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Kinematic Formulae for Support Measures of Convex Bodies

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Abstract. A new kinematic formula for the support measures of convex bodies is proved. The underlying operation is the convex hull operation for pairs of convex bodies. Further, the concept of mixed support measures is introduced and kinematic relations for these new functionals are indicated.

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1. Introduction

The main purpose of this paper is to establish a new kinematic formula for support measures (or generalized curvature measures) of convex bodies. The proof depends on a recent result of Schneider [11], which confirmed a conjecture of the author [4, 6]. For surveys on integral geometry of convex bodies, the reader is referred to Schneider and Wieacker [13], Schneider and Weil [12], and Section 4.5 in Schneider [9]. The latter book is the basic reference for the geometric notions used below. For the measure theoretic results we will need, see Ash [1] and Cohn [2], for example.

Let $\mathcal{K}$ be the set of all convex bodies, i.e. the set of all non-empty, compact, convex subsets of $\mathbf{R}^n$, and let $\mathcal{K}$ be endowed with the Hausdorff metric. Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product of $\mathbf{R}^n$. Let B^n be the unit ball in $\mathbf{R}^n$ and S^{n-1} its boundary. For $K \in \mathcal{K}$, let $\mathrm{Nor}K$ be the set of all support elements of K, i.e. the set of all pairs $(x, u) \in \Sigma := \mathbf{R}^n \times S^{n-1}$

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where x is a boundary point of K and u is an outer unit normal vector of K at x. The *support measures* (or generalized curvature measures) of a convex body K are common generalizations of Federer's curvature measures and the area measures of Aleksandrov-Fenchel-Jessen. They are the unique Borel measures $\Theta_0(K;\cdot),\dots,\Theta_{n-1}(K;\cdot)$ on Σ which are concentrated on $\mathrm{Nor}K$ and satisfy the Steiner-type relation

$$\int_{\mathbf{R}^n\setminus K} f\,d\lambda^n = \sum_{j=0}^{n-1}\binom{n-1}{j}\int_0^\infty\int_\Sigma t^{n-j-1} f(x+tu)\,d\Theta_j(K;(x,u))\,dt \tag{1}$$

for all Lebesgue integrable $f:\mathbf{R}^n\to\mathbf{R}$; here λ^n is the Lebesgue measure on $\mathbf{R}^n$. We will make essential use of the fact that the maps $K\mapsto\Theta_j(K;\cdot)$ are weakly continuous, i.e. if $\lim K_i = K$, then $\lim\int f\,d\Theta_j(K_i;\cdot) = \int f\,d\Theta_j(K;\cdot)$ for all bounded Borel functions $f:\Sigma\to\mathbf{R}$ that are continuous $\Theta_j(K;\cdot)$-almost everywhere.

We denote by $\mathcal{B}(X)$ the Borel σ-algebra of a topological space X. The measures defined by $C_j(K;A) := \Theta_j(K;A\times S^{n-1})$, $A\in\mathcal{B}(\mathbf{R}^n)$, are, apart from normalizing constants, Federer's curvature measures, and the equation $S_j(K;\omega) := \Theta_j(K;\mathbf{R}^n\times\omega)$, $\omega\in\mathcal{B}(S^{n-1})$, defines the area measures of Alexandrov-Fenchel-Jessen. The global measure $\Theta_j(K;\Sigma)$ equals, up to constant multipliers, the quermassintegral $W_{n-j}(K)$ and the intrinsic volume $V_j(K)$.

The Haar measure on the unimodular group G_n of all proper rigid motions of $\mathbf{R}^n$ is denoted by μ, where the normalization is chosen so that $\mu(\{g\in G_n : g(0)\in B^n\})$ equals κ_n, the volume of the unit ball B^n. For $\eta,\eta'\subset\Sigma$ and a motion $g\in G_n$, let $g\eta := \{(gx, g_0u)\in\Sigma : (x,u)\in\eta\}$, where g_0 is the rotational part of g, and

$$\begin{aligned}\eta\wedge\eta' := \{(x,u)\in\Sigma \ :\ & \text{there are } u_1,u_2\in S^{n-1} \text{ with } (x,u_1)\in\eta,\\ & (x,u_2)\in\eta', \text{ and } u\in\mathrm{pos}\{u_1,u_2\}\}\,,\end{aligned}$$

where pos denotes the positive hull operation.

The following theorem can be viewed as an extension of a special case of Federer's principal kinematic formula [3]. In Theorems 1 and 3 below, we assume that the domains of the measures $\Theta_j(K;\cdot)$ are extended to the respective completions of $\mathcal{B}(\Sigma)$.

Theorem 1. *Let $K,K'\in\mathcal{K}$, $\eta,\eta'\in\mathcal{B}(\Sigma)$, and $j\in\{0,\dots,n-2\}$. Then we have*

$$\begin{aligned}&\int_{\{g\in G_n : K\cap gK'\neq\emptyset\}} \Theta_j(K\cap gK';(\eta\cap\mathrm{Nor}K)\wedge g(\eta'\cap\mathrm{Nor}K'))\,d\mu(g)\\ &= \frac{\kappa_{n-j}}{n\kappa_n\kappa_j}\sum_{k=j+1}^{n-1}\binom{n-j}{k-j}\frac{\kappa_k\kappa_{n+j-k}}{\kappa_{n-k}\kappa_{k-j}}\ \Theta_k(K;\eta)\Theta_{n+j-k}(K';\eta')\,.\end{aligned}$$

Theorem 1 was proved in Glasauer [5] under some restrictions on the bodies K,K' (see [5], Theorem 3.1), which can now be removed by the following recent result of Schneider [11]. Recall that the normal cone $N(K,x)$ of a convex body K at a boundary point x is the set of all outer normal vectors of K at x. The boundary of $K\in\mathcal{K}$ is denoted by $\mathrm{bd}K$.

Theorem 2. (Schneider) *Let $K, K' \in \mathcal{K}$. Then for μ-almost all $g \in G_n$, the linear hulls of the normal cones $N(K,x)$, $N(gK',x)$ have trivial intersection for all $x \in \mathrm{bd}K \cap \mathrm{bd}gK'$.*

Remark. There is an extension of Theorem 1 to finite unions of convex bodies, see [5], Theorem 3.3. It is likely that there are extensions to more general sets, but this remains an open problem.

The main purpose of this paper is to prove a "dual" counterpart to Theorem 1, which can be stated as follows. For $\eta, \eta' \subset \Sigma$ let

$$\eta \vee \eta' := \{(x,u) \in \Sigma \;:\; \text{there are } x_1, x_2 \in \mathbf{R}^n \text{ with } x_1 - x_2 \perp u,\ (x_1,u) \in \eta,\ (x_2,u) \in \eta', \text{ and } x \in [x_1,x_2]\},$$

where $[x_1,x_2]$ is the closed line segment with endpoints x_1, x_2. The convex hull of the union of two convex bodies K, K' is denoted by $K \vee K'$.

Theorem 3. *Let $j \in \{1,\dots,n-1\}$. Then we have*

$$\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{\{g \in G_n : gK' \subset rB^n\}} \Theta_j(K \vee gK'; (\eta \cap \mathrm{Nor}K) \vee g(\eta' \cap \mathrm{Nor}K'))\, d\mu(g)$$
$$= \frac{\kappa_{n-1}}{(n+1)n\kappa_n} \sum_{k=0}^{j-1} \Theta_k(K;\eta)\Theta_{j-k-1}(K';\eta')$$

uniformly for all $\eta, \eta' \in \mathcal{B}(\Sigma)$ and all $K, K' \in \mathcal{K}$ contained in a fixed ball.

The proof of Theorem 3 will be given in the next section. An essential ingredient is the following recent result of Schneider [11] (the formulation there is equivalent to the one given here).

Theorem 4. (Schneider) *Let $K, K' \in \mathcal{K}$. Then for μ-almost all $g \in G_n$, the following is true. For each point $x \in (\mathrm{bd}(K \vee gK')) \setminus (K \cup gK')$, there are unique points $y \in K$ and $z \in gK'$ with $x \in [y,z]$.*

Both Theorem 3 and Theorem 4 were conjectured by the author in [6], where it was stated that Theorem 3 can be proved if Theorem 4 holds true. Special cases of Theorems 2 and 4 were proved in [4, 5, 6]. Counterparts to Theorems 1–4 in spherical space were established in [4].

The setup of the paper is as follows. Section 2 contains the proof of Theorem 3. In Section 3, the concept of mixed support measures is introduced and generalizations of Theorems 1 and 3 for these new functionals are stated.

2. Proof of Theorem 3

Let us first observe that for strictly convex K, K', Theorem 3 follows from a known result.

Lemma 1. *Theorem 3 holds if K and K' are strictly convex.*

Proof. The following is a special case of Theorem 5 in [6]: Let $j \in \{0, \ldots, n-1\}$. Then

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n : gK'\subset rB^n\}} S_j(K \vee gK'; \omega \cap g_0\omega')\, d\mu(g) \\
&= \frac{\kappa_{n-1}}{(n+1)n\kappa_n} \sum_{k=0}^{j-1} S_k(K;\omega) S_{j-k-1}(K';\omega')
\end{aligned} \tag{2}$$

uniformly for all $\omega, \omega' \in \mathcal{B}(S^{n-1})$ and all $K, K' \in \mathcal{K}$ which are contained in a fixed ball.

For strictly convex $K \in \mathcal{K}$ and $\eta \in \mathcal{B}(\Sigma)$ we define $\omega_{K,\eta} := \{u \in S^{n-1} : (x,u) \in \eta \cap \mathrm{Nor}K$ for an $x \in \mathbf{R}^n\}$. Let $h_K : S^{n-1} \to \mathbf{R}$ be the support function of K, i.e. $h_K(u) := \max_{x\in K}\langle x, u\rangle = \langle \tau_K(u), u\rangle$ for $u \in S^{n-1}$, where $\tau_K(u)$ is the boundary point of K with outer normal vector u.

Now let $K, K' \in \mathcal{K}$ be strictly convex and let $\eta \subset \mathrm{Nor}K$ and $\eta' \subset \mathrm{Nor}K'$ be Borel sets. We have

$$\eta \vee \eta' = (\mathbf{R}^n \times (\omega_{K,\eta} \cap \omega_{K',\eta'} \cap \{h_K = h_{K'}\})) \cap \mathrm{Nor}(K \vee K')\,.$$

In fact, let $(x,u) \in \eta \vee \eta'$. Let $x_1, x_2 \in \mathbf{R}^n$ with $\langle x_1, u\rangle = \langle x_2, u\rangle$, $(x_1,u) \in \eta$, $(x_2,u) \in \eta'$, and $x \in [x_1, x_2]$. We have $h_K(u) = h_{K'}(u) = \langle x_1, u\rangle = \langle x, u\rangle$. Thus $(x,u) \in \mathrm{Nor}(K \vee K')$ and $u \in \omega_{K,\eta} \cap \omega_{K',\eta'} \cap \{h_K = h_{K'}\}$. Now let $(x,u) \in \mathrm{Nor}(K \vee K')$ with $u \in \omega_{K,\eta} \cap \omega_{K',\eta'}$ and $h_K(u) = h_{K'}(u)$. Let $x_1 := \tau_K(u)$, $x_2 := \tau_{K'}(u)$. It follows that $(x_1,u) \in \eta$, $(x_2,u) \in \eta'$, and $x \in [x_1, x_2]$, as required.

We infer from this equation that

$$\begin{aligned}
&S_j(K \vee K'; \omega_{K,\eta} \cap \omega_{K',\eta'}) - S_j(K; S^{n-1}) - S_j(K'; S^{n-1}) \\
&\quad\le\; S_j(K \vee K'; \omega_{K,\eta} \cap \omega_{K',\eta'} \cap \{h_K = h_{K'}\}) \\
&\quad=\; \Theta_j(K \vee K'; \eta \vee \eta') \\
&\quad\le\; S_j(K \vee K'; \omega_{K,\eta} \cap \omega_{K',\eta'})
\end{aligned}$$

for all $j \in \{0, \ldots, n-1\}$. We also have

$$\Theta_j(K;\eta) = S_j(K;\omega_{K,\eta}), \qquad \Theta_j(K';\eta') = S_j(K';\omega_{K',\eta'})\,.$$

Now the assertion follows from equation (2). □

We now want to extend our formula to general convex bodies by approximation.

We collect some notations which will be used below. Let $K, K' \in \mathcal{K}$. We define $\Sigma_1(K,K')$ as the set of all $(x,u) \in \mathrm{Nor}K \cup \mathrm{Nor}K'$ such that there is a hyperplane which supports both K and K', with both bodies on the same side of it, and which is orthogonal to u and contains x. We let $\Sigma_2(K,K') := \mathrm{Nor}(K \vee K') \backslash (\mathrm{Nor}K \cup \mathrm{Nor}K')$. We define $G(K,K')$ to be the set of all motions $g \in G_n$ such that for each $x \in (\mathrm{bd}(K \vee gK')) \backslash (K \cup gK')$ there are unique points $y \in K$, $z \in gK'$ with $x \in [y,z]$. Theorem 4 says that for all $K, K' \in \mathcal{K}$ there is a Borel subset of $G(K,K')$ whose complement has μ-measure zero. For $g \in G(K,K')$ we define two maps, $\pi_{g,K,K'}$ and $\pi'_{g,K,K'}$, on $\Sigma_2(K, gK')$ by the requirement that $\pi_{g,K,K'}(x,u) := (y,u)$, $\pi'_{g,K,K'}(x,u) := (g^{-1}z, g_0^{-1}u)$, where y, z are the unique points in K, gK', respectively, with $x \in [y,z]$. The images of $\pi_{g,K,K'}$, $\pi'_{g,K,K'}$ are contained in $\mathrm{Nor}K$, $\mathrm{Nor}K'$, respectively.

Lemma 2. *For μ-almost all $g \in G_n$, we have $\Theta_j(K \vee gK'; \Sigma_1(K, gK')) = 0$ for all $j \in \{0, \dots, n-1\}$.*

Proof. Let $j \in \{0, \dots, n-1\}$. Let $\mathcal{E}$ be the set of all hyperplanes in $\mathbf{R}^n$, and let $\mathcal{E}(K)$ be the subset of all support planes of K (these sets are endowed with their natural topologies). For $A \subset \mathcal{E}$, let $\eta_A := \{(x, u) \in \Sigma : x \in H, H \perp u \text{ for an } H \in A\}$. Define a measure $\rho(K; \cdot)$ on $\mathcal{E}$ by

$$\rho(K; A) := \Theta_j(K; \eta_A), \qquad A \in \mathcal{B}(\mathcal{E}).$$

We have $\Sigma_1(K, gK') \subset \eta_{\mathcal{E}(K) \cap \mathcal{E}(gK')} \cap (\mathrm{Nor}K \cup \mathrm{Nor}gK')$ and therefore

$$\Theta_j(K \vee gK'; \Sigma_1(K, gK')) \leq \rho(K; \mathcal{E}(K) \cap \mathcal{E}(gK')) + \rho(gK'; \mathcal{E}(K) \cap \mathcal{E}(gK')) .$$

We will show that both terms in this sum are zero for μ-almost all g. We denote the rotation invariant probability measure on the rotation group SO_n by ν. The line through 0 orthogonal to a hyperplane H is written as $H^\perp$. The Fubini theorem shows that

$$\begin{aligned}
&\int_{G_n} \rho(K; \mathcal{E}(K) \cap \mathcal{E}(gK'))\, d\mu(g) \\
&= \int_{SO_n} \int_{\mathbf{R}^n} \int_{\mathcal{E}} \mathbf{1}_{\mathcal{E}(K)}(H) \mathbf{1}_{\mathcal{E}(\vartheta K') + x}(H)\, d\rho(K; H) d\lambda^n(x) d\nu(\vartheta) \\
&= \int_{SO_n} \int_{\mathcal{E}} \mathbf{1}_{\mathcal{E}(K)}(H) \int_{H} \int_{H^\perp} \mathbf{1}_{\mathcal{E}(\vartheta K')}(H + y + z)\, d\lambda^1(y) d\lambda^{n-1}(z) d\rho(K; H) d\nu(\vartheta) \\
&= 0 ,
\end{aligned}$$

since $H + y \in \mathcal{E}(\vartheta K')$ for at most two values of $y \in H^\perp$. Hence $\rho(K; \mathcal{E}(K) \cap \mathcal{E}(gK')) = 0$ for μ-almost all g, and the same reasoning applies to the second term. $\square$

Lemma 3. *Let $j \in \{0, \dots, n-1\}$. Then we have*

$$\begin{aligned}
&\lim_{r \to \infty} \frac{1}{r^{n+1}} \int_{\{g \in G_n : gK' \subset rB^n\}} \int_{\Sigma_2(K, gK')} (f \circ \pi_{g,K,K'})(f' \circ \pi'_{g,K,K'})\, d\Theta_j(K \vee gK'; \cdot) d\mu(g) \\
&= \frac{\kappa_{n-1}}{(n+1) n \kappa_n} \sum_{k=0}^{j-1} \int_{\Sigma} f\, d\Theta_k(K; \cdot) \int_{\Sigma} f'\, d\Theta_{j-k-1}(K'; \cdot)
\end{aligned}$$

uniformly for all continuous $f, f' : \Sigma \to \mathbf{R}$ with $0 \leq f, f' \leq 1$ and all strictly convex bodies $K, K' \in \mathcal{K}$ contained in a fixed ball.

Proof. Let $j \in \{0, \dots, n-1\}$, let $K, K' \in \mathcal{K}$ be strictly convex, and let $\eta, \eta' \in \mathcal{B}(\Sigma)$. We have $G(K, K') = G_n$. By using Lemma 2 we obtain

$$\begin{aligned}
&\Theta_j(K \vee gK'; (\eta \cap \mathrm{Nor}K) \vee g(\eta' \cap \mathrm{Nor}K')) \\
&\quad = \Theta_j(K \vee gK'; ((\eta \cap \mathrm{Nor}K) \vee g(\eta' \cap \mathrm{Nor}K')) \setminus \Sigma_1(K, gK')) \\
&\quad = \Theta_j(K \vee gK'; ((\eta \cap \mathrm{Nor}K) \vee g(\eta' \cap \mathrm{Nor}K')) \cap \Sigma_2(K, gK'))
\end{aligned}$$

$$
\begin{aligned}
&= \int\limits_{\Sigma_2(K,gK')} \mathbf{1}_{(\eta\cap \operatorname{Nor}K)\vee g(\eta'\cap\operatorname{Nor}K')}\, d\Theta_j(K\vee gK';\cdot) \\
&= \int\limits_{\Sigma_2(K,gK')} (\mathbf{1}_\eta \circ \pi_{g,K,K'})(\mathbf{1}_{\eta'}\circ \pi'_{g,K,K'})\, d\Theta_j(K\vee gK';\cdot)
\end{aligned}
$$

for μ-almost all $g \in G_n$. From this and Lemma 1 it follows that

$$
\begin{aligned}
&\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n : gK'\subset rB^n\}} \int\limits_{\Sigma_2(K,gK')} (f\circ \pi_{g,K,K'})(f'\circ\pi'_{g,K,K'})\, d\Theta_j(K\vee gK';\cdot) d\mu(g) \\
&= \frac{\kappa_{n-1}}{(n+1)n\kappa_n} \sum_{k=0}^{j-1} \int\limits_{\Sigma} f\, d\Theta_k(K;\cdot) \int\limits_{\Sigma} f'\, d\Theta_{j-k-1}(K';\cdot)
\end{aligned}
$$

uniformly for all strictly convex $K, K' \in \mathcal{K}$ contained in a fixed ball and all simple functions $f, f' : \Sigma \to \mathbf{R}$ with $0 \le f, f' \le 1$, i.e. functions of the type $\sum_{i=1}^m c_i \mathbf{1}_{\eta_i}$ with $m \in \mathbf{N}$, $0 \le c_i \le 1$, and pairwise disjoint Borel sets $\eta_1, \ldots, \eta_m \subset \Sigma$. Since every non-negative continuous function is the pointwise limit of an increasing sequence of simple functions, the monotone convergence theorem shows the assertion. □

Lemma 4. *Lemma* 3 *holds also for general convex bodies* $K, K' \in \mathcal{K}$.

Proof. Let $K, K' \in \mathcal{K}$. Let $(K_i)_{i\in\mathbf{N}}$, $(K'_i)_{i\in\mathbf{N}}$ be sequences of strictly convex bodies such that $K_{i+1} \subset \operatorname{int}K_i$, $K'_{i+1} \subset \operatorname{int}K'_i$ for all i, where int denotes the interior, and $\lim K_i = K$, $\lim K'_i = K'$. (The existence of such sequences follows easily from Theorem 3.3.1 in Schneider [9].) Let $g \in G(K, K')$ be such that $\Theta_j(K \vee gK'; \Sigma_1(K, gK')) = 0$. Since K_i, K'_i are strictly convex, we have $G(K_i, K'_i) = G_n$ for $i \in \mathbf{N}$. Let $K_0 := K$, $K'_0 := K'$. Let $f, f' \in C(\Sigma)$. Let $m \in \mathbf{N}$. We define

$$
\begin{aligned}
A_m &:= \{(x,u) \in \bigcup_{i\in\mathbf{N}_0} \operatorname{Nor}(K_i \vee gK'_i) : |x-y| \ge \frac{1}{m} \text{ for all } (y,v) \in \bigcup_{i\in\mathbf{N}_0} \Sigma_1(K_i, gK'_i)\}, \\
B_m &:= \{(x,u) \in \Sigma : |x-y| \ge \frac{1}{m} \text{ for all } (y,v) \in \bigcup_{i\in\mathbf{N}_0} \Sigma_1(K_i, gK'_i)\}.
\end{aligned}
$$

The set A_m is a closed subset of B_m, since $\bigcup_{i\in\mathbf{N}_0} \operatorname{Nor}(K_i \vee gK'_i)$ is closed, and also B_m is a closed set, hence a normal subspace of Σ.

Define a function $g_1 : A_1 \to \mathbf{R}$ as follows. If $(x,u) \in A_1 \cap \Sigma_2(K_i, gK'_i)$ for one (and only one) $i \in \mathbf{N}_0$, let $g_1(x,u) := f(\pi_{g,K_i,K'_i}(x,u)) f'(\pi'_{g,K_i,K'_i}(x,u))$. For the remaining $(x,u) \in A_1$, we define $g_1(x,u) := 0$.

The function $g_1 : A_1 \to \mathbf{R}$ thus defined is continuous: Let $(x,u) \in A_1$, and let $((x_m,u_m))_{m\in\mathbf{N}}$ be a sequence in A_1 converging to (x,u). Then $(x,u) \in \operatorname{Nor}(K_i \vee gK'_i)$ for a unique $i \in \mathbf{N}_0$. The pair (x,u) is either in $\operatorname{Nor}K_i$, in $\operatorname{Nor}gK'_i$, or in $\Sigma_2(K_i, gK'_i)$. If $(x,u) \in \operatorname{Nor}K_i$, then $(x_m,u_m) \in \bigcup_{i\in\mathbf{N}_0} \operatorname{Nor}K_i$ for almost all $m \in \mathbf{N}$ and therefore $g_1(x,u) = 0$ and $g_1(x_m,u_m) = 0$ for almost all m. The case $(x,u) \in \operatorname{Nor}gK'_i$ is analogous. If $(x,u) \in \Sigma_2(K_i, gK'_i)$, then there is a sequence $(i_m)_{m\in\mathbf{N}}$ of non-negative integers such that $(x_m,u_m) \in \Sigma_2(K_{i_m}, gK'_{i_m})$ for almost all m. Let y_m and z_m be the unique points in

K_{i_m}, gK'_{i_m}, respectively, with $x_m \in [y_m, z_m]$ and let y, z be the points in K_i, gK'_i, respectively, with $x \in [y, z]$. Assume that y_m does not converge to y for $m \to \infty$. Then there is a convergent subsequence $(y_{m_k})_{k\in\mathbf{N}}$ with $\bar{y} := \lim_{k\to\infty} y_{m_k} \neq y$. Let $\bar{z}$ be a limit point of the sequence $(z_{m_k})_{k\in\mathbf{N}}$. We conclude that $x \in [\bar{y}, \bar{z}]$. Since $\bar{y} \in K_i$ and $\bar{z} \in gK'_i$, this contradicts the fact that $g \in G(K_i, gK'_i)$. Thus $\lim y_m = y$, and analogously we get $\lim z_m = z$. Hence $\lim \pi_{g,K_{i_m},K'_{i_m}}(x_m, u_m) = \pi_{g,K_i,K'_i}(x, u)$ and $\lim \pi'_{g,K_{i_m},K'_{i_m}}(x_m, u_m) = \pi'_{g,K_i,K'_i}(x, u)$, and therefore $\lim g_1(x_m, u_m) = g_1(x, u)$, as required.

We can now apply Tietze's extension theorem to extend g_1 to a continuous function h_1 on B_1. If $m \in \mathbf{N}$ and $h_m : B_m \to \mathbf{R}$ is already defined, let g_{m+1} be the extension of h_m to the set $B_m \cup A_{m+1}$ which is defined in the same way as g_1. Then apply Tietze's theorem to extend g_{m+1} to a continuous function h_{m+1} on B_{m+1}. In this way, we obtain continuous functions $h_m : B_m \to \mathbf{R}$ with $h_{m+1}|B_m = h_m$ for all $m \in \mathbf{N}$.

Now define a function $h : \Sigma \to \mathbf{R}$ as follows. If $(x, u) \in \Sigma \setminus \bigcup_{i\in\mathbf{N}_0} \Sigma_1(K_i, gK'_i)$, then $(x, u) \in B_m$ for a sufficiently large m, since $\bigcup_{i\in\mathbf{N}_0} \Sigma_1(K_i, gK'_i)$ is closed; we let $h(x, u) := h_m(x, u)$. If $(x, u) \in \bigcup_{i\in\mathbf{N}_0} \Sigma_1(K_i, gK'_i)$, we let $h(x, u) := 0$. So if $(x, u) \in \operatorname{Nor}(K_i \vee gK'_i)$ for an $i \in \mathbf{N}_0$, then $h(x, u) = f(\pi_{g,K_i,K'_i}(x, u)) f'(\pi'_{g,K_i,K'_i}(x, u))$ in the case $(x, u) \in \Sigma_2(K_i, gK'_i)$ and $h(x, u) = 0$ if $(x, u) \in \operatorname{Nor} K_i \cup \operatorname{Nor} gK'_i$. Since $\Theta_j(K \vee gK'; \Sigma_1(K, gK')) = 0$, the function h is continuous $\Theta_j(K \vee gK'; \cdot)$-almost everywhere, and it is Borel measurable and bounded. Now the weak continuity of the support measures implies that

$$
\begin{aligned}
&\lim_{i\to\infty} \int\limits_{\Sigma_2(K_i,gK'_i)} (f \circ \pi_{g,K_i,K'_i})(f' \circ \pi'_{g,K_i,K'_i})\, d\Theta_j(K_i \vee gK'_i; \cdot) \\
&\quad = \lim_{i\to\infty} \int\limits_{\Sigma} h \, d\Theta_j(K_i \vee gK'_i; \cdot) \\
&\quad = \int\limits_{\Sigma} h \, d\Theta_j(K \vee gK'; \cdot) \\
&\quad = \int\limits_{\Sigma_2(K,gK')} (f \circ \pi_{g,K,K'})(f' \circ \pi'_{g,K,K'})\, d\Theta_j(K \vee gK'; \cdot).
\end{aligned}
$$

The dominated convergence theorem, Lemma 2, and Theorem 4 show that

$$
\begin{aligned}
&\lim_{i\to\infty} \int\limits_{\{g\in G_n : gK'_i \subset rB^n\}} \int\limits_{\Sigma_2(K_i,gK'_i)} (f \circ \pi_{g,K_i,K'_i})(f' \circ \pi'_{g,K_i,K'_i})\, d\Theta_j(K_i \vee gK'_i; \cdot) d\mu(g) \\
&\quad = \int\limits_{\{g\in G_n : gK' \subset rB^n\}} \int\limits_{\Sigma_2(K,gK')} (f \circ \pi_{g,K,K'})(f' \circ \pi'_{g,K,K'})\, d\Theta_j(K \vee gK'; \cdot) d\mu(g) \qquad (3)
\end{aligned}
$$

for all $r > 0$. According to Lemma 3, for a given $\epsilon > 0$, there is an $r_0 > 0$ such that for all $r \geq r_0$ we have

$$
\begin{aligned}
&\Bigg| \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n : gK'_i \subset rB^n\}} \int\limits_{\Sigma_2(K_i,gK'_i)} (f \circ \pi_{g,K_i,K'_i})(f' \circ \pi'_{g,K_i,K'_i})\, d\Theta_j(K_i \vee gK'_i; \cdot) d\mu(g) \\
&\quad - \frac{\kappa_{n-1}}{(n+1)n\kappa_n} \sum_{k=0}^{j-1} \int\limits_{\Sigma} f \, d\Theta_k(K_i; \cdot) \int\limits_{\Sigma} f' \, d\Theta_{j-k-1}(K'_i; \cdot) \Bigg| \leq \epsilon
\end{aligned}
$$

for all continuous $f, f' : \Sigma \to \mathbf{R}$ with $0 \le f, f' \le 1$ and all $i \in \mathbf{N}$. In addition, r_0 depends only on the radius of the smallest centered ball containing all K_i, K_i' and not on the particular choices of the bodies K_i, K_i'. Now (3) shows that

$$\Bigg| \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n : gK' \subset rB^n\}} \int\limits_{\Sigma_2(K,gK')} (f\circ \pi_{g,K,K'})(f'\circ \pi'_{g,K,K'})\, d\Theta_j(K\vee gK';\cdot) d\mu(g) - \frac{\kappa_{n-1}}{(n+1)n\kappa_n} \sum_{k=0}^{j-1} \int\limits_{\Sigma} f\, d\Theta_k(K;\cdot) \int\limits_{\Sigma} f'\, d\Theta_{j-k-1}(K';\cdot) \Bigg| \le \epsilon$$

for all $r \ge r_0$, and the proof of Lemma 4 is complete. □

Proof of Theorem 3. Let $K, K' \in \mathcal{K}$, and let $\eta \in \mathcal{B}(\mathrm{Nor}K)$, $\eta' \in \mathcal{B}(\mathrm{Nor}K')$. It can easily be shown that $(\eta \vee g\eta')\backslash\Sigma_1(K, gK')$ is a Borel set for all $g \in G(K, K')$. Hence the map $F_{\eta,\eta'} : g \mapsto \Theta_j(K \vee gK'; \eta \vee g\eta')$ is defined on a set of full μ-measure. We want to show that $F_{\eta,\eta'}$ is measurable. Assume that η and η' are compact. Then there are decreasing sequences $(f_i)_{i\in\mathbf{N}}$, $(f_i')_{i\in\mathbf{N}}$ in $C(\Sigma)$ with $0 \le f_i, f_i' \le 1$ and $\lim f_i = \mathbf{1}_\eta$, $\lim f_i' = \mathbf{1}_{\eta'}$ in the sense of pointwise convergence. The limit $\lim_{i\to\infty} \mathbf{1}_{\Sigma_2(K,gK')}(f_i \circ \pi_{g,K,K'})(f_i' \circ \pi'_{g,K,K'})$ equals the indicator function of the set $(\eta\vee g\eta')\backslash\Sigma_1(K, K')$ for all $g \in G(K, K')$. Thus the monotone convergence theorem, Lemma 2, Theorem 4, and the proof of Lemma 4 show that $F_{\eta,\eta'}$ is the limit of measurable functions μ-almost everywhere and hence is measurable on a set of full measure. Let $\eta' \subset \mathrm{Nor}K'$ still be compact. The set of all $\eta \in \mathcal{B}(\mathrm{Nor}K)$ such that $F_{\eta,\eta'}$ is measurable on a set of full measure can easily be shown to be a Dynkin system. Since it contains all compact sets, it must coincide with $\mathcal{B}(\mathrm{Nor}K)$, so $F_{\eta,\eta'}$ has the required property for compact η'. We can now apply the same argument to see that $F_{\eta,\eta'}$ is measurable on a set of full measure for all $\eta \in \mathcal{B}(\mathrm{Nor}K)$, $\eta' \in \mathcal{B}(\mathrm{Nor}K')$.

Let us define

$$\begin{aligned}\varphi(r, K, K', \eta, \eta') \;:=\; & \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n : gK'\subset rB^n\}} \Theta_j(K\vee gK'; (\eta\cap \mathrm{Nor}K)\vee g(\eta'\cap\mathrm{Nor}K'))\, d\mu(g) \\ & - \frac{\kappa_{n-1}}{(n+1)n\kappa_n} \sum_{k=0}^{j-1} \Theta_k(K;\eta)\Theta_{j-k-1}(K';\eta')\end{aligned}$$

for all $r > 0$, $K, K' \in \mathcal{K}$, and $\eta, \eta' \in \mathcal{B}(\Sigma)$. It follows from Lemma 2, Theorem 4, and the monotone convergence theorem that $\varphi(r, K, K', \eta, \eta')$ is a finite signed measure in both η and η'. Approximating the indicator functions of compact sets $\eta, \eta' \subset \Sigma$ by monotone sequences of continuous functions as before, we deduce from Lemma 4 that for all $\epsilon > 0$ there is an $r_0 > 0$ with

$$|\varphi(r, K, K', \eta, \eta')| \le \epsilon$$

for all $r \ge r_0$, all compact $\eta, \eta' \subset \Sigma$, and all $K, K' \in \mathcal{K}$ that are contained in a fixed ball. Using the regularity of finite signed measures on Σ, we deduce that this inequality also holds for Borel sets η and compact sets η'. In the same way we finally obtain it for arbitrary $\eta, \eta' \in \mathcal{B}(\Sigma)$. This concludes the proof of Theorem 3. □

3. Mixed support measures and extensions of Theorems 1 and 3

In this section, we want to introduce the concept of mixed support measures and state generalizations of Theorems 1 and 3 for these new functionals.

Let $S_{n-1}(K;\cdot) = \Theta_{n-1}(K;\mathbf{R}^n \times \cdot)$ be the area measure of $K \in \mathcal{K}$. For n-dimensional convex bodies K and Borel subsets ω of the sphere, $S_{n-1}(K;\omega)$ equals the $(n-1)$-dimensional Hausdorff measure of the set of all $x \in \mathrm{bd}K$ such that there exists an outer unit normal vector of K at x that belongs to ω. The mixed area measures $S(K_1,\dots,K_{n-1};\cdot)$ appear as coefficients in the polynomial expansion of the area measure of a Minkowski linear combination of convex bodies: we have

$$S_{n-1}(\lambda_1 K_1 + \ldots + \lambda_m K_m;\cdot) = \sum_{i_1,\dots,i_{n-1}=1}^{m} \lambda_{i_1}\cdots\lambda_{i_{n-1}} S(K_{i_1},\dots,K_{i_{n-1}};\cdot) \tag{4}$$

for all $m \in \mathbf{N}$, $K_1,\dots,K_m \in \mathcal{K}$, $\lambda_1,\dots,\lambda_m \geq 0$, see Schneider [9], p. 275.

Let $\mathcal{S}$ be the set of all strictly convex bodies. For $M \in \mathcal{S}$ and $u \in S^{n-1}$, we defined $\tau_M(u)$ to be the boundary point of M with outer normal vector u. Let $s_M : S^{n-1} \to \Sigma$, $u \mapsto (\tau_M(u), u)$. For a positive real number λ, we define a map $t_M^\lambda : \Sigma \to \Sigma$ by means of $t_M^\lambda(x,u) := ((x - \tau_M(u))/\lambda, u)$. Both s_M and t_M^λ are continuous and hence Borel measurable.

Theorem 5. *For $j \in \{0,\dots,n-2\}$, $K \in \mathcal{K}$, and $M_1,\dots,M_{n-j-1} \in \mathcal{S}$, there are unique Borel measures $\Theta_j(K;M_1,\dots,M_{n-j-1};\cdot)$ on Σ, such that they are symmetric in $M_1,\dots,M_{n-j-1}$ and satisfy*

$$\begin{aligned} t^{\lambda}_{\lambda_1 M_1+\ldots+\lambda_m M_m}(\Theta_{n-1}(\lambda K + \lambda_1 M_1 + \ldots + \lambda_m M_m;\cdot)) &= \lambda^{n-1}\Theta_{n-1}(K;\cdot) + \\ + \sum_{j=0}^{n-2} \binom{n-1}{j} \lambda^j \sum_{i_1,\dots,i_{n-j-1}=1}^{m} &\lambda_{i_1}\cdots\lambda_{i_{n-j-1}} \Theta_j(K;M_{i_1},\dots,M_{i_{n-j-1}};\cdot) \end{aligned} \tag{5}$$

for all $m \in \mathbf{N}$, $M_1,\dots,M_m \in \mathcal{S}$, $\lambda > 0$, and $\lambda_1,\dots,\lambda_m \geq 0$. These measures are concentrated on $\mathrm{Nor}K$, *and they depend weakly continuously on $K \in \mathcal{K}$.*

We call the functional $\Theta_j(K;\cdot;\cdot)$ the *j-th mixed support measure of K*. It can easily be deduced from (1) and (5) that $\Theta_j(K;B^n,\dots,B^n;\cdot) = \Theta_j(K;\cdot)$ for all $K \in \mathcal{K}$ and $j \in \{0,\dots,n-2\}$.

Proof. The uniqueness statement being clear, we need only prove existence. Let $j \in \{0,\dots,n-2\}$. From (4) we can infer that there are $\ell \in \mathbf{N}$, $a_i \in \mathbf{R}$, $\lambda^{(i)} > 0$, $\lambda_1^{(i)},\dots,\lambda_{n-j-1}^{(i)} \geq 0$ $(i \in \{1,\dots,\ell\})$ with

$$S(K,\dots,K,M_1,\dots,M_{n-j-1};\cdot) = \sum_{i=1}^{\ell} a_i\, S_{n-1}(\lambda^{(i)}K + \lambda_1^{(i)}M_1 + \ldots + \lambda_{n-j-1}^{(i)}M_{n-j-1};\cdot)$$

for all $K, M_1,\dots,M_{n-j-1} \in \mathcal{K}$, compare the proofs of Hilfssatz 15.1 in [8] or of Lemma 5.1.3 in [9], for example. Now let $K \in \mathcal{K}$ and let $M_1,\dots,M_{n-j-1} \in \mathcal{S}$. Define the signed measure

$$\begin{aligned} &\Theta_j(K;M_1,\dots,M_{n-j-1};\cdot) \\ &:= \sum_{i=1}^{\ell} a_i\, t^{\lambda^{(i)}}_{\lambda_1^{(i)}M_1+\ldots+\lambda_{n-j-1}^{(i)}M_{n-j-1}}(\Theta_{n-1}(\lambda^{(i)}K + \lambda_1^{(i)}M_1 + \ldots + \lambda_{n-j-1}^{(i)}M_{n-j-1};\cdot)). \end{aligned}$$

It is concentrated on $\mathrm{Nor} K$ and weakly continuous in $K \in \mathcal{K}$. If $K \in \mathcal{S}$, then

$$\Theta_j(K; M_1, \ldots, M_{n-j-1}; \cdot) = s_K(S(K, \ldots, K, M_1, \ldots, M_{n-j-1}; \cdot))$$

because of $\Theta_{n-1}(M; \cdot) = s_M(S_{n-1}(M; \cdot))$ and $t_M^\lambda \circ s_{\lambda K+M} = s_K$ for all $M \in \mathcal{S}$, $\lambda > 0$. This shows that $\Theta_j(K; M_1, \ldots, M_{n-j-1}; \cdot)$ is a positive measure, which is symmetric in $M_1, \ldots, M_{n-j-1}$, and that the asserted relation (5) holds if $K \in \mathcal{S}$. Now the general result follows by weak continuity. □

We are now in a position to formulate our extension of Theorem 3. Let $f : S^{n-1} \to \mathbf{R}$ be a non-negative continuous function. Let $\delta > 0$. Define a measure α on $\mathcal{B}(\mathbf{R}^n)$ by means of

$$\alpha(A) := \int_0^\infty t^{\delta-1} \int_{S^{n-1}} \mathbf{1}_A(tu) f(u)\, d\sigma(u) dt,$$

where σ is the spherical Lebesgue measure. Let ν be the invariant probability measure on the rotation group SO_n. Let μ_α be the image of the product measure $\alpha \otimes \nu$ on $\mathbf{R}^n \times SO_n$ under the map $\mathbf{R}^n \times SO_n \to G_n$, $(x, \vartheta) \mapsto g_{x,\vartheta}$, where the motion $g_{x,\vartheta}$ is defined by $g_{x,\vartheta}(y) := x + \vartheta y$. There is a unique convex body $Z_\alpha \in \mathcal{K}$ with support function $\int_{B^n} |\langle x, \cdot\rangle|\, d\alpha(x)$, see [9], Theorem 1.7.1. A straightforward computation shows that this support function is of class C^1, so Z_α is strictly convex (cf. [9], Corollary 1.7.3). Let $j \in \{0, \ldots, n-1\}$, and let $M_1, \ldots, M_{n-j-1} \in \mathcal{S}$ be strictly convex bodies. As before, we extend the domain of the measure $\Theta_j(K; M_1, \ldots, M_{n-j-1}; \cdot)$, $K \in \mathcal{K}$, by replacing this measure by its completion. Now the following can be stated.

Theorem 6. *Under the above assumptions, we have*

$$\lim_{r\to\infty} \frac{1}{r^{\delta+1}} \int_{\{g \in G_n : gK' \subset rB^n\}} \Theta_j(K \vee gK'; M_1, \ldots, M_{n-j-1}; (\eta \cap \mathrm{Nor}K) \vee g(\eta' \cap \mathrm{Nor}K'))\, d\mu_\alpha(g)$$

$$= \frac{1}{2n\kappa_n} \sum_{k=0}^{j-1} \Theta_k(K; M_1, \ldots, M_{n-j-1}, B^n, \ldots, B^n, Z_\alpha; \eta)\Theta_{j-k-1}(K'; \eta')$$

uniformly for all $\eta, \eta' \in \mathcal{B}(\Sigma)$ and all $K, K' \in \mathcal{K}$ contained in a fixed ball.

We omit the proof of Theorem 6, since it follows exactly the same lines as the proof of Theorem 3 but requires a clumsier notation.

We remark that it is the continuity of f that insures the existence of the limit for all Borel sets η, η', see the proof of Theorem 4 in [6]. Of course, Z_α is the zonoid whose generating measure has the density $\frac{1}{\delta+1} f$ with respect to σ.

Finally, we want to state a generalization of Theorem 1 without proof. This is a result in the spirit of Hadwiger's general kinematic formula ([7], Section 6.3.5) and Schneider's [10] local version of it (see [4], 105–108, for a simpler proof of the latter result).

Denote by $\mathcal{E}_j^n$ the set of all j-dimensional affine subspaces of $\mathbf{R}^n$, endowed with its usual topology, and let μ_j be the Haar measure on $\mathcal{E}_j^n$, normalized by $\mu_j(\{E \in \mathcal{E}_j^n : E \cap B^n \neq \emptyset\}) = \kappa_{n-j}$. For $E \in \mathcal{E}_j^n$ and $\eta \subset \Sigma$, we let

$$\begin{aligned} \eta \wedge E := \{(x,u) \in \Sigma \quad : \quad & \text{there are } u_1, u_2 \in S^{n-1} \text{ with } (x,u_1) \in \eta, \\ & (x,u_2) \in E \times E^\perp, \text{ and } u \in \operatorname{pos}\{u_1,u_2\}\}, \end{aligned}$$

where $E^\perp$ is the $(n-j)$-dimensional linear subspace orthogonal to E. Now the following can be stated.

Theorem 7. *Let $\Psi : \mathcal{K} \times \mathcal{B}(\Sigma) \to \mathbf{R}$ be a map with the following properties.*

1. *$\Psi(K;\cdot)$ is a measure, concentrated on* $\operatorname{Nor}K$ *for all $K \in \mathcal{K}$.*
2. *For $K, K' \in \mathcal{K}$, $\eta \in \mathcal{B}(\Sigma)$ with $\eta \cap \operatorname{Nor}K = \eta' \cap \operatorname{Nor}K'$, we have $\Psi(K;\eta) = \Psi(K';\eta')$.*
3. *The map $K \mapsto \Psi(K;\cdot)$ is weakly continuous.*

Denote the completion of the measure $\Psi(K;\cdot)$ by $\overline{\Psi}(K;\cdot)$. For $k \in \{1,\ldots,n-1\}$, $K \in \mathcal{K}$, $\eta \in \mathcal{B}(\Sigma)$, the integral

$$\Psi_k(K;\eta) := \int\limits_{\{E \in \mathcal{E}_{n-k}^n : K \cap E \neq \emptyset\}} \overline{\Psi}(K \cap E; (\eta \cap \operatorname{Nor}K) \wedge E)\, d\mu_{n-k}(E)$$

exists, and we have

$$\int\limits_{\{g \in G_n : K \cap gK' \neq \emptyset\}} \overline{\Psi}(K \cap gK'; (\eta \cap \operatorname{Nor}K) \wedge g(\eta' \cap \operatorname{Nor}K'))\, d\mu(g)$$

$$= \sum_{k=1}^{n-1} \frac{\binom{n}{k}}{n\kappa_k} \Psi_k(K;\eta)\Theta_{n-k}(K';\eta')$$

for all $K, K' \in \mathcal{K}$ and all $\eta, \eta' \in \mathcal{B}(\Sigma)$.

The proof of Theorem 7 combines techniques developed in [4], [5], and in the present paper.

Remarks. 1. Examples for the measure $\Psi(K;\cdot)$ are the mixed support measures $\Theta_j(K; M_1, \ldots, M_{n-j-1};\cdot)$. In the case $\Psi(K;\cdot) := \Theta_j(K;\cdot)$, the Crofton formula proved in [5], Theorem 3.2, shows that Theorem 7 contains Theorem 1 as a special case.

2. It is easy to see that a map $K \mapsto \Psi(K;\cdot)$ satisfying conditions 1 and 2 of Theorem 7 is a valuation. Now one can apply Groemer's extension theorem for continuous valuations (see, e.g., [12], Satz 2.4.2) to extend Theorem 7 to finite unions of convex bodies. See [5] for a definition of the set $\operatorname{Nor}K$ if K is a finite union of convex bodies and for additional arguments required to establish this extension.

References

[1] Ash, R. B.: *Measure, Integration, and Functional Analysis.* Academic Press, New York 1972.

[2] Cohn, D. L.: *Measure Theory.* Birkhäuser, Boston 1980.

[3] Federer, H.: *Curvature measures.* Trans. Amer. Math. Soc. **93** (1959), 418–491.

[4] Glasauer, S.: *Integralgeometrie konvexer Körper im sphärischen Raum.* Thesis, University of Freiburg 1995. Summary: Diss. Summ. Math. **1** (1996), 219–226.

[5] Glasauer, S.: *A generalization of intersection formulae of integral geometry.* Geom. Dedicata **68** (1997), 101–121.

[6] Glasauer, S.: *Translative and kinematic integral formulae concerning the convex hull operation.* Math. Z. (to appear).

[7] Hadwiger, H.: *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie.* Springer, Berlin 1957.

[8] Leichtweiß, K.: *Konvexe Mengen.* Springer, Berlin 1980.

[9] Schneider, R.: *Convex Bodies: the Brunn-Minkowski Theory.* Cambridge University Press, Cambridge 1993.

[10] Schneider, R.: *An extension of the principal kinematic formula of integral geometry.* Suppl. Rend. Circ. Mat. Palermo II **35** (1994), 275–290.

[11] Schneider, R.: *Convex bodies in exceptional relative positions.* J. London Math. Soc. (to appear).

[12] Schneider, R., Weil, W.: *Integralgeometrie.* Teubner, Stuttgart 1992.

[13] Schneider, R., Wieacker, J. A.: *Integral geometry.* In: Handbook of Convex Geometry (editors P.M. Gruber, J.M. Wills), North-Holland, Amsterdam 1993, 1349–1390.

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