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Correct Soltan-Vasiloi Criterion for Convex Cones

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Abstract. Let S be a nonempty closed convex locally compact subset of a real locally convex topological linear space L ($\dim L \geq 2$), S different from L and a single point. It is proved among others that if S contains no straight line or L has the complement property, then S is a cone if and only if every three or fewer closed supporting hyperplanes of S have a nonempty intersection. This is a correct version of an absurd theorem of Soltan and Vasiloi established for a convex body in a real quasicompact topological linear space. The possibility of proving such a result under other assumptions on L and S is also discussed.

1. Introduction

Let S be a nonempty set in a real linear space $\mathcal{L}$. S is a cone if it is the union of a nonempty set of halflines having the common endpoint called the apex of the cone and the set of all such apices, denoted by $\ker_R S$, is called the *R-kernel* of S (cf. [5]–[7]). For distinct points x and y in S, y is *R-visible* from x via S if and only if there is in S a closed halfline $R(x,y)$ emanating from x through y. We extend this definition to the case when x and y coincide and require then that S contains some closed halfline emanating from x. A convex set in L, a real topological linear space, which has an interior point is called a *convex body* [8, Def.1.10]. A closed hyperplane $\mathfrak{h}$ in L is said to *support* S at a point s in $\mathrm{cl}S$ if and only if $\mathfrak{h}$ passes through s and S is contained in one of closed halfspaces determined by $\mathfrak{h}$ in L. Following [1] and some earlier works, for a closed convex subset S of L and a point u in S, we denote by S_u^R the union of all closed halflines emanating from u in S and call it the *characteristic cone* of S at u. It is readily verified that if S_u^R is nonempty, then it is a closed convex cone with apex u, and that $S_v^R = S_u^R + (v-u)$ whenever $v \in S$. The characteristic cone is also called the *recession cone* or the *asymptotic cone*. It is easily seen that $S_u^R = u + \bigcap_{\lambda>0} \lambda(S-u)$ [11, Chap.II, Exc.§2.14a]. We say that a topological linear space L has the *complement property* if and only if every closed linear subspace of L

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has a topological complement [13, §15.8.(1)]. Standard examples of such spaces, including Hilbert spaces, are listed in [13, §31.4]. By a *flat* in L we mean a translate of a linear subspace of L. Two closed flats in L are called topologically complementary if and only if the corresponding linear subspaces are topologically complementary. (xy) will represent a straight line determined by two distinct points x, y.

Soltan and Vasiloi established in [2] (cf. MR **45**(1973) #1041) several characteristics of a convex cone in a finite-dimensional Euclidean space. [2, Th.5] (cf.[12, Th.22.20]) states that a convex body in R^d is a cone if and only if any three or fewer of its supporting hyperplanes have a nonempty intersection. [2, Th.4] (cf.[12, Th.22.22]) states that a convex body in R^d ($d \geq 3$) is a cone if and only if its projection onto any two-dimensional hyperplane is a cone. In a subsequent paper [3] (cf. MR **53**(1977) #6285) these authors tried to generalize the first of these statements and proved that a convex body in a quasicompact topological linear space is a cone if and only if any three or fewer of its supporting hyperplanes have a nonempty intersection. Following [10, Chap.I, §9.1], recall here that a topological space X is *quasicompact* if and only if every filter on X has an adherent point or, equivalently, every family of closed subsets of X with empty intersection contains a finite subfamily with empty intersection. It follows immediately from these definitions that the topology of a quasicompact topological linear space is antidiscrete, that is, it consists only of the whole space and the empty set, so that this generalization is an absurd. It is the main purpose of this paper to give a correct form of the Soltan-Vasiloi result. We also generalize the second of the above mentioned criteria. Next, the characteristics of a convex cone are obtained using the notion of R-visibility. Finally, we discuss briefly the possibility of proving these results under other assumptions on L and S. The reader is referred to [5]–[7] for other characterizations of cones in finite and infinite dimensions and to [8], [9], [11] and [13] for techniques employed in the paper.

2. Main results

We start with the following lemma.

Lemma. *If S is a closed convex proper cone in a real locally convex topological linear space L, then* $\ker_R S$ *is the intersection of all closed supporting hyperplanes of S.*

Proof. It is well-known [8, Th.2.16] that there exists at least one closed hyperplane supporting S at any chosen apex, i.e. the collection $\mathcal{H}$ of all closed supporting hyperplanes of S is nonempty. It is immediate that every member of $\mathcal{H}$ must contain all apices of S, whence $\ker_R S \subseteq \bigcap_{\mathfrak{h} \in \mathcal{H}} \mathfrak{h}$. To prove the converse inclusion select an arbitrary point $x \in \bigcap_{\mathfrak{h} \in \mathcal{H}} \mathfrak{h}$. If $x \notin S$, then, by the separation theorem [13, §20.7.(1)], there exists a closed hyperplane $\mathfrak{g}$ which separates x and S strictly. Then however the hyperplane $\mathfrak{g}'$ parallel to $\mathfrak{g}$ and passing through any apex of S supports S and of course does not contain x, a contradiction. Hence, $x \in S$. Assume in turn that $x \notin \ker_R S$. Fix an apex $a \in \ker_R S$. If there exists a point $y \in L \sim S$ such that $a \in (xy)$, then we separate strictly y and S by a closed hyperplane $\mathfrak{f}$ and conclude that its translate $\mathfrak{f}'$ through a supports S and does not contain x, a contradiction. Hence, $(xa) \subseteq S$. Now choose in S an arbitrary point u different from a and x. Since $R(a, u) \subseteq S$, the convexity of S implies $R(x, u) \subseteq$

$\mathrm{conv}((xa) \cup R(a,u)) \subseteq S$, whence $x \in \ker_R S$, a contradiction. Consequently, $\bigcap_{\mathfrak{h} \in \mathcal{H}} \mathfrak{h} \subseteq \ker_R S$ which completes the proof. □

Here is the main result of the paper.

Theorem. *Let S be a nonempty closed convex locally compact subset of a real locally convex topological linear space L ($\dim L \geq 2$), S different from L and a single point. If S contains no straight line or L has the complement property, then*

(i) *S is a cone if and only if every three or fewer closed supporting hyperplanes of S have a nonempty intersection;*

(ii) *S is a cone if and only if, for every topologically complementary decomposition of L into two closed linear subspaces H and G with $\dim G = 2$, the projection of S onto G is a cone;*

(iii) *$\ker_R S$ is the intersection of all closed supporting hyperplanes of S and, moreover,*

$$\ker_R S = \bigcap\nolimits_{z \in \mathrm{bdry} S} B_z^R = \bigcap\nolimits_{z \in \mathrm{bdry} S} \mathrm{conv} B_z^R = \bigcap\nolimits_{z \in \mathrm{bdry} S} \mathrm{claff} B_z^R$$

where $B_z^R = \{y \in S : z$ is R-visible from y via $S\}$.

Proof. Consider the assertion (i). If S is a cone, then the Lemma implies that all closed supporting hyperplanes of S intersect. Conversely, for S not being a cone, we indicate three closed supporting hyperplanes $\mathfrak{h}_i$ $(i = 1, 2, 3)$ of S with empty intersection. First suppose that S contains no straight line. If S contains no halfline, then, by [13, §25.5.(2)], it is compact. In this case, select in S two distinct points x, y and separate them strictly by a closed hyperplane $\mathfrak{h}$ in L. By [13, §20.7.(8)], there are translates $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of $\mathfrak{h}$ which support S and are distinct in view of the choice of $\mathfrak{h}$. Choosing as $\mathfrak{h}_3$ any other closed supporting hyperplane of S, we are done. Hence, in the sequel let S contain some halfline, i.e. the characteristic cone S_u^R of S at some point $u \in S$ is nonempty, and of course closed, convex and locally compact, so that, by [13, §25.4.(2)], there is in L a closed hyperplane $\mathfrak{h}$ not containing u such that $\mathfrak{h} \cap S_u^R$ is nonempty and compact. Then the intersection of S with this closed halfspace determined by $\mathfrak{h}$ in L, which contains u, is closed, convex and locally compact, and contains no halfline, so that again by [13, §25.5.(2)] it is compact. Hence, there is a translate $\mathfrak{h}_1$ of $\mathfrak{h}$ supporting S at some point v. Obviously, $M = \mathfrak{h} \cap S_v^R$ is a convex compact set. But S is, by assumption, different from S_v^R, so that there must exist a point $y \in \mathfrak{h} \cap S \sim M$. In virtue of the separation theorem, there exists a closed hyperplane $\mathfrak{k}$ in $\mathfrak{h}$ which separates strictly M and y in $\mathfrak{h}$. Denote $\mathfrak{m} = \mathrm{aff}(\{v\} \cup \mathfrak{k})$. As easily seen, $\mathfrak{m}$ is a closed hyperplane in L and $\mathfrak{m} \cap S_v^R = \{v\}$. Hence, the intersection of S with this closed halfspace determined by $\mathfrak{m}$ in L which contains y cannot contain any halfline and, moreover, is closed, convex and locally compact. Hence, it is compact and there is a translate $\mathfrak{h}_2$ of $\mathfrak{m}$ supporting S. Since $y \notin \mathfrak{m}, v \notin \mathfrak{h}_2$ and $\mathfrak{h}_1 \cap \mathfrak{h}_2$ is a closed flat of codimension 2 in L. Next, since M is compact, again there is a translate $\mathfrak{k}'$ of $\mathfrak{k}$ in $\mathfrak{h}$ such that M lies in this relatively open halfspace determined by $\mathfrak{k}'$ in $\mathfrak{h}$ which contains y. Denote $\mathfrak{p} = \mathrm{aff}(\{v\} \cup \mathfrak{k}')$. It is a closed hyperplane in L and $\mathfrak{p} \cap S_v^R = \{v\}$. As twice above, we deduce that there exists a translate $\mathfrak{h}_3$ of $\mathfrak{p}$ which supports S. $\mathfrak{h}_1 \cap \mathfrak{h}_3$ is a closed hyperplane in $\mathfrak{h}_1$ parallel to $\mathfrak{h}_1 \cap \mathfrak{h}_2$ and different from it, whence $\mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3 = \varnothing$, as desired. Now consider the case of an arbitrary closed convex locally compact set S in L

having the complement property. In view of the first part above, we can assume that S contains some straight lines. For nontriviality, assume also that $\dim L \geq 3$. Select any point a in S and consider the family $\mathcal{F}$ of all straight lines in S passing through a. Since S is convex, it contains with every two such distinct straight lines the plane spanned by them, whence immediately $\mathfrak{H} = \bigcup_{\mathfrak{l} \in \mathcal{F}} \mathfrak{l}$ is a flat in L. Since S is closed, an easy argument reveals that $\mathfrak{H}$ is closed too. Hence, $\mathfrak{H}$ is the maximal flat through a contained in S. An alternative proof may be constructed via Zorn's lemma and an elementary fact that the closure of a flat in L is also a flat [13, §15.2.(7)]. By assumption, there is in L a closed flat $\mathfrak{G}$ topologically complementary to $\mathfrak{H}$ with a continuous linear projection π from L onto $\mathfrak{G}$. For nontriviality, assume further that $\dim \mathfrak{G} \geq 2$. It is easily seen that $\pi(S) = S \cap \mathfrak{G}$ is a closed convex locally compact subset of $\mathfrak{G}$ which contains no straight line. It follows from the assumption that $S \cap \mathfrak{G}$ is not a cone, so that, by the first part above, there exist three closed supporting hyperplanes $\mathfrak{h}'_i$ $(i = 1, 2, 3)$ of $S \cap \mathfrak{G}$ in $\mathfrak{G}$ with empty intersection. Then however $\mathfrak{h}_i = \pi^{-1}(\mathfrak{h}'_i)$ $(i = 1, 2, 3)$ are closed hyperplanes in L supporting S and having empty intersection. This completes the proof of (i).

Invoking (i), we easily justify (ii). If S is a cone, then any linear projection of S onto an arbitrary two-dimensional flat is a cone, so that it is sufficient to prove the converse. In the sequel, suppose thus that S is not a cone. For nontriviality, let $\dim L \geq 3$. Consider first the case that S contains no straight line. If S is compact, then its continuous linear projection is compact too and therefore not a cone. Let further S contain some halfline. Take as $\mathfrak{G}$ any two-dimensional flat which is a topological, by [13, §15.8.(2)], complement of the closed flat $\mathfrak{k}$ of codimension 2, specified in the first part of the proof of (i). Assume, without loss of generality, that $\mathfrak{k} \cap \mathfrak{G} = \{\circ\}$. We see that the natural projection of S onto $\mathfrak{G}$ is unbounded, convex, not necessarily closed and supported by three straight lines with empty intersection, projections of hyperplanes $\mathfrak{h}_i$ $(i = 1, 2, 3)$. As easily seen, such a set cannot be a cone, which establishes (ii) when S contains no straight line. Now let L have the complement property. In the second part of the proof of (i) we specified two topologically complementary closed flats $\mathfrak{H}$ and $\mathfrak{G}$ with a continuous linear projection π of L onto $\mathfrak{G}$, such that $\pi(S) = S \cap \mathfrak{G}$ is closed, convex, locally compact, contains no straight line and is not a cone. If $\dim \mathfrak{G} = 2$, then we are done, so let $\dim \mathfrak{G} \geq 3$. We established above that there are two closed topologically complementary flats $\mathfrak{G}'$ and $\mathfrak{G}''$ in $\mathfrak{G}$ with $\dim \mathfrak{G}' = 2$ and the continuous linear projection σ of $\mathfrak{G}$ onto $\mathfrak{G}'$ such that $\sigma(S \cap \mathfrak{G})$ is not a cone. Without loss of generality, one can assume that $\mathfrak{H} \cap \mathfrak{G}' \cap \mathfrak{G}'' = \{\circ\}$. Hence, $L = (\mathfrak{H} + \mathfrak{G}'') + \mathfrak{G}'$ is a topologically complementary decomposition of L into two closed linear subspaces (note that $\mathfrak{H} + \mathfrak{G}'' = \pi^{-1}(\mathfrak{G}''))$ with the continuous linear projection $\sigma \circ \pi$ of L onto $\mathfrak{G}'$. But $\sigma \circ \pi(S) = \sigma(S \cap \mathfrak{G})$ is not a cone, as desired. This proves (ii).

Finally, we establish (iii). If $\ker_R S$ is nonempty, then, in view of the Lemma, it is the intersection of all closed supporting hyperplanes of S. On the other hand, if $\ker_R S$ is empty, i.e. S is not a cone, then, in view of the part (i) of the Theorem, there are three closed supporting hyperplanes of S with empty intersection. This proves the first assertion of (iii) and it remains to justify the given intersection formulae. We are concerned only with the proof of the inclusion $\bigcap_{z \in \mathrm{bdry} S}$ cl aff$B_z^R \subseteq \ker_R S$. Select a point $x \in \bigcap_{z \in \mathrm{bdry} S}$ cl affB_z^R and suppose that $x \notin \ker_R S$. It follows from the first assertion just proved that there is a closed hyperplane $\mathfrak{h}$ supporting S at a point a such that $x \notin \mathfrak{h}$. However, $B_a^R \subseteq \mathfrak{h}$, whence $x \in \mathrm{cl}\ \mathrm{aff} B_a^R \subseteq \mathfrak{h}$, a contradiction which ends the proof of (iii). $\square$

Proposition. *If S is a closed convex proper body in a real topological linear space L, then the assertion (iii) of the Theorem holds.*

Proof. In view of [13, §17.5.(1)], S is supported at every boundary point by a closed hyperplane. Assume first that $\ker_R S \neq \varnothing$, i.e. S is a cone. An easy argument reveals that $\ker_R S \subseteq \bigcap_{\mathfrak{h}\in\mathcal{H}} \mathfrak{h}$, where $\mathcal{H}$ is the family of all closed supporting hyperplanes of S. Now select any point $x \in \bigcap_{\mathfrak{h}\in\mathcal{H}} \mathfrak{h}$ and suppose that $x \notin \ker_R S$. If $R(x,z) \subseteq S$ for every $z \in \mathrm{int} S$, then $R(x,z) \subseteq S$ for every $z \in S$ because, by [8, Th.1.11], every boundary point of S is the limit point of $\mathrm{int} S$, implying $x \in \ker_R S$, a contradiction. Hence, assume that $R(x,y) \nsubseteq S$ for some $y \in \mathrm{int} S$, $y \neq x$. This means that there must exist in $R(x,y)$ a point $w \in \mathrm{bdry} S, w \neq x$. Then however a closed hyperplane supporting S at w cannot pass through x, a contradiction. Consequently, $\ker_R S = \bigcap_{\mathfrak{h}\in\mathcal{H}} \mathfrak{h}$. The remaining intersection formulae for $\ker_R S$ involving sets B_z^R can be justified as in the Theorem. The proof of the proposition is complete. $\square$

The final corollary below generalizes a part of the main result of [5].

Corollary. *Let S be a nonempty closed convex locally compact proper subset of a real locally convex topological linear space $L\,(\dim L \geq 2)$, and α a cardinal number. If S contains no straight line or L has the complement property, then $\dim \ker_R S \geq \alpha$ if and only if every two boundary points of S are R-visible via S from a common α-dimensional subset of S.*

Proof. The necessity is trivial. Suppose thus that every two boundary points of S are R-visible via S from a common subset of dimension α. Let first $\alpha = 0$. We have to show that S is a cone. Select arbitrarily in L three closed hyperplanes $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ supporting S at points p_1, p_2, p_3, respectively. By assumption, there are some boundary points p_{12}, p_{23}, p_{31} in S such that all closed rays $R(p_{12}, p_1), R(p_{12}, p_2), R(p_{23}, p_2), R(p_{23}, p_3), R(p_{31}, p_3), R(p_{31}, p_1)$ lie in S, whence $p_{ij} \in \mathfrak{h}_i \cap \mathfrak{h}_j$ for $i, j = 1, 2, 3 (i \neq j)$. If all points p_{12}, p_{23}, p_{31} coincide, then $\mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3 \neq \varnothing$, so, by case (i) of the Theorem, S is a cone. Let for example $p_{12} \neq p_{23}$. By the imposed combinatorial condition, there exists a point p such that rays $R(p, p_{12})$ and $R(p, p_{23})$ lie in S, which is possible only if $p \in \mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3 \neq \varnothing$. Case (i) of the Theorem implies again that S is a cone, i.e. $\dim \ker_R S \geq 0$, as desired. In the sequel, let $\alpha \geq 1$. Select any points $r \in \ker_R S \subseteq \mathrm{bdry} S$ and $s \in \mathrm{bdry} S$, $r \neq s$. By assumption, r and s are R-visible via S from some α-dimensional subset T of S. Let $t \in T \sim \ker_R S$. It follows from the Lemma that there exists a closed supporting hyperplane $\mathfrak{h}$ of S such that $t \notin \mathfrak{h}$. But $r \in \ker_R S \subseteq \mathfrak{h}$, so that $R(t, r) \nsubseteq S$, a contradiction. Consequently, $T \subseteq \ker_R S$, i.e. $\dim \ker_R S \geq \alpha$ which completes the proof. $\square$

3. Concluding remarks

The main result of the present paper is the part (i) of the Theorem. Its proof depends on some assumptions imposed on the space L and the set S, and it should be reexamined in order to carry it out possibly in other settings. Suppose thus that a closed convex set S in a real locally convex topological linear space L is given, S not being a cone, and we want to indicate three or fewer closed supporting hyperplanes of S with an empty intersection. What geometric-topological conditions must S and L fulfill to make this possible? To fix attention assume that S does not contain a line and is unbounded. The argument in the Theorem depends basically on three facts. Firstly, S should contain a halfline. For this, it is, for example, enough that S be weakly locally compact, because a nested sequence of weakly compact sets in L has a nonempty intersection (cf. the reasoning in [13, §25.5.(2)]). Secondly, S must have, say, the *bounded cut property*, i.e. a closed hyperplane $\mathfrak{h}$ in L must exist which separates strictly some points of S and such that $\mathfrak{h} \cap S$ is bounded. This fact seems to be of independent interest. Thirdly, S must have a closed supporting hyperplane parallel to each which cuts off a bounded portion of S. For this it is enough that L be quasicomplete and semireflexive (cf. [13, §24.4.(3)]). It remains to give an example of a pair S and L already existing in the literature which satisfies these three properties but is not covered by the Theorem. Let L be a real reflexive Banach space and S a closed convex unbounded set in L having the drop property. By virtue of [4, Prop.1], S has the bounded cut property, and of course the drop property implies the absence of lines in S.

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