Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 1, 195-202.

Symmetric Models of the Real Projective Plane

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Abstract. We show that the symmetry group of a stable immersion of the real projective plane P in E^3 is either trivial or is cyclic of order 3, and that of a stable map of P in E^3 is conjugate to a subgroup of the full tetrahedral group. Thus Boy's surface, in its 'standard' form, is the most symmetrical stable immersion of P in E^3, and Steiner's surface is given by the most symmetrical stable map of P in E^3. We also construct a smooth embedding of P in E^4 with symmetry group $SO(2)$ by orthogonal projection of the Veronese surface.

1. Stable maps of surfaces in E^3

Let M be a smooth (say C^2) closed connected surface, and let $f : M \to E^3$ be a smooth map of M into Euclidean 3-space E^3. Then f is said to be *stable* iff there exists a neighbourhood W_f of f in $C^2(M, E^3)$ such that for all $g \in W_f$ there exist diffeomorphisms $h_1 : M \to M$ and $h_2 : E^3 \to E^3$ such that $h_2 \circ f = g \circ h_1$. We point out that if f is a stable map then the only multiple points of f occur as self-transverse intersections and the only singularities of f are Whitney umbrellas. Thus for each point $p \in f(M)$, there is a diffeomorphism $\phi : U \to V$ from an open neighbourhood U of $O \in E^3$ to an open neighbourhood V of p in E^3, such that $f(O) = p$ and $f(M) \cap V$ is the image under ϕ of

* Partially supported by FCT and Praxis/2/2.1/MAT/63/94

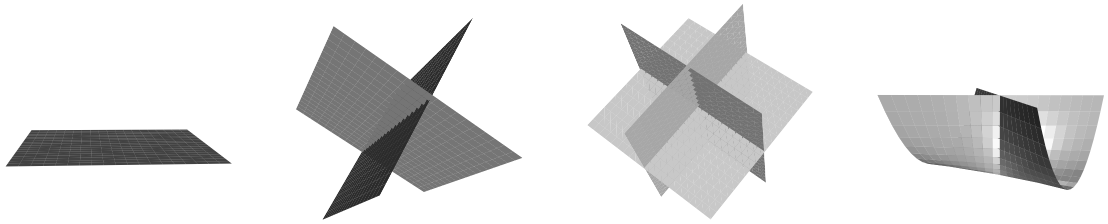

Figure 1. Local models of stable maps

the intersection with U of the set given by $x = 0$, or by $xy = 0$, or by $xyz = 0$, or by $x^2 = zy^2, z \geq 0$ (the 'Whitney umbrella'), as illustrated in Figure 1.

A point $u \in f^{-1}(p)$, or p itself, is then said to be *simple* (or *regular*), *double, triple* or *singular* respectively. We may now partition M into subsets M_1, M_2, M_3, M_σ accordingly. Thus M_1 is the set of simple points, and so on. Notice that M_σ and M_3 are finite sets, while M_2 is a union of finitely many mutually disjoint smoothly embedded simple closed curves and open arcs which terminate at the points of M_σ and M_3. Thus M_1 is open and dense in M. Note also that the number of triple points in M is divisible by 3, and that M_2 admits an involutory self-diffeomorphism without fixed points.

We shall try to determine the constraints that are imposed on the symmetry behaviour of f by the fact that any symmetry of f must respect this partition and its internal structure. Our first task, however, is to define what we mean by symmetry in this context. Suppose, then, that $f : M \to E^3$ is a stable map in the above sense. A *symmetry* of f is an isometry ϕ of E^3 to itself such that $\phi(W) = W$, where $W = f(M)$. Any such ϕ determines a unique diffeomorphism ϕ^* of M to itself, such that $f \circ \phi^* = \phi \circ f$. The map ϕ^* is obviously well-defined on M_1, and is extended to $M \backslash M_1$ by continuity. The set of all symmetries of f is a compact subgroup G_f of the Euclidean group $E(3)$, and the map sending $\phi \in G_f$ to ϕ^* is a monomorphism onto a subgroup G_f^* of the self-diffeomorphism group of M. Since M_σ is finite, f induces a Riemannian metric on M and G_f^* is a subgroup of the isometry group of M for this metric. We may assume without loss of generality that O is the centroid of W, and hence that G_f is a subgroup of the orthogonal group $O(3)$. If M is neither a torus nor a sphere, then G_f is finite.

2. Symmetry and the Euler number for stable immersions

We now prove some propositions that relate the symmetry group G_f of a stable immersion of a smooth closed connected surface M into E^3 to the Euler number $e(M)$ of M. Let $\mathcal{I}(M)$ denote the set of all such immersions. As above, we shall assume that the centroid of the image of $f \in \mathcal{I}(M)$ is O.

Proposition 1. *Let $f \in \mathcal{I}(M)$. Then either $G_f < SO(3)$ or $e(M)$ is even.*

Proof. Suppose that G_f is not a subgroup of $SO(3)$. Then it contains either a reflexion in a plane through O, or the reflexion $J = -I_3$ in O. Let us examine these two possibilities in turn.

(i) Suppose that A is a plane through O, and that $\alpha \in G_f$, where α denotes reflexion in A. Then A must be transverse to f, since f is stable. Thus α acts on M as an involutory diffeomorphism whose fixed point set F is a union of finitely many disjoint simple closed curves forming the common boundary of two compact connected surfaces M', M'', where $M' \cup M'' = M$ with disjoint interiors, such that $\alpha(M') = M''$ and vice-versa. Hence $e(M) = e(M') + e(M'') = 2e(M')$.

(ii) Now suppose that $J \in G_f$. If $O \notin f(M)$, then J determines an involution on M without fixed points, so $e(M)$ is even. Let us suppose, therefore, that $O \in f(M)$. Then $O \in M_2$, since some line through O is transverse to f , and so intersects $f(M)$ in an even number of simple and of triple points. It follows again that $e(M)$ is even.

Recall next that any finite subgroup of $SO(3)$ is conjugate in $SO(3)$ to the rotation group T of the regular tetrahedron, K of the cube, or I of the icosahedron, or, for some $n \in \mathbf{N}$, to the cyclic group C_n of rotations through multiples of $2\pi/n$ about some line, or to the dihedral group D_n obtained by extending C_n by rotation through π about a line that intersects the rotational axis of C_n orthogonally. For $n > 2$, the groups C_n, D_n are the groups of rotational symmetries of a rightcone on a regular convex plane n-sided polygon and of a right prism on such a base, respectively. We note that each orbit of T contains 1, 4, 6 or 12 points, each orbit of K contains 1, 6, 8, 12 or 24 points, and each orbit of I contains 1, 12, 20, 30 or 60 points. Thus each of these orbits, except in the case of the unique fixed point O, contains an even number of points.

Proposition 2. *Let $f \in \mathcal{I}(M)$. If G_f is conjugate to T, K or I, then $e(M)$ is even.*

Proof. For such f, $O \notin f(M)$. So each orbit of the action of G_f on M contains an even number of points. Hence $e(M)$ is even.

Proposition 3. *Let $f \in \mathcal{I}(M)$. If $e(M)$ is odd, then G_f is conjugate to C_n, for some $n \in \mathbf{N}$.*

Proof. For such f and M, G_f must be finite, so by Propositions 1 and 2, G_f is conjugate to either C_n or D_n, for some $n \in \mathbf{N}$. In the latter case, if $n > 0$ then $O \in f(M)$, for otherwise every orbit of the action of G_f on M would have an even number of elements, contradicting the assumption that $e(M)$ is odd. Let L be the line through O that is fixed setwise by G_f. Then an even number of points of M map into L, and so O is a double point of f, which implies that $e(M)$ is even. We conclude that G_f is conjugate to C_n, as claimed.

3. Stable immersions of the real projective plane

We turn now to the special case in which $M = P$ is the real projective plane. Since $e(P) = 1$, Proposition 3 implies that G_f is conjugate to C_n for some $n \in \mathbf{N}$. We show now that either $n = 1$ or $n = 3$.

Of course the best-known stable immersion of the real projective plane is that discovered in 1901 by Werner Boy[1,5]. We can present this as a smooth immersion $\beta : P \to E^3$

for which the group G_β is cyclic of order three about an axis that meets $\beta(P)$ in just one simple point and one triple point. Figure 2 indicates the curves of double points and the three triple points on P itself, and their images in E^3 under β. The force of Proposition 4 below is that no stable immersion of P in E^3 is more symmetrical than this form of Boy's surface.

Proposition 4. *Let $f \in \mathcal{I}(P)$. Then either G_f is trivial, or it is cyclic of order 3, generated by rotation through $2\pi/3$ about a straight line on which there lies exactly one simple point, no double points, and an odd number of triple points.*

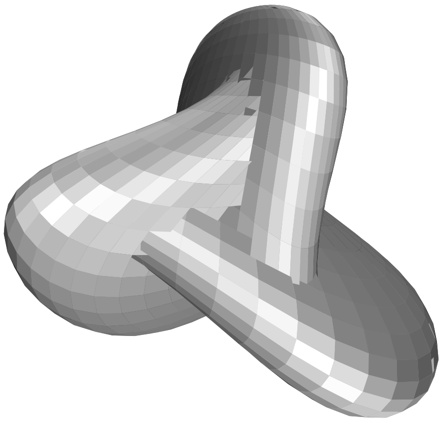

Figure 2. Boy's surface with 3-fold cyclic symmetry

Proof. Suppose that f is such an immersion, and that G_f is nontrivial. Thus G_f is conjugate to C_n, for some $n > 1$. Let L be the axis of rotation for G_f. Then an even number, say $2r$, of points of P are mapped by f into L. If $r = 0$, then $n|e(P) = 1$, a contradiction. So $r \geq 1$. Let $L \cap f(M)$ comprises m_1 simple points, m_2 double points, and m_3 triple points. Then for some integer s,

$$1 = e(P) = ns + m_1 + 2m_2 + 3m_3$$

where $m_1 + 2m_2 + 3m_3$ is even and positive. It follows that $m_1 + 3m_3$ is even, and that ns is odd and negative. Now, by a theorem of Eilenberg [9], the action of G_f on M has just one fixed point, so $m_1 = 1$. Hence m_3 is odd. Also G_f acts effectively on $f^{-1}(q)$, for each triple point $q \in f(M)$. It follows that $n = 3$, and hence that $m_2 = 0$.

4. Symmetry of stable maps of surfaces in E^3

Suppose now that $\mathcal{M}(M)$ denotes the set of all smooth stable maps of M in E^3. We find that the arguments used above for stable immersions can be adapted to yield analogous results for $f \in \mathcal{M}(M)$, as follows. As before, we shall assume for simplicity that the centroid of $f(M)$ is O.

Proposition 5. *Let $f \in \mathcal{M}(M)$. Then either $J \notin G_f$ or $e(M)$ is even.*

Proof. If $J \in G_f$, then O is not a singular point of f. Part (ii) of the proof given for Proposition 1 then applies.

Proposition 6. *Let $f \in \mathcal{M}(P)$, and suppose that $H < G_f$ is cyclic of order $n > 1$. Then $n = 2$ or $n = 3$.*

Proof. As in the proof of Proposition 4, H acts on P with just one fixed point, whose image under f lies on the axis of rotation L, say, of H in E^3. If there is a triple point p on L, then either $n = 3$, with all nearby points of L simple, or $n = 2$ and there is a line segment S of L, of which one end-point is p and the other end-point is either triple or singular, such that every point of the relative interior of S is double. Again, if there is no triple point on $f(M)$, then L is not transverse to $f(M)$, since otherwise an even number of points of M would be mapped to L, with $2r$ simple and $2s$ double: thus if $n > 2$, then $2r = 1$, by the theorem of Eilenberg referred to above, which is impossible; and if $n = 2$, then $e(M)$ is even, again a contradiction. We conclude that if there is no triple point on L, then there is a segment of L with singular end-points whose relative interior consists of double points, and so $n = 2$.

Let T^* denote the group of symmetries of a regular tetrahedron in E^3 with centroid O. Then T^* is a finite subgroup of $O(3)$, of order 24, isomorphic to the symmetric group S_4.

Proposition 7. *Let $f \in \mathcal{M}(P)$. Then G_f is conjugate in $O(3)$ to a subgroup of T^*.*

Proof. Propositions 4 and 5 imply that G_f is conjugate either to a subgroup of T^* or to the symmetry group of a right prism on an equilateral triangular base. But in the latter case, each reflexion in a plane that belongs to G_f contains the axis of rotation of the cyclic subgroup of G_f of order 3, since by Eilenberg's theorem there is just one fixed point of this cyclic subgroup. This contradiction completes the proof.

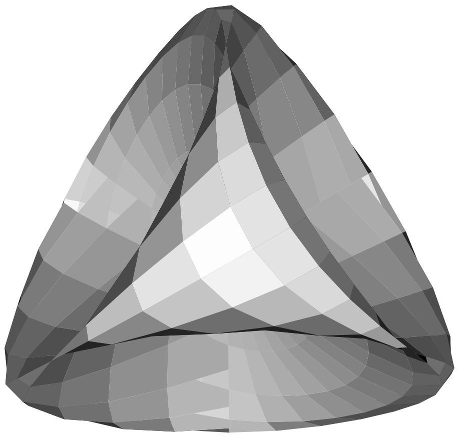

Figure 3. Steiner's surface

5. Steiner's surface

In a short note that was first published in his Collected Works [17], Steiner described a certain quartic surface that he studied during a visit to Rome in 1844, and which has become known as *Steiner's surface* or the *Roman surface*. A convenient algebraic equation for Steiner's surface is

$$y^2z^2 + z^2x^2 + x^2y^2 - xyz = 0$$

and the surface may be regarded as the image of a stable map $\sigma : P \to E^3$, given in terms of homogeneous coordinates on S^2 by $\sigma[\xi, \eta, \zeta] = (\eta\zeta, \zeta\xi, \xi\eta)$. (See, for example, Sommerville [16], and for a more detailed modern treatment Apéry [1].) The map σ has six singular points (Whitney umbrellas), each of which is joined in the image by a line segment of double points to its unique triple point. The group G_σ is conjugate in $O(3)$ to the symmetry group T^* of a regular tetrahedron. Thus we deduce from Proposition 7 that Steiner's surface is the most symmetrical of all images of stable maps of P in E^3.

6. A symmetrical embedding of P in E^4

For technical reasons, let us identify E^6 with the space of 3×3 real symmetric matrices, and regard each point of E^3 as a real 3×1 column matrix. As before, we denote the point of P with homogeneous coordinates x, y, z, where $x^2 + y^2 + z^2 = 1$ by $[x, y, z]$. Now define the *Veronese map* $\nu : P \to E^6$ by $\nu[x, y, z] = XX^t$, where $X = col(x, y, z)$. Then ν is a smooth embedding of P, and its image V is known as the *Veronese surface*. Since $X \in S^2, V$ lies in the affine 5-plane F with equation

$$x_{11} + x_{22} + x_{33} = 1,$$

so ν may be regarded as a smooth embedding of P in E^5. This embedding is maximally symmetrical: its symmetry group G_ν is isomorphic to $SO(3)$, the largest isometry group for any Riemannian metric on P. In this sense, the Veronese surface plays a rôle in E^5 analogous to that established above for Boy's surface and Steiner's surface in E^3. We now describe an embedding of P into E^4 which has some claim to a status alongside these three.

Suppose, then, that $f : P \to E^4$ is a smooth embedding. Then $G_f < O(3)$. So if G_f is isomorphic to $SO(3)$, it must be conjugate to $SO(3)$ itself, and then $f(P) \subset E^3$, which is impossible, since f is an embedding. Therefore G_f is a proper subgroup of $SO(3)$, and hence dim $G_f \leq 1$. But if dim $G_f = 1$, then the orbits are simple closed curves with the exception of a single fixed point. Hence G_f is isomorphic to $SO(2)$, or else is finite.

Recall that $SO(3)$ acts on E^6 (interpreted in the above way) by $A.Q = AQA^t$, for $A \in SO(3)$, $Q \in E^6$, and each orbit of this action contains a unique point of the form $diag(a, b, c)$, where $a \geq b \geq c$, the numbers a, b, c being of course the eigenvalues of each point of the orbit. The isotropy subgroup S_Q of this action at Q is isomorphic to $SO(3)$ if $a = b = c$, is isomorphic to $SO(2)$ if either $a > b = c$ or $a = b > c$, and is trivial if $a > b > c$. Now let $Q = [x_{ij}] \in E^6$, where $x_{ij} = 1 - \delta_{ij}$ for all i, j, and let π_Q be orthogonal projection of E^6 to the orthogonal complement $Q^\perp$ of Q. The eigenvalues of Q are -1, -1 and 2, so S_Q is isomorphic to $SO(2)$.

Let $f = \pi_Q \circ \nu$. Then f is a smooth map of P into (a copy of) E^4, and $G_f = S_Q$ is isomorphic to $SO(2)$. We now check that f has no singularities, and is injective. To establish the first of these claims, it is sufficient to show that no nonzero tangent vector to V is a multiple of Q. Now any tangent vector T to V at $\nu[x, y, z]$ is of the form $[t_{ij}]$, where

$$t_{11} = 2x\alpha,\ t_{22} = 2y\beta,\ t_{33} = 2z\gamma,\ t_{12} = x\beta + y\alpha,\ t_{13} = x\gamma + z\alpha, \text{ and } t_{23} = y\gamma + z\beta.$$

So T is a multiple of Q iff $T = O$. Secondly, for any $p = [x, y, z]$, $q = [u, v, w] \in P$,

$$f(p) = f(q) \text{ iff } x^2 = u^2,\ y^2 = v^2,\ z^2 = w^2,\ yz = vw,\ zx = wu,\ xy = uv,$$

that is to say iff $p = q$.

A different approach to an analogous example can be found in [3].

It remains to determine which finite subgroups of $O(4)$ can occur as the symmetry group of a smooth embedding of P in E^4. We already know that any such group must be isomorphic to a subgroup of $SO(3)$ and hence to $C_n,\ D_n,\ T,\ K$ or I. An obvious topological embedding of P in E^4 with cyclic symmetry of any odd order $m = 2n - 1$ can be derived from the embedding μ of the Möbius band $M = \mathbf{R} \times [-1, 1]/\mathbf{Z}$,

$$m.(x, y) = (x + m, (-1)^m y)$$

in E^3, where

$$\mu(x, y) = ((2 + cos2m\pi y)cos2\pi x, (2 + cos2m\pi y)sin2\pi x, sin2m\pi),$$

by taking the cone over $\mu(M) \subset E^3 \subset E^4$ with apex $a = (0, 0, 0, 1)$. However, it can be shown that this embedding cannot be smoothed at a [4].

The analogous situation in which P is replaced by a closed connected surface of some other topological type offers attractive problems, some of which (like that of P in E^4), involve questions about regular homotopy. It is interesting to recall that Morin's eversion of the 2-sphere in E^3 provides a halfway immersion with 4-fold symmetry, having a single quadruple point. Of course this does not contradict Proposition 4, since Morin's immersion is not stable. For further information on stable immersions, symmetry, regular homotopy, and quadruple points, see references below.

We thank David Singerman, Brian Bowditch and Thomas Banchoff for helpful comments.

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Received April 4, 1997; revised versions December 30, 1997 (1st), April 20, 1998 (2nd)