\cdot,\cdot)_{K^m}$ on K^m by $(\lambda, \mu)_{K^m} := \lambda G \mu^T \in K$. The signature of $(K^m, (\cdot,\cdot)_{K^m})$ is (s, t), and the radical of K^m is

$$\begin{aligned} \operatorname{rad} K^m &:= \{\lambda \in K^m \mid (\lambda, \mu)_{K^m} = 0 \ \forall \mu \in K^m\} \\ &= \{\lambda \in K^m \mid \lambda G = 0\}. \end{aligned}$$

We take an orthogonal basis $u_1, \dots, u_m$ of K^m such that $(u_i, u_i)_{K^m} = 1$ if $i \le s$, $(u_i, u_i)_{K^m} = -1$ if $s < i \le s + t$, and such that $u_{s+t+1}, \dots, u_m$ is a basis of the radical of K^m.
Let $x_1, \dots, x_m \in V$ satisfy (4) and (5). The linear map $g : K^m \to V$ defined by (3) is metric, and $W = \ker g < \operatorname{rad} K^m$. The space V_1 spanned by $g(u_1), \dots, g(u_{s+t})$ is a regular subspace of V with signature (s, t). The space $V_1^\perp$ orthogonal to V_1 is a regular subspace of V and has

signature $(k-s, n-k-t)$. The space V_2 spanned by $g(u_{s+t+1}), \ldots, g(u_m)$ is an isotropic subspace of $V_1^{\perp}$. Hence $\delta = \dim V_2 \leq \min\{k-s, n-k-t\}$, and (6) is satisfied.
We now assume that (6) is satisfied. Let $v_1, \ldots, v_n \in V$ be an orthogonal basis of V such that $(v_i, v_i)_V = 1$ if $i \leq k$ and $(v_i, v_i)_V = -1$ if $i > k$. Take a basis $w_1, \ldots, w_{m-s-t}$ of $\operatorname{rad} K^m$ such that $w_{\delta+1}, \ldots, w_{m-s-t}$ is a basis of W. We define the linear map $g : K^m \to V$ by

$$\begin{array}{lll} u_i & \mapsto v_i, & i = 1, \ldots, s, \\ u_{s+i} & \mapsto v_{k+i}, & i = 1, \ldots, t, \\ w_i & \mapsto v_{s+i} + v_{k+t+i}, & i = 1, \ldots, \delta, \\ w_{\delta+i} & \mapsto 0, & i = 1, \ldots, m-s-t-\delta. \end{array}$$

g is metric, its kernel is W. Hence the vectors $x_1, \ldots, x_m \in V$ defined by (3) satisfy (4) and (5). $\square$

Corollary 1. *The pairs* (G, W) *in Theorem 2 which satisfy* (6) *correspond to the orbits of m-tuples of vectors of V under the isometries of* $(V, (\cdot,\cdot)_V)$.

Corollary 2. *Let K be an ordered commutative field whose positive numbers are squares. Let V be an n-dimensional vector space over K with a symmetric bilinear form $(\cdot,\cdot)$ and signature $(k, n-k)$. Let G be the Gram matrix of $x_1, \ldots, x_m \in V$ and let s and t be the numbers of positive and negative eigenvalues of G, respectively.*

a) *G alone characterizes the orbit of $(x_1, \ldots, x_m)$ under the isometries of $(V, (\cdot,\cdot))$ (i.e., whenever G is the Gram matrix of $x'_1, \ldots, x'_m \in V$, then there is an isometry which takes x_i to x'_i, $i = 1, \ldots, m$) if, and only if,* $\det G \neq 0$ *or $s = k$ or $t = n-k$.*

b) *G and the maximal number r of independent vectors in $\{x_1, \ldots, x_m\}$ characterize the orbit of $(x_1, \ldots, x_m)$ if and only if $r = m$ or* $r = \operatorname{rank} G$.

Proof. Let $\kappa := \{\lambda \in \mathbb{R}^m \mid \lambda G = 0\}$.
a) If $m-s-t = 0$, then $W = \{0\}$ is the only subspace of $\mathbb{R}^m$ with (6). If $s = k$ or $t = n-k$, then $W = \kappa$ is the only subspace of $\mathbb{R}^m$ with (6). If $m-s-t > 0$ and $s < k$ and $t < n-k$, then any $(m-s-t-\delta)$-dimensional subspace W of κ satisfies (6), $\delta \in \{0, 1\}$.
b) Let W be defined by (5). Then W is a subspace of κ. There is no other subspace of κ which has the same dimension as W, iff $m - r = \dim W \in \{0, m-s-t\}$. $\square$

In the case $K = \mathbb{R}$, Corollary 2b) is stated in [2, §32], the right side of the equivalence is: $r = n$ or $r = m$ or $r = \operatorname{rank} G$. We note that $r = n$ implies $r = \operatorname{rank} G$.

4. Euclidean vector spaces

Let $m, n \in \mathbb{N}$. Let $(V, (\cdot,\cdot))$ be an n-dimensional real euclidean vector space. Its signature is $(n, 0)$. The Gram matrix of m vectors $x_1, \ldots, x_m \in V$ satisfies

$$G = G^T \text{ is positive semi-definite}, \quad \operatorname{rank} G \leq n. \tag{7}$$

On the other hand, any real $m \times m$ matrix which satisfies (7) is the Gram matrix of m vectors of V. Thus all $m \times m$ matrices with (7) correspond to the orbits of m-tuples of vectors under the isometries of V; cf. e.g. [3, §II.12].

A motion of n-dimensional euclidean geometry $E_n = V$ is a mapping $E_n \to E_n$, $x \mapsto g(x) + t$ where g is an isometry of $(V, (\cdot,\cdot))$ and $t \in V$. It is trivial that the orbit of $m+1$ vectors $x_0, \dots, x_m \in E_n$ under the group of motions of E_n corresponds to the Gram matrix G of $x_1 - x_0, \dots, x_m - x_0$, which satisfies (7).[1]

Let $n \geq 2$. The set of points of $(n-1)$-dimensional spherical geometry is $\mathbb{S}^{n-1} := \{x \in V \mid (x,x) = 1\}$. The spherical motions are the restrictions of the isometries of $(V, (\cdot,\cdot))$ to $\mathbb{S}^{n-1}$. The spherical distance between $x, y \in \mathbb{S}^{n-1}$ is $d(x,y) = \arccos(x,y)$. The orbit of m points $x_1, \dots, x_m \in \mathbb{S}^{n-1}$ corresponds to their Gram matrix $G = (g_{ij}) = (\cos d(x_i, x_j))$, which satisfies (7) and

$$g_{11} = \ldots = g_{mm} = 1. \tag{8}$$

If a real $m \times m$ matrix $G = (g_{ij})$ satisfies (7) and (8), then it is the Gram matrix of m spherical points.

5. Minkowski vector spaces

Let $m, n \in \mathbb{N}$, $n \geq 2$. Let $(V, (\cdot,\cdot))$ be an n-dimensional real vector space whose signature is $(n-1, 1)$. A vector $v \in V \setminus \{0\}$ is called *timelike*, *lightlike*, *spacelike* if $(v,v) < 0$, $= 0$, > 0, respectively. By $\mathfrak{L}$ we denote the set of all subspaces of V which contain lightlike, but no timelike vectors. According to Theorem 1 and 2, the pairs (G, W) which satisfy (6) correspond to the orbits of m-tuples of vectors under the isometries of V; we now have $\delta \in \{0, 1\}$. Let $\kappa := \{\lambda \in \mathbb{R}^m \mid \lambda G = 0\}$. Let $x_1, \dots, x_m$ be vectors with (4) and (5), and let $X < V$ be the subspace spanned by $x_1, \dots, x_m$. In the case $\delta = 0$ we have $X \notin \mathfrak{L}$, and $W = \kappa$ is determined by G. In the case $\delta = 1$ we have $X \in \mathfrak{L}$, and W is a hyperplane of κ. The orbit of $(x_1, \dots, x_m)$ may likewise be characterized by the pair (G, W') where $W' = \mathbb{R}\mu \in \mathbb{P}(\kappa)$ is defined by $W\mu^T = 0$. For all $\lambda \in \mathbb{R}^m$,

$$\sum_{i=1}^{m} \lambda_i x_i = 0 \quad \Leftrightarrow \quad \lambda G = 0 \text{ and } \lambda \mu^T = 0 \tag{5'}$$

Let $t \in V$ be a fixed timelike vector. The set of points of $(n-1)$-dimensional hyperbolic geometry in Weierstraß's model is (cf. e.g. [1])

$$\mathcal{W}^{n-1} := \{x \in V \mid (x,x) = -1 \ \wedge \ (x,t) < 0\}.$$

A hyperbolic motion is the restriction $f|_{\mathcal{W}^{n-1}}$ of an isometry f of $(V, (\cdot,\cdot))$ which is *orthochronous*, $(t, f(t)) < 0$. The hyperbolic distance between $x, y \in \mathcal{W}^{n-1}$ is $d(x,y) := \operatorname{arch}(-(x,y))$. Let $x_1, \dots, x_m$ be m hyperbolic points. Their Gram matrix $G = (g_{ij}) = -(\operatorname{ch} d(x_i, x_j))$ satisfies

$$\begin{aligned} &G = G^T, \quad G \text{ has exactly one negative eigenvalue,} \\ &\operatorname{rank} G \leq n, \quad g_{ij} \leq -1 = g_{ii} \text{ for all } i, j = 1, \dots, m. \end{aligned} \tag{9}$$

[1] If $(V, (\cdot,\cdot))$ has signature $(k, n-k)$, then the orbits of $(m+1)$-tuples of vectors in the corresponding inhomogeneous orthogonal geometry are obtained analogously, by Theorem 1 and 2.

If $x'_1, \ldots, x'_m \in \mathcal{W}^{n-1}$ have the same Gram matrix G, then there is an isometry of $(V, (\cdot,\cdot))$ which takes x_i to x'_i, $i = 1, \ldots, m$ and which is orthochronous. On the other hand, let G be a real $m \times m$ matrix with (9) and let $W := \{\lambda \in \mathbb{R}^m \mid \lambda G = 0\}$. From Theorem 2 there are $x_1, \ldots, x_m \in V$ with (4) and (5). Without loss of generality we may assume that $(x_1, t) < 0$. Then $(x_1, x_i) < 0$ implies $(x_i, t) < 0$ and $x_i \in \mathcal{W}^{n-1}$, $i = 1, \ldots, m$. Hence the orbits of m-tuples of points correspond to the $m \times m$-matrices G which satisfy (9).

The set of points of $(n-1)$-dimensional de-Sitter geometry is

$$\mathcal{S}^{n-1} := \{x \in V \mid (x, x) = 1\}.$$

The motions are restrictions of the isometries of $(V, (\cdot,\cdot))$ to $\mathcal{S}^{n-1}$. Let $x_1, \ldots, x_m$ be m points of $\mathcal{S}^{n-1}$ with Gram matrix G and subspace $W < \mathbb{R}^m$ defined by (5). If the space $X < V$ spanned by $x_1, \ldots, x_m$ is not in $\mathfrak{L}$, then $W = \{\lambda \in \mathbb{R}^m \mid \lambda G = 0\}$, and the orbit of $(x_1, \ldots, x_m)$ corresponds to the matrix G with

$$\begin{array}{l} G = G^T, \quad g_{11} = \ldots = g_{mm} = 1, \\ G \text{ has at most one negative and at most } n-1 \text{ positive eigenvalues.} \end{array}$$

If $X \in \mathfrak{L}$, then there is a uniquely determined projective point $\mathbb{R}\mu \in \mathbb{P}(V)$ with (5'). The orbit of $(x_1, \ldots, x_m)$ corresponds to the pair $(G, \mathbb{R}\mu)$ where

$$\begin{array}{l} G = G^T, \quad g_{11} = \ldots = g_{mm} = 1, \\ G \text{ is positive semi-definite}, \quad \operatorname{rank} G < n-1, \quad \mu G = 0. \end{array}$$

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