Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 1, 163-183.

Hyperplane Sections of Convex Bodies in Isotropic Position

Matthieu Fradelizi

Université de Marne-la-Vallée, Equipe d'Analyse et de Mathématiques Appliquées
Cité Descartes, 5 Bd Descartes, Champs sur Marne, 77454 Marne-la-Vallée Cedex 2, France
e-mail: fradeliz@math.univ-mlv.fr

Abstract. We prove sharp inequalities showing that the volume of the hyperplane sections of a convex body in isotropic position through its centroid, as well as those of maximal volume in a fixed direction, does not depend much of the hyperplane. This generalizes a result of K. Ball to non-symmetric convex bodies. For the sections through its centroid, the extremal bodies are the same as in the symmetric case (cylinders and double-cones). For the sections of maximal volume parallel to a fixed hyperplane, the extremal bodies are cylinders and cones. We deduce that the cross-section body and the intersection body of a convex body with respect to its centroid are close to its Binet ellipsoid; we generalize results of J. Bourgain on maximal functions to the case of non-symmetric convex bodies and we prove that some equivalences between different forms of the hyperplane conjecture still hold in the non-symmetric case.

MSC 1991: 52A20, 52A40
Keywords: volume, section, centroid, isotropic, ellipsoid, inertia, hyperplane, intersection body, cross-section body, maximal function

Introduction

In [8], D. Hensley established the relation between the volume of the central hyperplane sections of a centrally symmetric convex body K and its covariance. More precisely, he proved that for some constants c_1 and c_2 independent of n, and for any unit vector u,

$$c_1 \frac{|K|}{|K \cap u^{\perp}|} \leq \left(\frac{1}{|K|} \int_K < x, u >^2 dx \right)^{\frac{1}{2}} \leq c_2 \frac{|K|}{|K \cap u^{\perp}|}.$$

In particular, he deduced that the the volume of the central hyperplane sections of a centrally symmetric convex body in isotropic position is almost constant. Then, K. Ball [1] sharpened and extended this result. He deduced some Khinchine type inequalities and established some equivalent forms of the hyperplane conjecture. In [11], V. Milman and A. Pajor gave a systematic treatment of this question and proved that the intersection body of K and some classical ellipsoids and zonoids related to K like the Binet ellipsoid of K and the centroid body of K are very close to each other. In [2] and [3], J. Bourgain used this estimate to prove that for $p > \frac{3}{2}$, the maximal function associated to K is bounded on $L_p(\mathbb{R}^n)$ by a constant independent of K and n.

In this paper, we generalize all these results to general convex bodies. In fact, we study a somewhat more general problem. Let ϕ be a convex function (sometimes assumed to be even) and u be a unit vector. We find the extremum values of

$$\int_K \phi(< x, u >)dx$$

over the set of convex bodies K whose centroids are at the origin, in terms of the volume of K and of some of its hyperplane sections orthogonal to u. Applying this with $\phi(t) = |t|^p$, $p \geq 1$, this allows to estimate the volume of the hyperplane sections of a general convex body K through its centroid and the one of maximal volume orthogonal to a fixed direction in terms of the p-th moments of inertia of K.

This paper contains four sections. In Section 1, we state the general results (Theorems 1 and 2) and investigate their geometric consequences. We obtain sharp inequalities in the comparison between the p-th moments of inertia of a convex body K and the volume of its hyperplane sections through its centroid as well as those of maximal volume. Applying this in the case $p = 2$, we obtain that if K is in isotropic position, the volume of these sections are almost constant. This also gives the sharp distance between the Binet ellipsoid of K and respectively the intersection body of K with respect to its centroid and the cross-section body of K. Then we generalize the results of Bourgain ([2] and [3]) on maximal functions to the case of non-symmetric convex bodies and we prove that some equivalences between different forms of the hyperplane conjecture hold in the non-symmetric case.

In Section 2, we express Theorems 1 and 2 in terms of $\frac{1}{n}$-concave functions and we obtain some Marshall-Olkin-Proschan-type inequalities [10]. We prove them in Sections 3 and 4 by stating some sharp inequalities.

1. Geometric results

All the results of this section will follow from the general results contained in Theorems 1 and 2.

1.1. General results

By $< \ , >$ we denote the canonical scalar product on $\mathbb{R}^n$. The Euclidean ball and sphere of $\mathbb{R}^n$ are denoted by B_2^n and S^{n-1}. If $u \in S^{n-1}$, $u^\perp = \{x \in \mathbb{R}^n; < x, u >= 0\}$ denotes the hyperplane orthogonal to u through the origin. If $A \subset \mathbb{R}^n$, $|A|$ denotes the volume of A

relative to its affine hull. The convex hull of A is denoted by $\mathrm{conv}(A)$. Recall that if $K \subset \mathbb{R}^n$ is a convex body, its centroid is $\frac{1}{|K|}\int_K x dx$.

Theorem 1. *Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an even convex function. Let $u \in S^{n-1}$ and set $\alpha = \frac{|K|}{2|K\cap u^\perp|}$. Then*

$$\frac{1}{2}\int_{-1}^{1} \phi(\alpha t)dt \leq \frac{1}{|K|}\int_K \phi(< x, u >)dx \leq \frac{1}{2}\int_{-n}^{n} \phi(\alpha t)\left(1 - \frac{|t|}{n}\right)^{n-1} dt. \tag{1}$$

Moreover, the right hand side inequality holds for any convex function ϕ.

Theorem 2. *Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $u \in S^{n-1}$. Set $\gamma = \frac{|K|}{2\max_{x\in\mathbb{R}^n}|(K+x)\cap u^\perp|}$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an even convex function. Then*

$$\frac{1}{2}\int_{-1}^{1} \phi(\gamma t)dt \leq \frac{1}{|K|}\int_K \phi(< x, u >)dx \leq \frac{1}{\left(1+\frac{1}{n}\right)^n}\int_{-1}^{n} \phi\left(\frac{2n\gamma t}{n+1}\right)\left(1 - \frac{t}{n}\right)^{n-1} dt. \tag{2}$$

Moreover, the left hand side inequality holds for any convex function ϕ.

Equality cases. The inequalities of Theorems 1 and 2 are sharp, but the equality cases can only be characterized if ϕ is strictly convex. Then there is equality

- on the left hand side of (1) or (2) if and only if K is cylindrical in the direction u, i.e. there exists $x \in \mathbb{R}^n$ such that $K = K \cap u^\perp + [-x, x]$.
- on the right hand side of (1) if and only if K is a double-cone in the direction u, i.e. there exist x_1 and x_2 in $\mathbb{R}^n$ such that $< x_1, u >= - < x_2, u >$ and $K = \mathrm{conv}\left(K \cap u^\perp, x_1, x_2\right)$.
- on the right hand side of (2) if and only if K is a cone in the direction u, i.e. there exists x_1 and x_2 in $\mathbb{R}^n$ such that $K = \mathrm{conv}\left(K \cap (u^\perp + x_1), x_2\right)$.

Remark. In the special case when $\phi(t) = |t|^p$ and for symmetric convex bodies K, Theorem 1 was proved in [1], with the same extremal cases.

1.2. Notations and preliminaries

If $K \subset \mathbb{R}^n$ is a convex body with the origin as an interior point, the *intersection body* of K with respect to the origin, IK is the star body whose radial function is $\rho_{IK}(u) = |K \cap u^\perp|$ for all $u \in S^{n-1}$. The *cross-section body* of K, CK is the star body whose radial function is $\rho_{CK}(u) = \max_{t\in\mathbb{R}}|K\cap(u^\perp+tu)|$ for all $u \in S^{n-1}$. The *support function* of K is $h_K(y) = \sup\{< z, y > ; z \in K\}$. The *polar* of K with respect to the origin is $K^* = \{y \in \mathbb{R}^n ; < z, y > \leq 1 , \forall z \in K\}$.

Let $K \subset \mathbb{R}^n$ be a convex body whose centroid is at the origin. The *inertia matrix* of K is the $n \times n$ matrix defined by

$$M_K = \frac{1}{|K|^{1+\frac{2}{n}}}\left(\int_K x_i x_j dx\right)_{1\leq i,j\leq n}.$$

The number $L_K = (\det M_K)^{\frac{1}{2n}}$ is an affine invariant for K called the *isotropic constant* of K. For all convex bodies $K \subset \mathbb{R}^n$, there exists an affine image $\widetilde{K}$ of K, such that the inertia matrix of $\widetilde{K}$ is a multiple of the identity matrix. In this case and if the centroid of $\widetilde{K}$ is at the origin, we say that $\widetilde{K}$ is *in isotropic position.* The body K is in isotropic position if and only if for all $u \in S^{n-1}$,

$$\int_K < x, u >^2 dx = |K|^{1+\frac{2}{n}} L_K^2 .$$

For $p \geq 1$, the *p-th centroid body* $\Gamma_p K$ of K is the convex body whose support function is

$$h_{\Gamma_p K}(u) = \left(\frac{1}{|K|} \int_K | < x, u > |^p dx \right)^{\frac{1}{p}}$$

for all $u \in S^{n-1}$. Observe that the second centroid body $\Gamma_2 K$ is the inertia ellipsoid of K and its polar is often called the *Binet ellipsoid* of K and denoted by BK (we refer to R. J. Gardner [6] for more details).

1.3. Moments of inertia and ϕ-centroid regions

The following results generalize Corollary 3.1 of [11] to the non centrally symmetric convex bodies. They are the direct consequences of Theorems 1 and 2 applied with the even convex function $\phi(t) = |t|^p$, $p \geq 1$.

Theorem 3. *Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $u \in S^{n-1}$ and $p \geq 1$. Then*

$$c_2(p,n) \left(\frac{1}{|K|} \int_K | < x, u > |^p dx \right)^{\frac{1}{p}} \leq \frac{|K|}{|K \cap u^\perp|} \leq c_1(p) \left(\frac{1}{|K|} \int_K | < x, u > |^p dx \right)^{\frac{1}{p}}$$

where $c_1(p) = 2(p+1)^{\frac{1}{p}}$ and $c_2(p,n) = \frac{2}{n} \left(\frac{(p+1)\dots(p+n)}{n!} \right)^{\frac{1}{p}}$ decreases toward $2\,\Gamma(p+1)^{-\frac{1}{p}}$ when $n \to \infty$. Moreover, there is equality in the right hand side if and only if K is cylindrical in the direction u and in the left hand side if and only if K is a double-cone in the direction u.

Theorem 4. *Let K be a convex body in $\mathbb{R}^n$ whose centroid is at the origin. Let $u \in S^{n-1}$ and $p \geq 1$. Then*

$$c_3(p,n) \leq \frac{|K|}{\max_{x \in \mathbb{R}^n} |(K+x) \cap u^\perp|} \left(\frac{1}{|K|} \int_K | < x, u > |^p dx \right)^{-\frac{1}{p}} \leq c_1(p)$$

where $c_1(p) = 2(p+1)^{\frac{1}{p}}$ and $c_3(p,n) = \left(\left(\frac{n}{n+1}\right)^{n+p} \int_{-1}^{n} |t|^p \left(1 - \frac{t}{n}\right)^{n-1} dt \right)^{-\frac{1}{p}}$ decreases to $\left(\int_0^{+\infty} |t-1|^p e^{-t} dt \right)^{-\frac{1}{p}}$ when $n \to \infty$. Moreover, there is equality in the right hand side if and only if K is cylindrical in the direction u and in the left hand side if and only if K is a cone in the direction u.

Remarks. 1) Theorem 3 and 4 may be seen as the following inclusions:

$$\frac{|K|}{c_1(p)}(\Gamma_p K)^* \subset IK \subset \frac{|K|}{c_2(p,n)}(\Gamma_p K)^* \quad \text{and} \quad \frac{|K|}{c_1(p)}(\Gamma_p K)^* \subset CK \subset \frac{|K|}{c_3(p,n)}(\Gamma_p K)^* \ ,$$

for all $p \geq 1$ and all convex bodies K in $\mathbb{R}^n$, whose centroid is at the origin. Hence, the distance between IK and $(\Gamma_p K)^*$ is less than $\Gamma(p+2)^{\frac{1}{p}}$ and the distance between CK and $(\Gamma_p K)^*$ is less than $2\left((p+1)\int_0^{+\infty}|t-1|^p e^{-t}dt\right)^{\frac{1}{p}}$ and these bounds are sharp.

2) In the case of a general non-negative convex function ϕ such that $\phi(0)=0$, instead of the function $t \mapsto |t|^p$, we define the *ϕ-centroid regions of* K: for $t \geq 0$

$$\Gamma_\phi K(t) = \{x \in \mathbb{R}^n; \int_K \phi(<x,y>)dy \leq t\} \ .$$

Since ϕ is convex, the function $x \mapsto \int_K \phi(<x,y>)dy$ is convex. Hence $\Gamma_\phi K(t)$ is a convex body, which is symmetric if K is symmetric or ϕ is even. For $t \geq 0$, let

$$I_\phi K(t) = \left\{x \in \mathbb{R}^n; \int_{-\frac{|K|}{2}}^{\frac{|K|}{2}} \phi\left(\frac{s}{\rho_{IK}(x)}\right)ds \leq t\right\} \ ,$$

$$J_\phi K(t) = \left\{x \in \mathbb{R}^n; \int_{-\frac{n|K|}{2}}^{\frac{n|K|}{2}} \phi\left(\frac{s}{\rho_{IK}(x)}\right)\left(1-\frac{2|s|}{|K|}\right)^{n-1} ds \leq t\right\} \ .$$

Since ϕ is convex and $\rho_{IK}(\lambda x) = \lambda^{-1}\rho_{IK}(x)$, the function $\lambda \mapsto \phi\left(s(\rho_{IK}(\lambda x))^{-1}\right)$ is non-decreasing on $\mathbb{R}^+$. Hence $I_\phi K(t)$ and $J_\phi K(t)$ are star bodies. Moreover, if K is symmetric, then from H. Busemann [4], IK is convex, i.e. $x \mapsto \rho_{IK}(x)^{-1}$ is convex, hence $I_\phi K(t)$ and $J_\phi K(t)$ are convex bodies.

Finally, Theorem 6 may be seen as the following inclusion: If $K \subset \mathbb{R}^n$ is a convex body whose centroid is at the origin and ϕ is an even convex function, then

$$I_\phi K(t) \subset \Gamma_\phi K(t) \subset J_\phi K(t) \ , \tag{3}$$

for all $t \geq 0$, and the right hand side inclusion holds for any convex function ϕ.

3) In [12], M. Meyer and E. Werner used this result to investigate some properties of the Santaló regions of K

$$S(K,t) = \left\{x \in K; \frac{|K||K^{*x}|}{v_n^2} \leq t\right\} \ ,$$

where $K^{*x} = (K-x)^*$ is the polar of K with respect to x. Actually, we have

$$|K^{*x}| = \frac{1}{n}\int_{S^{n-1}} \frac{d\sigma(u)}{(h_K(u)-<x,u>)^n} = \int_{K^*} \frac{dy}{(1-<x,y>)^{n+1}} \ .$$

Hence, with the convex function $\phi_n(t) = (1-t)_+^{-(n+1)}$, we get

$$S(K,t) = \left\{x \in K; \int_{K^*} \phi_n(<x,y>)dy \leq \frac{v_n^2 t}{|K|}\right\} \ .$$

Thus $S(K,t)$ may be viewed as some $\Gamma_\phi K^*(t)$ and therefore satifies the right hand side of the double inclusion (3).

1.4. Hyperplane sections of convex bodies in isotropic position

If we apply Theorems 3 and 4 with $p = 2$, we obtain that the intersection and cross-section bodies of any convex body whose centroid is at the origin are close to the Binet ellipsoid in the sense of the Banach-Mazur distance. This also means that in isotropic position, the volume of the hyperplane sections of a convex body through its centroid and of maximal volume are almost constant. These are the following corollaries.

Corollary 1. *Let $K \subset \mathbb{R}^n$ be a convex body whose centroid is at the origin. Then*

$$\frac{|K|}{2\sqrt{3}} BK \subset IK \subset \frac{n|K|}{\sqrt{2(n+1)(n+2)}} BK \subset \frac{|K|}{\sqrt{2}} BK \ .$$

This means that for any convex K in isotropic position and $u \in S^{n-1}$,

$$\frac{1}{2\sqrt{3}} \frac{1}{L_K} \leq \frac{|K \cap u^\perp|}{|K|^{\frac{n-1}{n}}} \leq \frac{n}{\sqrt{2(n+1)(n+2)}} \frac{1}{L_K} < \frac{1}{\sqrt{2}} \frac{1}{L_K} \ .$$

Moreover, there is equality in the left hand side if and only if K is cylindrical in the direction u and in the right hand side if and only if K is a double-cone.

Corollary 2. *Let $K \subset \mathbb{R}^n$ be a convex body whose centroid is at the origin. Then*

$$\frac{|K|}{2\sqrt{3}} BK \subset CK \subset \frac{|K| n^{\frac{3}{2}}}{(n+1)\sqrt{(n+2)}} BK \subset |K| BK \ .$$

This means that for any convex K in isotropic position and $u \in S^{n-1}$,

$$\frac{1}{2\sqrt{3}} \frac{1}{L_K} \leq \frac{\max_{x \in \mathbb{R}^n} |(K+x) \cap u^\perp|}{|K|^{\frac{n-1}{n}}} \leq \frac{n^{\frac{3}{2}}}{(n+1)\sqrt{n+2}} \frac{1}{L_K} < \frac{1}{L_K} \ .$$

Moreover, there is equality in the left hand side if and only if K is cylindrical in the direction u and in the right hand side if and only if K is a cone in the direction u.

Remarks. As a consequence of Corollaries 1 and 2, if K is in isotropic position then for every hyperplanes H_1 and H_2

$$\frac{1}{\sqrt{6}} \leq \frac{|K \cap H_1|}{|K \cap H_2|} \leq \sqrt{6} \quad \text{and} \quad \frac{1}{\sqrt{12}} \leq \frac{\max_{x \in \mathbb{R}^n} |(K+x) \cap H_1|}{\max_{x \in \mathbb{R}^n} |(K+x) \cap H_2|} \leq \sqrt{12} \ .$$

In the symmetric case, this was first observed by D. Hensley [8] and by K. Ball [1] for sharp constants. In the non-symmetric case, C. Schütt [13] gave a version of Corollary 1 with non-optimal constants and used it to get some estimate related to the floating body and the illumination body.

We now deduce results concerning extremal sections of some convex bodies in isotropic position. Statement 4) of the following corollary is due to S. Webb [14].

Corollary 3. 1) *Let $K \subset \mathbb{R}^n$ be a cylinder in the direction u in isotropic position. Then for all $v \in S^{n-1}$, $|K \cap v^\perp| \geq |K \cap u^\perp|$.*

2) *Let $K = \mathrm{conv}(K \cap (u^\perp + x_1), x_2)$ be a cone in isotropic position. Then for all $v \in S^{n-1}$ and $x \in \mathbb{R}^n$, $|K \cap (v^\perp + x)| \leq |K \cap (u^\perp + x_1)|$.*

3) *Let $K \subset \mathbb{R}^n$ be a double-cone in the direction u in isotropic position. Then for all $v \in S^{n-1}$, $|K \cap v^\perp| \leq |K \cap u^\perp|$.*

4) *Among the hyperplane sections through the centroid of the regular simplex, the sections passing through $n-1$ vertices have maximal volume.*

Proof. 1) Since K and $|K \cap u^\perp|$ satisfy the case of equality of the left hand side inequality of Corollary 1, $|K \cap v^\perp| \geq |K|^{\frac{n-1}{n}} \frac{1}{2\sqrt{3}} \frac{1}{L_K} = |K \cap u^\perp|$, for all $v \in S^{n-1}$.

2) From the equality case of the right hand side inequality of Corollary 2, $\max_{x \in \mathbb{R}^n} |K \cap (v^\perp + x)| \leq |K|^{\frac{n-1}{n}} \frac{n^{\frac{3}{2}}}{(n+1)\sqrt{n+2}} \frac{1}{L_K} = |K \cap (u^\perp + x_1)|$, for all $v \in S^{n-1}$.

3) Since K satisfies the equality case of the right hand side inequality of Corollary 1, $|K \cap v^\perp| \leq |K|^{\frac{n-1}{n}} \frac{n}{\sqrt{2(n+1)(n+2)}} \frac{1}{L_K} = |K \cap u^\perp|$, for all $v \in S^{n-1}$.

4) A regular simplex $S = \mathrm{conv}(x_i)_{1 \leq i \leq n+1}$ with the origin as centroid is in isotropic position. Moreover, S is a double-cone in the direction $u_{i,j} := \frac{x_i - x_j}{|x_i - x_j|}$, for all $1 \leq i < j \leq n+1$. Since $u_{i,j}^\perp = \mathrm{vect}(x_k)_{k \neq i, k \neq j}$, the result follows from 3). □

The following geometric result is the immediate consequence of the forthcoming Lemmas 4 and 6, used in Sections 2 and 3 to prove the Theorems 1 and 2.

Proposition 1. *Let $K \subset \mathbb{R}^n$ be a convex body whose centroid is at the origin, then for all $u \in S^{n-1}$,*

$$\frac{2h_K(u)}{n} |K \cap u^\perp| \leq |K| \leq 2h_K(u) \max_{x \in \mathbb{R}^n} |(K + x) \cap u^\perp| \ ,$$

with equality in the left hand side if and only if K is a double-cone in the direction u and in the right if and only if K is a cylinder. This may be seen as :

$$\frac{2}{n|K|} IK \subset K^* \cap (-K^*) \subset K^* \cup (-K^*) \subset \frac{2}{|K|} CK \ .$$

1.5. Maximal functions and non-symmetric convex bodies

In [2] and [3] J. Bourgain applied the symmetric version of Corollaries 1 and 2 to get a result on maximal functions, which answered to a question of E. Stein. Our results immediately give a generalization of this to the non-symmetric case:

Theorem 5. *Let $p > \frac{3}{2}$ and $K \subset \mathbb{R}^n$ be a convex body. Then the maximal function*

$$M_K f(x) := \sup_{t > 0} \frac{1}{|K|} \int_K |f(x + ty)| dy$$

has its $L_p(\mathbb{R}^n)$-norm bounded by $c_p ||f||_p$ for all $f \in L_p(\mathbb{R}^n)$, where c_p is independent of K and n.

Using the same proof as J. Bourgain in [3], it is enough to prove the following estimate for a general convex body.

Lemma 1. *Let $K \subset \mathbb{R}^n$ be a convex body in isotropic position, such that $|K| = 1$. Then there is some numerical constant c such that for all $u \in \mathbb{R}^n$,*

1) $\displaystyle \Big|\int_K e^{2i\pi <x,u>} dx\Big| \le \frac{c}{L_K|u|}$, 2) $\displaystyle \Big|\int_K \left(1 - e^{2i\pi<x,u>}\right) dx\Big| \le cL_K|u|$,

3) $\displaystyle \Big|\int_K < x, u > e^{2i\pi<x,u>} dx\Big| \le c$.

Proof. 1) Let $u \in \mathbb{R}^n$, $u \neq 0$ and set $f(t) = |\{x \in K; < x, \frac{u}{|u|} >= t\}|$. Then $\int_{-\infty}^{+\infty} f(t)dt = |K| = 1$. Integrating by parts and using Corollary 2, we get

$$\begin{aligned} \Big|\int_K e^{2i\pi<x,u>} dx\Big| &= \Big|\int_{-\infty}^{+\infty} f(t)e^{2i\pi|u|t} dt\Big| = \frac{1}{2\pi|u|}\Big|\int_{-\infty}^{+\infty} f'(t)e^{2i\pi|u|t} dt\Big| \\ &\le \frac{1}{2\pi|u|}\int_{-\infty}^{+\infty} |f'(t)|dt \le \frac{\max f}{\pi|u|} \le \frac{1}{\pi|u|L_K} . \end{aligned}$$

2) Using Cauchy-Schwarz's inequality, we have

$$\begin{aligned} \Big|\int_K \left(1 - e^{2i\pi<x,u>}\right) dx\Big| &\le \int_{-\infty}^{+\infty} f(t)\left|1 - e^{2i\pi|u|t}\right| dt \le \int_{-\infty}^{+\infty} f(t)2\pi|u||t|dt \\ &\le 2\pi|u|\left(\int_{-\infty}^{+\infty} t^2 f(t)dt\right)^{\frac{1}{2}} = 2\pi|u|L_K . \end{aligned}$$

3) We have

$$I(u) := \int_K < x, u > e^{2i\pi<x,u>} dx = \int_{-\infty}^{+\infty} |u|tf(t)e^{2i\pi|u|t} dt = \frac{-1}{2i\pi}\int_{-\infty}^{+\infty} (tf)'(t)e^{2i\pi|u|t} dt .$$

Hence $|I(u)| \le \frac{1}{2\pi}\int_{-\infty}^{+\infty} |(tf)'(t)|dt \le \frac{1}{2\pi}\left(1 + \int_{-\infty}^{+\infty} |tf'(t)|dt\right)$. Let t_0 such that $f(t_0) = \max f$. Since $f^{\frac{1}{n-1}}$ is concave, it follows that $\int_{-\infty}^{+\infty} |tf'(t)|dt \le 2t_0 \max f + 1$. Since $f(0) \ge \frac{\max f}{e}$, we have $[0, t_0] \subset \{t; f(t) \ge \frac{\max f}{e}\}$. Hence we get

$$t_0 \le |\{t \; ; \; f(t) \ge \frac{\max f}{e}\}| \le \frac{e}{\max f}\int_{-\infty}^{+\infty} f(t)dt = \frac{e}{\max f} .$$

Putting together the preceding inequalities, we get 3). □

1.6. The hyperplane conjecture in the non-symmetric case

The following problem is one of the major open problems of the local theory of Banach Spaces. For more details, we refer to A. A. Giannopoulos and M. Papadimitrakis [7]. In [11], V. Milman and A. Pajor established some equivalent reformulations. Here we prove that they still hold in the non-symmetric case. We begin with the following lemma.

Lemma 2. *Let $K \subset \mathbb{R}^n$ be a convex body whose centroid is at the origin, then*

$$\frac{|K|^{\frac{n-1}{n}}}{2\sqrt{3}\,L_K} \le \left(\frac{|IK|}{|B_2^n|}\right)^{\frac{1}{n}} = \left(\int_{S^{n-1}} |K\cap u^{\perp}|^n d\sigma_{n-1}(u)\right)^{\frac{1}{n}} \le \frac{n}{\sqrt{2(n+1)(n+2)}}\frac{|K|^{\frac{n-1}{n}}}{L_K}$$

and

$$\frac{1}{2\sqrt{3}}\frac{|K|^{\frac{n-1}{n}}}{L_K} \le \left(\frac{|CK|}{|B_2^n|}\right)^{\frac{1}{n}} \le \frac{n^{\frac{3}{2}}}{(n+1)\sqrt{n+2}}\frac{|K|^{\frac{n-1}{n}}}{L_K}\ .$$

Proof. Since $BK = |K|^{-\frac{1}{n}} M_K^{-\frac{1}{2}}(B_2^n)$, we have $\left(\frac{|BK|}{|B_2^n|}\right)^{\frac{1}{n}} = \frac{1}{L_K|K|^{\frac{1}{n}}}$. We now use Corollaries 1 and 2 and we express the volume of IK and CK in terms of their radial functions to get the result. □

Proposition 2. *The following assertions are equivalent:*

1) *There exists c such that for all $n \in \mathbb{N}$, for all convex bodies $K \subset \mathbb{R}^n$, $L_K \le c$.*
2) *There exists c such that for all $n \in \mathbb{N}$, for all convex bodies $K_1, K_2 \subset \mathbb{R}^n$ whose centroids are at the origin, which satisfy $|K_1\cap u^{\perp}| \le |K_2\cap u^{\perp}|$ for all $u \in S^{n-1}$, then $|K_1| \le c|K_2|$.*
3) *There exists c such that for all $n \in \mathbb{N}$, for all convex bodies $K_1, K_2 \subset \mathbb{R}^n$, which satisfy $\max_{x\in\mathbb{R}^n} |(K_1+x)\cap u^{\perp}| \le \max_{x\in\mathbb{R}^n} |(K_2+x)\cap u^{\perp}|$ for all $u \in S^{n-1}$, then $|K_1| \le c|K_2|$.*
4) *There exists c such that for all $n \in \mathbb{N}$, for all convex bodies $K \subset \mathbb{R}^n$ whose centroid is at the origin $\max_{u\in S^{n-1}} |K\cap u^{\perp}| \ge c|K|^{\frac{n-1}{n}}$.*
5) *There exists c such that for all $n \in \mathbb{N}$, for all convex bodies $K \subset \mathbb{R}^n$, we have*

$$\max_{u\in S^{n-1}, x\in\mathbb{R}^n} |K\cap(u^{\perp}+x)| \ge c|K|^{\frac{n-1}{n}}\ .$$

Proof. We only prove 1) ⇒ 2) ⇒ 4) ⇒ 1), since 1) ⇒ 3) ⇒ 5) ⇒ 1) goes exactly the same way.

1) ⇒ 2): From Lemma 2, we have $\frac{|K_1|^{\frac{n-1}{n}}}{L_{K_1}} \le \sqrt{6}\frac{|K_2|^{\frac{n-1}{n}}}{L_{K_2}}$. Since L_{K_2} is bounded from below and L_{K_1} from above, the result follows.

2) ⇒ 4): We apply 2) with $K_1 = K$ and $K_2 = rB_2^n$, where r is chosen such that $r^{n-1}|B_2^{n-1}| = \max_{u\in S^{n-1}} |K\cap u^{\perp}|$ so that K_1 and K_2 satisfy $|K_1\cap u^{\perp}| \le |K_2\cap u^{\perp}|$ for all $u \in S^{n-1}$. Hence by an elementary calculation we get that for some $c > 0$,

$$|K| \le |rB_2^n| = \max_{u\in S^{n-1}} |K\cap u^{\perp}|^{\frac{n}{n-1}} v_n(v_{n-1})^{-\frac{n}{n-1}} \le c \max_{u\in S^{n-1}} |K\cap u^{\perp}|^{\frac{n}{n-1}}.$$

4) ⇒ 1): It comes immediately from Corollary 1. □

In the symmetric case, K. Ball [1] proved that the hyperplane conjecture, i.e. the truth of the assertions of Proposition 2, is equivalent to the following assertion concerning k-codimensional subspaces:

6) *There exists c such that for all $n \in \mathbb{N}$, for all $1 \leq k < n$, for all vector subspaces E_1, E_2 of $\mathbb{R}^n$ of codimension k, for all symmetric convex bodies $K \subset \mathbb{R}^n$*

$$\left(\frac{|K \cap E_1|}{|K \cap E_2|}\right)^{\frac{1}{k}} \leq c \ .$$

In the non-symmetric case, this equivalence remains an open problem. Nevertheless, we prove that the both expected generalizations to the non-symmetric case are equivalent to each other.

Proposition 3. *Let $K \subset \mathbb{R}^n$ be a convex body, whose centroid is at the origin. Let E be a k-codimensional subspace of $\mathbb{R}^n$. Then*

$$\left(\frac{\max_{x \in K} |K \cap (E + x)|}{|K \cap E|}\right)^{\frac{1}{k}} \leq e \ ,$$

and the constant e is asymptotically optimal.

Proof. From [5] and an obvious calculation, the sharp inequality follows

$$\left(\frac{\max_{x \in K} |K \cap (E + x)|}{|K \cap E|}\right)^{\frac{1}{k}} \leq \left(\frac{n+1}{n-k+1}\right)^{\frac{n-k}{k}} \leq e. \qquad \square$$

Corollary 4. *Let $K \subset \mathbb{R}^n$ be a convex body, whose centroid is at the origin. Let E_1 and E_2 be two k-codimensional subspaces of $\mathbb{R}^n$. Then*

$$\frac{1}{e}\left(\frac{|K \cap E_1|}{|K \cap E_2|}\right)^{\frac{1}{k}} \leq \left(\frac{\max_{x \in K} |K \cap (E_1 + x)|}{\max_{x \in K} |K \cap (E_2 + x)|}\right)^{\frac{1}{k}} \leq e\left(\frac{|K \cap E_1|}{|K \cap E_2|}\right)^{\frac{1}{k}} \ .$$

2. Functional results and method of proofs

We start with some notations. If $A \subset \mathbb{R}^n$, we denote by $\chi_A : \mathbb{R} \to \mathbb{R}$, the characteristic function of A ($\chi_A(t) = 1$ if $t \in A$ and $\chi_A(t) = 0$ if $t \notin A$). If $f : \mathbb{R} \to \mathbb{R}$, let $f_+ = \max(f, 0)$ and $\mathrm{supp}(f)$ be its support ($\mathrm{supp}(f) = \overline{\{f \neq 0\}}$). A function $f : \mathbb{R} \to \mathbb{R}^+$ is said *log-concave* (resp. $\frac{1}{n}$-*concave*) if $\mathrm{supp}(f)$ is convex and $\log f$ (resp. $f^{\frac{1}{n}}$) is concave on $\mathrm{supp}(f)$. If f has a limit at $+\infty$, we denote $f(+\infty) = \lim_{t \to +\infty} f(t)$.

Let K be a convex body. We set $f(t) := |K \cap \{x \in \mathbb{R}^n ; < x, u >= t\}|$. We can express the quantities appearing in Theorems 1 and 2 in terms of f.

$$|K \cap u^\perp| = f(0) \ , \quad \max_{x \in \mathbb{R}^n} |(K + x) \cap u^\perp| = \max f \quad \text{and} \quad |K| = \int_{-\infty}^{+\infty} f(t)dt \ ,$$

$$\alpha = \frac{\int_{-\infty}^{+\infty} f(t)dt}{2f(0)} \ , \ \gamma = \frac{\int_{-\infty}^{+\infty} f(t)dt}{2 \max f} \ \text{and} \ \int_K \phi(< x, u >)dx = \int_{-\infty}^{+\infty} \phi(t) f(t)dt \ .$$

From the theorem of Brunn-Minkowski, $f^{\frac{1}{n-1}}$ is concave on its support and since the centroid of K is at 0, we have

$$\int_{-\infty}^{+\infty} tf(t)dt = \int_{-\infty}^{+\infty} t|K \cap \{x \in \mathbb{R}^n; < x, u >= t\}|dt = \int_K < x, u > dx = 0 \ .$$

We are now able to give the functional version of Theorems 1 and 2.

Theorem 6. *Let $n > 1$ and $f : \mathbb{R} \to \mathbb{R}^+$, $\frac{1}{n-1}$-concave, such that $\int_{-\infty}^{+\infty} tf(t)dt = 0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an even convex function. Set $\alpha = \frac{\int_{-\infty}^{+\infty} f(t)dt}{2f(0)}$. Then*

$$f(0)\int_{-\alpha}^{\alpha} \phi(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)f(t)dt \leq f(0)\int_{-n\alpha}^{n\alpha} \phi(t)\left(1 - \frac{|t|}{n\alpha}\right)^{n-1} dt \ . \tag{4}$$

Moreover, the right hand side inequality holds for any convex function ϕ.

Theorem 7. *Let $n > 1$ and $f : \mathbb{R} \to \mathbb{R}^+$, $\frac{1}{n-1}$-concave, such that $\int_{-\infty}^{+\infty} tf(t)dt = 0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an even convex function. Set $\gamma = \frac{\int_{-\infty}^{+\infty} f(t)dt}{2\max f}$ and $\beta = \frac{2n\gamma}{n+1}$. Then*

$$\max f \int_{-\gamma}^{\gamma} \phi(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)f(t)dt \leq \frac{\max f}{\left(1 + \frac{1}{n}\right)^{n-1}} \int_{-\beta}^{n\beta} \phi(t)\left(1 - \frac{t}{n\beta}\right)^{n-1} dt \ . \tag{5}$$

Moreover, the left hand side inequality holds for any convex function ϕ.

Remark. Theorem 6 may be seen as a generalization of Marshall-Olkin-Proschan's inequalities [10].

If f is log-concave instead of being $\frac{1}{n-1}$-concave, the following versions of Theorems 6 and 7 hold and can be proved either by an approximation of f by $\frac{1}{n}$-concave functions or by an adaptation of their proofs.

Theorem 8. *Let $f : \mathbb{R} \to \mathbb{R}^+$, integrable, log-concave such that $\int_{-\infty}^{+\infty} tf(t)dt = 0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be an even convex function. Set $\alpha = \frac{\int_{-\infty}^{+\infty} f(t)dt}{2f(0)}$ and $\gamma = \frac{\int_{-\infty}^{+\infty} f(t)dt}{2\max f}$. Then*

$$f(0)\int_{-\alpha}^{\alpha} \phi(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)f(t)dt \leq f(0)\int_{-\infty}^{+\infty} \phi(t)e^{-\frac{|t|}{\alpha}}dt \, , \tag{6}$$

$$\max f \int_{-\gamma}^{\gamma} \phi(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)f(t)dt \leq \frac{\max f}{e} \int_{-2\gamma}^{+\infty} \phi(t)e^{-\frac{t}{2\gamma}}dt \, . \tag{7}$$

Moreover, ϕ does not need to be even for the right hand side of (6) and the left hand side of (7).

Remark. The upper bound of the double inequality (6) in the case $\phi(t) = t^2$ was proved by S. Webb [14].

Methods of proof

We now explain the method that we use to prove (4) and (5): Let $f:\mathbb{R}\to\mathbb{R}^+$, integrable, $\frac{1}{n-1}$-concave, such that $\int_{-\infty}^{+\infty} tf(t)dt = 0$. Let $\phi:\mathbb{R}\to\mathbb{R}$ be convex. Set $\alpha := \frac{\int_{-\infty}^{+\infty} f(t)dt}{2f(0)}$, $\gamma := \frac{\int_{-\infty}^{+\infty} f(t)dt}{2\max f}$ and $\beta := \frac{2n\gamma}{n+1}$. Notice that since f is a $\frac{1}{n-1}$-concave integrable function, it is continuous on its support, which is a bounded interval, except maybe on its extremities. Since we are only interested in the values of f in the interior of its support and in integration's properties of f, in the following we shall always suppose that f is continuous on its support.

To prove (4) and (5), we define g_i and $h_i:\mathbb{R}\to\mathbb{R}$ for $i=1,2$ by

$$g_1(t) := f(0)\chi_{[-\alpha,\alpha]}(t) \quad \text{and } h_1(t) := f(0)\left(1-\frac{|t|}{n\alpha}\right)_+^{n-1},$$

$$g_2(t) := \max f\chi_{[-\gamma,\gamma]}(t) \quad \text{and } h_2(t) := \frac{\max f}{\left(1+\frac{1}{n}\right)^{n-1}}\left(1-\frac{t}{n\beta}\right)^{n-1}\chi_{[-\beta,n\beta]}\ .$$

Because of the choice of α, β and γ we have for $i=1,2$

$$\int_{-\infty}^{+\infty} g_i(t)dt = \int_{-\infty}^{+\infty} f(t)dt = \int_{-\infty}^{+\infty} h_i(t)dt \text{ and } \int_{-\infty}^{+\infty} tg_i(t)dt = \int_{-\infty}^{+\infty} th_i(t)dt = 0\,. \tag{8}$$

Moreover for $i=1,2$, (4) and (5) are equivalent to

$$\int_{-\infty}^{+\infty}\phi(t)g_i(t)dt \le \int_{-\infty}^{+\infty}\phi(t)f(t)dt \le \int_{-\infty}^{+\infty}\phi(t)h_i(t)dt\ . \tag{9}$$

To prove (9), we use the following lemma.

Lemma 3. *Let $v:\mathbb{R}\to\mathbb{R}$ be an integrable compactly supported function such that $\int_{-\infty}^{+\infty} v(t)dt = 0$ and $\int_{-\infty}^{+\infty} tv(t)dt = 0$. Set $V(t) := \int_{-\infty}^{t} v(s)ds$. Let ϕ be a convex function and μ be the positive Borel measure on $\mathbb{R}$ such that $\phi'' = \mu$. Then the function $W(t) := \int_{-\infty}^{t} V(s)ds$ is compactly supported and*

$$\int_{-\infty}^{+\infty}\phi(t)v(t)dt = \int_{-\infty}^{+\infty} W(t)d\mu(t)\ . \tag{10}$$

Proof. In the following, we call W "the second primitive" of v. Since v is compactly supported and $\int_{-\infty}^{+\infty} v(t)dt = 0$, V is well-defined and compactly supported. Moreover, integrating by parts, we have

$$\int_{-\infty}^{+\infty} V(s)ds = \Big[sV(s)\Big]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} sv(s)ds = -\int_{-\infty}^{+\infty} sv(s)ds = 0\ .$$

Thus, W is well-defined and compactly supported. Integrating by parts,

$$\int_{-\infty}^{+\infty}\phi(t)v(t)dt = -\int_{-\infty}^{+\infty}\phi'(t)V(t)dt = \int_{-\infty}^{+\infty} W(t)d\mu(t). \qquad \square$$

Now since $f^{\frac{1}{n-1}}$ is concave and f is integrable, f is compactly supported. Hence, using (8), we apply Lemma 3 to $v = f - g_i$ and $v = h_i - f$. From (5) it is enough to show that the "second primitive" of $f - g_i$ and $h_i - f$ are non-negative. We prove this in Section 2, for $i=1$ and in Section 3, for $i=2$.

Equality cases

If ϕ is an even strictly convex function and μ is the positive Borel measure on $\mathbb{R}$ such that $\phi'' = \mu$, then $\mathrm{supp}(\mu) = \mathbb{R}$ and the equality cases in any of the side of the double inequalities of Theorems 1, 2, 6 and 7 can be characterized.

In the functional form: if, for example there is equality in the left hand side of the double inequality (4) or (5). Then for $i = 1$ or 2, f satisfies

$$\int_{-\infty}^{+\infty} \phi(t)f(t)dt = \int_{-\infty}^{+\infty} \phi(t)g_i(t)dt \ ,$$

hence from (10), the second primitive W of $f - g_i$ satisfies $\int_{-\infty}^{+\infty} W(t)d\mu(t) = 0$. Since μ is positive, if we can prove that W is non-negative, it implies that $W = 0$ on $\mathrm{supp}(\mu) = \mathbb{R}$, thus $f = g_i$ almost everywhere. Since f and g_i are continuous on their support, it follows that $f = g_i$. With the same arguments, there is equality in the right hand side inequality (9) for $i = 1$ or 2 if and only if $f = h_i$.

In the geometrical form: If there is equality in one inequality of Theorems 1 or 2, then the section function of K in direction u, $f(t) := |K \cap \{x \in \mathbb{R}^n; < x, u >= t\}$, satisfies the equality case in the corresponding functional form. Using the theorem of Brunn-Minkowski, this proves the equality cases stated in the first section.

3. Sections through the centroid

We prove separately the upper and lower bound. Throughout this section, we denote g_1 by g and h_1 by h.

3.1. Upper bound

Lemma 4. *Let $f : \mathbb{R} \to \mathbb{R}^+$, $\frac{1}{n-1}$-concave, integrable such that $\int_{-\infty}^{+\infty} tf(t)dt = 0$. Set $\alpha := \frac{\int_{-\infty}^{+\infty} f(t)dt}{2f(0)}$. Then* $\mathrm{supp} f \subset [-n\alpha, n\alpha]$ *with equality if and only if and $f(t) = f(0)\left(1 - \frac{|t|}{n\alpha}\right)_+^{n-1}$.*

Proof. Let $h(t) = f(0)\left(1 - \frac{|t|}{n\alpha}\right)_+^{n-1}$ and $[-a, b] = \mathrm{supp}(f)$. We have to prove that $\mathrm{supp}(f) \subset \mathrm{supp}(h)$, with equality if and only if $f = h$. Set $v := f - h$ and $V(t) := \int_{-\infty}^{t} v(s)ds$. Since $\int_{-\infty}^{+\infty} v(t)dt = \int_{-\infty}^{+\infty} tv(t)dt = 0$, it follows that V is compactly supported and $\int_{-\infty}^{+\infty} V(t)dt = 0$. Hence if $v \neq 0$, v has to change sign at least twice. Since $f(0) = h(0)$, $f^{\frac{1}{n-1}}$ is concave on $[-a, b]$ and $h^{\frac{1}{n-1}}$ is affine on $[-n\alpha, 0]$ and $[0, n\alpha]$, it follows that $\mathrm{supp}(f) \subset \mathrm{supp}(h)$. Hence $\max(a, b) \leq n\alpha$. If $\max(a, b) = n\alpha$, then v cannot change sign twice, hence $f = h$. □

Proof of the upper bound in (4). From Lemma 3, it is enough to prove that the second primitive of $v := f - h$ is non-positive. Set $V(t) := \int_{-\infty}^{t} v(s)ds$ and $W(t) := \int_{-\infty}^{t} V(s)ds$. Let $[-a, b] = \mathrm{supp}(f)$; by Lemma 4 we have $\max(a, b) \leq n\alpha$ with equality if and only if $f = h$. Hence if $f \neq h$, we have $a < n\alpha$ and $b < n\alpha$. Since $f^{\frac{1}{n-1}}$ is concave, $h^{\frac{1}{n-1}}$ is affine on $]-n\alpha, 0]$ and on $[0, n\alpha[$ and $f(0) = h(0)$, there exist $t_0 \leq 0 \leq t_1$ such that $v \leq 0$ on $]-\infty, t_0]$, $v \geq 0$ on $[t_0, t_1]$, $v \leq 0$ on $[t_1, +\infty[$. Since $W'' = v$ and W is compactly supported it follows that $W \leq 0$. □

3.2. Lower bound

For the lower bound in (4), we assume that ϕ is even, but we may weaken the hypotheses on f. Since $\frac{1}{n}$-concave functions are log-concave, to prove it, we may replace the hypothesis of $\frac{1}{n}$-concavity by the weaker hypothesis that $\mathrm{supp}(f)$ is an interval and that the function $\varphi : t \mapsto \frac{\ln f(t)}{t}$ is non-increasing on $\mathrm{supp} f$.

The proof goes into two steps. We first show that there exists a log-affine function g (i.e. $\log g$ is affine on $\mathrm{supp}(g)$) such that $g(0) = f(0)$, $\int_{-\infty}^{+\infty} g(t)dt = \int_{-\infty}^{+\infty} f(t)dt$, $\int_{-\infty}^{+\infty} tg(t)dt = 0$ and $\int_{-\infty}^{+\infty} \phi(t)f(t)dt \geq \int_{-\infty}^{+\infty} \phi(t)g(t)dt$. Then, we prove that among log-affine functions g such that $g(0)$ and $\int_{-\infty}^{+\infty} g(t)dt$ are fixed and $\int_{-\infty}^{+\infty} tg(t)dt = 0$, $\int_{-\infty}^{+\infty} \phi(t)g(t)dt$ is minimal when g is constant on its support.

3.2.1. Reduction to the log-affine case

Proposition 4. *Let* $f : \mathbb{R} \to \mathbb{R}$, $f \geq 0$, *integrable such that* $\mathrm{supp}(f)$ *is an interval,* $f(0) = 1$, $\int_{-\infty}^{+\infty} tf(t)dt = 0$ *and suppose that the function* $\varphi : t \mapsto \frac{\ln f(t)}{t}$ *is non-increasing on* $\mathrm{supp}(f)$. *Then there exist* $a \in \mathbb{R}$, $b > 0$ *and* $c > 0$ *such that the function* $g(t) := e^{at}\chi_{[-c,b]}(t)$ *satisfies*

$$\int_0^{+\infty} g(t)dt = \int_0^{+\infty} f(t)dt, \ \int_{-\infty}^0 g(t)dt = \int_{-\infty}^0 f(t)dt, \ \int_{-\infty}^{+\infty} tg(t)dt = 0 \ . \tag{11}$$

Moreover, such a function g *satisfies* $\mathrm{supp}(g) \supset \mathrm{supp}(f)$ *and* $\forall \phi : \mathbb{R} \to \mathbb{R}$ *convex,*

$$\int_{-\infty}^{+\infty} \phi(t)g(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)f(t)dt \ . \tag{12}$$

Proof. Define $a_1 = \lim_{t\to 0^+} \varphi(t)$, $a_2 = -\left(\int_0^{+\infty} f(t)dt\right)^{-1}$ and $a_3 = \left(\int_{-\infty}^0 f(t)dt\right)^{-1}$. We have $a_2 \leq 0 \leq a_3$.

• Let us prove first that $a_2 \leq a_1 \leq a_3$: By hypothesis $\varphi(t) = \frac{\ln f(t)}{t}$ decreases on $\mathrm{supp}(f)$; hence $f(t) \leq e^{a_1 t}$ for all t. If $a_1 < 0$, then obviously $a_1 \leq a_3$ and

$$\frac{-1}{a_2} = \int_0^{+\infty} f(t)dt \leq \int_0^{+\infty} e^{a_1 t}dt = \frac{-1}{a_1} \ ,$$

hence $a_2 \leq a_1 \leq a_3$. If $a_1 > 0$, the same proof gives the result.

• For all $a \in [a_2, a_3]$, we define $b(a) := \frac{1}{a}\ln\left(1 - \frac{a}{a_2}\right)$ if $a \neq 0$ and $b(0) = -\frac{1}{a_2}$; $c(a) := \frac{1}{a}\ln\left(1 - \frac{a}{a_3}\right)$ if $a \neq 0$ and $c(0) = -\frac{1}{a_3}$. The functions b and c are continuous and decreasing on $[a_2, a_3]$. Define $g_a(t) := e^{at}\chi_{[c(a),b(a)]}(t)$. Then

$$\int_0^{+\infty} g_a(t)dt = \int_0^{+\infty} f(t)dt \quad \text{and} \quad \int_{-\infty}^0 g_a(t)dt = \int_{-\infty}^0 f(t)dt \ . \tag{13}$$

So that if $\Psi(a) = \int_{-\infty}^{+\infty} t\left(f(t) - g_a(t)\right)dt$, Ψ is continuous on $[a_2, a_3]$. Let $F(x) = \int_{-\infty}^x f(t)dt$ and $G_a(x) = \int_{-\infty}^x g_a(t)dt$; from (13), $F - G$ is compactly supported. Hence

$$\Psi(a) = \int_{-\infty}^{+\infty} t\Big(f(t) - g_a(t)\Big)dt = -\int_{-\infty}^{+\infty} \Big(F(x) - G_a(x)\Big)dx \ .$$

• Let us prove that $\Psi(a_2) \leq 0$: Since $b(a_2) = +\infty$, for all $t > 0$, we have $(f - g_{a_2})(t) = e^{t\varphi(t)} - e^{a_2 t}$, hence $f - g_{a_2}$ has the same sign as $\varphi(t) - a_2$, which is non-increasing. Since $\int_0^{+\infty}(f - g_{a_2})(t)dt = 0$, there exists $t_0 > 0$ such that the sign of the function $f - g_{a_2}$ changes at t_0. Therefore

– on $[0, t_0]$, $\varphi \geq a_2$, hence $f \geq g_{a_2}$, thus $F - G_{a_2}$ is non-decreasing,

– on $[t_0, +\infty[$, $\varphi \leq a_2$, hence $f \leq g_{a_2}$, thus $F - G_{a_2}$ is non-increasing.

For all $t \leq 0$, since $a_1 \geq a_2$, we have $f(t) \leq e^{a_1 t} \leq e^{a_2 t}$. It follows that

– on $[c(a_2), 0]$, $f \leq g_{a_2}$, thus $F - G_{a_2}$ is non-increasing,

– on $]-\infty, c(a_2)]$, $f - g_{a_2} = f \geq 0$, thus $F - G_{a_2}$ is non-decreasing.

Finally, since $(F - G_{a_2})(-\infty) = (F - G_{a_2})(0) = (F - G_{a_2})(+\infty) = 0$, we get $F - G_{a_2} \geq 0$, hence $\Psi(a_2) \leq 0$.

From the same proof, it follows that $\Psi(a_1) \geq 0$. Now, since Ψ is continuous, $\Psi(a_1) \geq 0$ and $\Psi(a_2) \leq 0$, there exists $a_0 \in [a_2, a_1]$ such that $\Psi(a_0) = 0$. Since g_{a_0} satisfies also (13), it satisfies (11). In the following we denote $g := g_{a_0}$.

• To prove (12), we apply Lemma 3 to $v = f - g$. It is enough to see that $W \geq 0$. Since $\int_0^{+\infty} v(t)dt = \int_{-\infty}^0 v(t)dt = 0$, the sign of v changes on $]0, +\infty[$ and on $]-\infty, 0[$. We consider two cases:

1) If v keeps the same sign on $[-c, b] = \mathrm{supp}(g)$.

Since outside $[-c, b]$, $v = f \geq 0$, we have $v \leq 0$ on $[-c, b]$. Since W is compactly supported and $W'' = v$, we get $W \geq 0$ and $\mathrm{supp}(g) \subset \mathrm{supp}(f)$.

2) If the sign of v changes at some $t_0 \in\ ]-c, b[$. By a symmetry argument, we may assume that $t_0 \geq 0$. On $]0, b[$, $v(t) = e^{t\varphi(t)} - e^{at}$ has the same sign as $\varphi - a$, which is non-increasing. We get that $v \geq 0$ on $[0, t_0]$ and $v \leq 0$, on $[t_0, b]$. On $[-c, 0]$, $v \leq 0$ as $a - \varphi$. Outside $[-c, b]$, $v = f \geq 0$. Since $W'' = v$ and W is compactly supported, we get $W \geq 0$ and $\mathrm{supp}(g) \subset \mathrm{supp}(f)$. □

3.2.2. Reduction to the constant case

Lemma 5. *Let $b, c > 0$ be such that $\int_{-c}^b te^t dt = 0$. Then $\frac{4bc}{b+c} \geq \int_{-c}^b e^t dt$.*

Proof. We have $be^b + ce^{-c} = e^b - e^{-c}$, so that if we set $x = \frac{b+c}{2}$, we get $c = \frac{xe^x}{\sinh x} - 1$ and $b = 1 - \frac{xe^{-x}}{\sinh x}$. Since $\int_{-c}^b e^t dt = 2\sinh x\ e^{1-\frac{x\ \cosh x}{\sinh x}}$, Lemma 5 is equivalent to $(\frac{xe^x}{\sinh x} - 1)(1 - \frac{xe^{-x}}{\sinh x}) \geq x\ \sinh x\ e^{1-\frac{x\ \cosh x}{\sinh x}}$, for all $x \geq 0$. It is enough to prove that for all $x \geq 0$,

$$F(x) := e^{\frac{x\ \cosh x}{\sinh x}} \frac{(xe^x - \sinh x)(\sinh x - xe^{-x})}{x\sinh^3 x} \geq e\ .$$

This is the case since $\lim_{x\to 0} F(x) = e$ and for all $x > 0$,

$$F'(x) = \frac{\left(\sinh^2 x - x^2\right)^2 F(x)}{x\sinh^2 x(xe^x - \sinh x)(\sinh x - xe^{-x})} \geq 0. \qquad \square$$

Proposition 5. *Let $a, b, c \in \mathbb{R}$ satisfying $\int_{-c}^b te^{at}dt = 0$. Set $d := \frac{1}{2}\int_{-c}^b e^{at}dt$. Then for every even convex function $\phi : \mathbb{R} \to \mathbb{R}$, $\int_{-c}^b \phi(t)e^{at}dt \geq \int_{-d}^d \phi(t)dt$.*

Proof. By a change of variable, we reduce to the case when $a = 1$. Hence, b and c satisfy the hypotheses of Lemma 5. And if we set $x := \frac{b+c}{2}$, then b, c, d are C^∞, positive functions of $x > 0$. Differentiating the equality $\int_{-c(x)}^{b(x)} te^t dt = 0$, valid for all $x \geq 0$, we obtain $c'(x)e^{-c(x)}c(x) = b'(x)e^{b(x)}b(x)$. Since $d(x) = \frac{1}{2}\int_{-c(x)}^{b(x)} e^t dt$, we have $d'(x) = \frac{1}{2}\Big(b'(x)e^{b(x)} + c'(x)e^{-c(x)}\Big) = \frac{1}{2}b'(x)e^{b(x)}\left(\frac{b(x)+c(x)}{c(x)}\right)$. Since the function $I(x) := \int_{-c(x)}^{b(x)} \phi(t)e^t dt - \int_{-d(x)}^{d(x)} \phi(t)dt$ satisfies $I(0) = 0$, it is enough to show that $I' = b'e^b\phi(b) + c'e^{-c}\phi(-c) - d'\left(\phi(d) + \phi(-d)\right) \geq 0$. Since ϕ is even, we have

$$I' = b'e^b\left(\frac{b+c}{c}\right)\left(\frac{c}{b+c}\phi(b) + \frac{b}{b+c}\phi(c) - \phi(d)\right).$$

Since ϕ is convex, even, and $b' \geq 0$, it follows from Lemma 5 that

$$I' \geq b'e^b\left(\frac{b+c}{c}\right)\left(\phi\Big(\frac{2bc}{b+c}\Big) - \phi(d)\right) \geq 0. \qquad \square$$

Proof of the lower bound in (4). It follows from Proposition 4 and 5. $\square$

4. Sections of maximal volume

We now prove the double inequality (9) for $i = 2$. As it was already noticed, it gives Theorem 7. We prove separately the upper and the lower bound.

4.1. Lower bound

Throughout this section, we set $g(t) = \max f\chi_{[-\gamma,\gamma]}(t)$.

Lemma 6. *Let $n \geq 1$ and $f : \mathbb{R} \to \mathbb{R}$, $f \geq 0$, integrable and bounded, such that $\int_{-\infty}^{\infty} tf(t)dt = 0$. Set $\gamma := \frac{\int_{-\infty}^{+\infty} f(t)dt}{2\max f}$. Then $[-\gamma,\gamma] \subset \mathrm{conv}(\mathrm{supp}(f))$ with equality if and only if $f = \max f\chi_{[-\gamma,\gamma]}$.*

Proof. Set $v := f - g$ and $V(t) := \int_{-\infty}^{t} v(s)ds$. Since $\int_{-\infty}^{+\infty} v(t)dt = \int_{-\infty}^{+\infty} tv(t)dt = 0$, it follows that V is compactly supported and $\int_{-\infty}^{+\infty} V(t)dt = 0$. Hence if $v \neq 0$, v has to change sign at least twice. Since $f \leq g = \max f$ on $[-\gamma,\gamma]$, it follows that $\mathrm{supp}(g) \subset \mathrm{supp}(f)$. Hence $\min(a,b) \geq \gamma$. If $\min(a,b) = \gamma$, then v cannot change sign twice, hence $f = g$. $\square$

Proof of the lower bound in (5). It is enough to see that the second primitive of $v := f - g$ is non-positive. Set $V(t) = \int_{-\infty}^{t} v(s)ds$ and $W(t) = \int_{-\infty}^{t} V(s)ds$. Let $[-a,b] = \mathrm{supp}(f)$. If $f \neq g$, then by Lemma 6 we have $a > \gamma$ and $b > \gamma$. It follows that $v = f \geq 0$ on $]-\infty,\gamma]$ and $[\gamma,+\infty[$; and $v = f - \max f \leq 0$ on $[-\gamma,\gamma]$. Since $W'' = v$ and W is compactly supported, we get that $W \leq 0$. The result follows now from Lemma 3. $\square$

Remark. It was not necessary to assume that ϕ is even. But in that case, since ϕ is even and convex, for all $t \in \mathbb{R}$, we have $\phi\left(\frac{t}{f(0)}\right) \geq \phi\left(\frac{t}{\max f}\right)$. Hence, we could have used Theorem 6, with $I = \frac{1}{2}\int_{-\infty}^{+\infty} f(t)dt$, we get

$$\int_{-\infty}^{+\infty} \phi(t)f(t)dt \geq \int_{-I}^{I} \phi\left(\frac{t}{f(0)}\right) dt \geq \int_{-I}^{I} \phi\left(\frac{t}{\max f}\right) dt = \max f \int_{\gamma}^{\gamma} \phi(t)dt. \qquad \square$$

4.2. Upper bound

We now prove the upper bound in (5). To this end, we proceed into two steps.
We first show that there exists a simple "piecewise $\frac{1}{n-1}$-affine" function g (i.e. such that $g^{\frac{1}{n-1}}$ is piecewise affine on $\mathrm{supp}(g)$) such that $\max g = \max f$, $\int_{-\infty}^{+\infty} g(t)dt = \int_{-\infty}^{+\infty} f(t)dt$, $\int_{-\infty}^{+\infty} tg(t)dt = 0$ and $\int_{-\infty}^{+\infty} \phi(t)f(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)g(t)dt$. Then, we show that among these "piecewise $\frac{1}{n-1}$-affine" functions g, $\int_{-\infty}^{+\infty} \phi(t)g(t)dt$ is maximal when $g = h := h_2$, where h_2 is defined in 2.2.

4.2.1. Reduction to "piecewise affine" functions

Lemma 7. *Let $n > 1$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$, integrable, non-increasing, $\frac{1}{n-1}$-concave such that $f(0) = 1$. Then there exist $\beta \geq \alpha > 0$ such that the function*

$$g(t) := \chi_{[0,\alpha]}(t) + \left(\frac{\beta - t}{\beta - \alpha}\right)^{n-1} \chi_{[\alpha,\beta]}(t)$$

satisfies 1) $\int_0^{+\infty} g(t)dt = \int_0^{+\infty} f(t)dt$, $\int_0^{+\infty} tg(t)dt = \int_0^{+\infty} tf(t)dt$,
2) $\mathrm{supp}(f) \subset \mathrm{supp}(g)$,
3) *for every convex function* $\phi : \mathbb{R} \to \mathbb{R}$, $\int_{-\infty}^{+\infty} \phi(t)f(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)g(t)dt$.

Proof. Set $\alpha_1 = \int_0^{+\infty} f(t)dt$. For $0 \leq \alpha \leq \alpha_1$, we define $\beta(\alpha) = n\alpha_1 - (n-1)\alpha$. Note that $\beta(\alpha) \geq \alpha$ and define $g_\alpha(t) := \chi_{[0,\alpha]}(t) + \left(\frac{\beta(\alpha)-t}{\beta(\alpha)-\alpha}\right)^{n-1} \chi_{[\alpha,\beta(\alpha)]}(t)$. Then

$$\int_0^{+\infty} g_\alpha(t)dt = \int_0^{+\infty} f(t)dt. \tag{14}$$

Define $\Psi(\alpha) = \int_0^{+\infty} t(f(t) - g_\alpha(t))dt$. The function Ψ is continuous on $[0, \alpha_1]$ and it follows from (14) that $f - g_\alpha$ changes sign at some point $t_\alpha \in \mathrm{supp}(f)$. Let us prove that $\Psi(0) \leq 0$ and $\Psi(\alpha_1) \geq 0$.

• Observe that $f(0) = g_0(0)$ and that $f^{\frac{1}{n-1}}$ and $g_0{}^{\frac{1}{n-1}}$ are respectively concave and affine on their support. It follows that $\mathrm{supp}(f) \subset \mathrm{supp}(g_0)$ and $(t - t_0)(f - g_0)(t) \leq 0$ for all $t \geq 0$. From (14), we get

$$\int_0^{+\infty} (t - t_0)(f - g_0)(t)dt = \int_0^{+\infty} t(f - g_0)(t)dt = \Psi(0) \leq 0 \ .$$

• Since $f(t) \le \max f = 1$, we have $\alpha_1 = \int_0^{+\infty} f(t)dt \le |\mathrm{supp}(f)|$. Thus $(t-\alpha_1)(f-g_{\alpha_1})(t) \ge 0$ for all $t \ge 0$. Using (14), it follows that

$$\int_0^{+\infty} (t-\alpha_1)(f-g_{\alpha_1})(t)dt = \int_0^{+\infty} t(f-g_{\alpha_1})(t)dt = \Psi(\alpha_1) \ge 0\ .$$

Since Ψ is continuous, there exists $\alpha_0 \in [0, \alpha_1]$ such that $\Psi(\alpha_0) = 0$. To simplify, we set $\alpha = \alpha_0$, $g = g_{\alpha_0}$, $\beta = \beta(\alpha_0)$ and $t_0 = t_{\alpha_0}$.

• Let us prove 2) and 3): Set $v := f - g$ and $V(t) := \int_{-\infty}^t v(s)ds$. Since V is compactly supported and satisfies $\int_{-\infty}^{+\infty} V(t)dt = 0$, if $v \ne 0$, then v has to change sign at least twice. On $[0, \alpha]$, we have $v = f - \max f \le 0$. On $[\alpha, \beta] \cap \mathrm{supp}(f)$, $f^{\frac{1}{n-1}}$ is concave and $g^{\frac{1}{n-1}}$ is affine, hence v changes sign at most twice. Therefore $\mathrm{supp}(f) \subset \mathrm{supp}(g)$ and v changes sign twice at t_0 and t_1, with $\alpha \le t_0 \le t_1 \le \beta$. Hence we have $v \le 0$ on $[0, t_0]$ and $[t_1, \beta]$ and $v \ge 0$ on $[t_0, t_1]$. Since $W'' = v$, $W(0) = 0$ and $W(+\infty) = 0$, we get $W \le 0$. The result follows from Lemma 3. □

Proposition 6. *Let $n > 1$, $f : \mathbb{R} \to \mathbb{R}$, $f \ge 0$, integrable, $\frac{1}{n-1}$-concave such that $\int_{-\infty}^{+\infty} tf(t)dt = 0$. Then there exist $a < 0$, $b \in \mathbb{R}$ and $c > 0$, $a < b < c$ such that the function $g(t) := \max f \left(\left(\frac{t-a}{b-a}\right)^{n-1} \chi_{[a,b]}(t) + \left(\frac{c-t}{c-b}\right)^{n-1} \chi_{[b,c]}(t) \right)$ satisfies*

1) *$\int_{-\infty}^{+\infty} g(t)dt = \int_{-\infty}^{+\infty} f(t)dt$, $\int_{-\infty}^{+\infty} tg(t)dt = 0$,*
2) *$\mathrm{supp}(f) \subset \mathrm{supp}(g) = [a, c]$,*
3) *for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, $\int_{-\infty}^{+\infty} \phi(t)f(t)dt \le \int_{-\infty}^{+\infty} \phi(t)g(t)dt$.*

Proof. We may assume that $\max f = 1 = f(t_0)$. Then, we apply Lemma 7 to $f(t - t_0)$ and $f(t_0 - t)$. We get two functions g_1 and g_2 defined on $[0, +\infty[$ and satisfying 1), 2), and 3) of Lemma 7. We define the function $\tilde{g} : \mathbb{R} \to \mathbb{R}$ by $\tilde{g}(t) := g_1(t - t_0)\chi_{[t_0,+\infty[}(t) + g_2(t_0 - t)\chi_{]-\infty,t_0[}(t)$. There are some $\beta' \le \alpha' \le t_0 \le \alpha \le \beta$ such that $\tilde{g}(t) = \left(\frac{t-\beta'}{\alpha'-\beta'}\right)^{n-1} \chi_{[\beta',\alpha']}(t) + \chi_{[\alpha',\alpha]}(t) + \left(\frac{\beta-t}{\beta-\alpha}\right)^{n-1} \chi_{[\alpha,\beta]}(t)$ satisfies $\mathrm{supp}(f) \subset \mathrm{supp}(\tilde{g})$,

$$\int_{t_0}^{+\infty} (f-\tilde{g})(t)dt = \int_{-\infty}^{t_0} (f-\tilde{g})(t)dt = \int_{t_0}^{+\infty} t(f-\tilde{g})(t)dt = \int_{-\infty}^{t_0} t(f-\tilde{g})(t)dt = 0\,,$$

and for every convex function $\phi : \mathbb{R} \to \mathbb{R}$, $\int_{-\infty}^{+\infty} \phi(t)f(t)dt \le \int_{-\infty}^{+\infty} \phi(t)\tilde{g}(t)dt$.

• For $b \in [\alpha', \alpha]$, we set $c = c(b) = b + n\int_b^{+\infty} \tilde{g}(t)dt$, $a = a(b) = b - n\int_{-\infty}^b \tilde{g}(t)dt$. The function $g_b : \mathbb{R} \to \mathbb{R}$ defined by $g_b(t) := \left(\frac{t-a}{b-a}\right)^{n-1} \chi_{[a,b]}(t) + \left(\frac{c-t}{c-b}\right)^{n-1} \chi_{[b,c]}(t)$ satisfies $\int_{-\infty}^b g_b(t)dt = \int_{-\infty}^b \tilde{g}(t)dt$ and $\int_b^{+\infty} g_b(t)dt = \int_b^{+\infty} \tilde{g}(t)dt$. Moreover, we have $(\tilde{g} - g_b)(b) = 0$, $\tilde{g}^{\frac{1}{n-1}}$ is concave on $[b, \beta]$ and $g_b^{\frac{1}{n-1}}$ is affine on $[b, c]$. Hence, if $c < \beta$ then $\tilde{g} > g_b$ on $]b, \beta[\supset]b, c]$. This is absurd since $\int_{-\infty}^b g_b(t)dt = \int_{-\infty}^b \tilde{g}(t)dt$. Thus $c \ge \beta$. For the same reasons, we have $a \le \beta'$. Hence $\mathrm{supp}(\tilde{g}) \subset \mathrm{supp}(g_b)$.

• We define now $\Psi(b) = \int_{-\infty}^{+\infty} tg_b(t)dt$. Let us prove that $\Psi(\alpha') \ge 0$ and $\Psi(\alpha) \le 0$:

1) $\Psi(\alpha') \geq 0$: For $b = \alpha'$, we have $a = \alpha' - n\int_{-\infty}^{\alpha'} \tilde{g}(t)dt = \beta'$. Hence $g_{\alpha'} = \tilde{g}$ on $[\beta', \alpha']$. Since $c > \beta$, $\tilde{g}^{\frac{1}{n-1}}$ is concave on $[\alpha', \beta]$ and $g_{\alpha'}{}^{\frac{1}{n-1}}$ is affine on $[\alpha', c]$, there exists $t_0 \in]\alpha', \beta[$ such that $(t - t_0)(g_{\alpha'}(t) - \tilde{g}(t)) \geq 0$. Thus

$$\int_{-\infty}^{+\infty} (t - t_0)(g_{\alpha'}(t) - \tilde{g}(t))dt = \int_{-\infty}^{+\infty} t(g_{\alpha'}(t) - \tilde{g}(t))dt = \int_{-\infty}^{+\infty} tg_{\alpha'}(t)dt = \Psi(\alpha') \geq 0 \ .$$

2) $\Psi(\alpha) \leq 0$: For $b = \alpha$, we have $c = \alpha + n\int_{\alpha}^{+\infty} \tilde{g}(t)dt = \beta$. Hence, using exactly the same arguments, we get $\Psi(\alpha) \leq 0$.

• Since Ψ is continuous on $[\alpha', \alpha]$, there exists $b_0 \in [\alpha', \alpha]$ such that $\Psi(b_0) = 0$. We denote $b = b_0$ and $g = g_{b_0}$. Then g satisfies

$$\int_{-\infty}^{b} g(t)dt = \int_{-\infty}^{b} \tilde{g}(t)dt \ , \int_{b}^{+\infty} g(t)dt = \int_{b}^{+\infty} \tilde{g}(t)dt \ , \int_{-\infty}^{+\infty} tg(t)dt = 0 \ ,$$

and $\mathrm{supp}(\tilde{g}) \subset \mathrm{supp} g$. Moreover $(g - \tilde{g})(b) = 0$, $\tilde{g}^{\frac{1}{n-1}}$ is concave on $[b, \beta]$ and $g^{\frac{1}{n-1}}$ is affine on $[b, c]$. Hence there exist $t_1 \in]\beta', \alpha'[$ and $t_2 \in]\alpha, \beta[$ such that $\tilde{g} \leq g$ on $[\beta', t_1]$, $\tilde{g} \geq g$ on $[t_1, t_2]$ and $\tilde{g} \leq g$ on $[t_2, \beta]$.

• Set $v = \tilde{g} - g$. Since $W'' = v$ and W is compactly supported, we get $W \leq 0$. Using Lemma 3, we obtain

$$\int_{-\infty}^{+\infty} \phi(t)\tilde{g}(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)g(t)dt \quad \text{hence} \quad \int_{-\infty}^{+\infty} \phi(t)f(t)dt \leq \int_{-\infty}^{+\infty} \phi(t)g(t)dt \ ,$$

and $\int_{-\infty}^{+\infty} g(t)dt = \int_{-\infty}^{+\infty} f(t)dt$, $\int_{-\infty}^{+\infty} tg(t)dt = 0$ and $\mathrm{supp}(f) \subset \mathrm{supp}(g)$. □

4.2.2. Reduction to the "affine" case

Using that ϕ is even, we reduce the "affine by parts" case to the "affine" case.

Proposition 7. *Let $n > 1$, $a > 0$, $c > 0$ and $b \in \mathbb{R}$, satisfying $c - a + (n-1)b = 0$ and $-a \leq b \leq c$. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(t) := (\frac{t+a}{b+a})^{n-1}\chi_{[-a,b]}(t) + (\frac{c-t}{c-b})^{n-1}\chi_{[b,c]}(t)$. Set $d := \frac{c+a}{n+1}$ and define $h(t) := (\frac{n}{n+1})^{n-1}(1 + \frac{t}{nd})^{n-1}\chi_{[-nd,d]}$. Then for every even function $\phi : \mathbb{R} \to \mathbb{R}$ non-decreasing on $[0, +\infty[$,*

$$\int_{-a}^{c} \phi(t)g(t)dt \leq \int_{-nd}^{d} \phi(t)h(t)dt \ .$$

Proof. Since ϕ is even, by symmetry, we may assume that $b \geq 0$. By the definition of a, b, c and d, the functions g and h satisfy

$$\int_{-\infty}^{+\infty} h(t)dt = \int_{-\infty}^{+\infty} g(t)dt \ , \int_{-\infty}^{+\infty} tg(t)dt = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} th(t)dt = 0 \ .$$

Moreover, since $\max g = 1$, using Fubini's theorem and a change of variable,

$$\int_{-\infty}^{+\infty} \phi(t)g(t)dt \ = \ (n-1)\int_{0}^{1} u^{n-2}\Big(\int_{\{g^{\frac{1}{n-1}} \geq u\}} \phi(t)dt\Big)du \ .$$

From the definition of g, we have $\{g^{\frac{1}{n-1}} \geq u\} = [x(u), y(u)]$, where $x(u) = (a+b)u - a$ and $y(u) = c - (c-b)u$. Denote $z(u) = d((n+1)u - n)$. We also have $\int_{-\infty}^{+\infty} \phi(t)h(t)dt = (n-1)\int_0^1 u^{n-2}\Big(\int_{z(u)}^d \phi(t)dt\Big)du$. Thus it is enough to prove that $\int_{x(u)}^{y(u)} \phi(t)dt \leq \int_{z(u)}^d \phi(t)dt$, for all $u \in [0,1]$. From the definition of a, b, c and d, we have $\frac{y(u)-x(u)}{2} = \frac{d-z(u)}{2} := \delta(u)$. Fix $u \in [0,1]$ and define $\Psi(\alpha) = \int_{\alpha-\delta(u)}^{\alpha+\delta(u)} \phi(t)dt$. We have $\Psi\left(\frac{y(u)+x(u)}{2}\right) = \int_{x(u)}^{y(u)} \phi(t)dt$ and $\Psi\left(\frac{d+z(u)}{2}\right) = \int_{z(u)}^d \phi(t)dt$. Since ϕ is even and non-decreasing on $[0,+\infty[$, we have $\phi(\alpha) \leq \phi(\beta)$ for $|\alpha| \leq |\beta|$. If $\alpha > 0$, since $\delta > 0$, we have $|\alpha-\delta| \leq |\alpha+\delta|$. Hence $\Psi'(\alpha) = \phi(\alpha+\delta) - \phi(\alpha-\delta) \leq 0$. Thus Ψ is non-decreasing on $[0,+\infty[$. Moreover since ϕ is even, Ψ is even. Therefore if $|\alpha| \leq |\beta|$, we have $\Psi(\alpha) \leq \Psi(\beta)$. For all $u \in [0,1]$, since $0 \leq b \leq c$, we have

$$\frac{|y(u)+x(u)|}{2} = \left|u - \frac{n-1}{n+1}\right|\left(\frac{n+1}{2}\right)b \leq \left|u - \frac{n-1}{n+1}\right|\left(c + \frac{n-1}{2}b\right) = \frac{|d+z(u)|}{2}\ .$$

It follows that $\int_{x(u)}^{y(u)} \phi(t)dt \leq \int_{z(u)}^d \phi(t)dt$ for all $u \in [0,1]$. □

Proof of the upper bound in (5). It follows from Propositions 6 and 7. □

References

[1] Ball, K. M.: *Logarithmically concave functions and sections of convex sets in* $\mathbb{R}^n$. Studia Math. **88** (1988), 69–84.

[2] Bourgain, J.: *On high dimensional maximal functions associated to convex bodies.* Amer. J. Math. **108** (1986), 1467–1476.

[3] Bourgain, J.: *On the* L^p*-bounds for maximal functions associated to convex bodies in* $\mathbb{R}^n$. Israel J. Math. **54** (1986), 257–265.

[4] Busemann, H.: *A theorem on convex bodies of Brunn-Minkowski type.* Amer. J. Math. **71** (1949), 743–762.

[5] Fradelizi, M.: *Sections of convex bodies through their centroid.* Arch. Math. **69** (1997) 515–522.

[6] Gardner, R. J.: *Geometric tomography.* Cambridge University Press, New-York 1995.

[7] Giannopoulos, A. A.; Papadimitrakis, M.: *The Busemann-Petty problem.* Preprint.

[8] Hensley, D.: *Slicing convex bodies – bounds for slice area in terms of the body's covariance.* Proc. Amer. Math. Soc. **79** (1980) (4), 619–625.

[9] Kannan, R.; Lovász, L.; Simonovits, M.: *Isoperimetric problems for convex bodies and a localization lemma.* Discrete Comput. Geom. **13** (1995) (3–4), 541–559.

[10] Marshall, A. W.; Olkin, I.; Proschan, F.: *Monotonicity of ratios of means and other applications of majorization.* In: Inequalities (ed. by O. Shisha), Acad. Press, New York, London (1967), 177–190.

[11] Milman, V. D.; Pajor, A.: *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space.* In: Geometric Aspects of Functional Analysis (ed. by J. Lindenstrauss and V. Milman), Lecture Notes in Mathematics **1376**, Springer, Heidelberg 1989, 64–104.

[12] Meyer, M.; Werner, E.: *The Santaló regions of a convex body.* Trans. Amer. Math. Soc., to appear.

[13] Schütt, C.: *Floating body, illumination body, and polytopal approximation.* C. R. Acad. Sci. Paris Serie I **324** (2) (1997), 201–203.

[14] Webb, S.: *Central slices of the regular simplex.* Geom. Dedicata **61** (1) (1996), 19–28.

Received March 12, 1998