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Immersions into the Hyperbolic Space Invariant by Reflections

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Abstract. In this paper we prove that a nicely curved full submanifold of the hyperbolic space with parallel higher fundamental forms is 2-symmetric. This allows to give examples of 2-symmetric submanifolds of the hyperbolic space. Moreover we prove a splitting theorem for 2-symmetric submanifolds of a space of constant curvature.

Since the seventies there was an extensive study of *(extrinsic) symmetric submanifolds* of spaces of constant curvature, i.e. the submanifolds which are locally invariant under reflections with respect to their normal spaces. These submanifolds are characterized by the fact that their second fundamental form is parallel [9], [17] and are known to be essentially the so-called standard embeddings of symmetric R-spaces [9], [18], [1].

A generalization of these submanifolds was given by O. Kowalski and I. Kulich in [11], who defined the notion of k-symmetric submanifold (k any integer). In particular one has the *2-symmetric submanifolds*, which can be equivalently defined as follows [3]: *a nicely curved substantial submanifold* $f : M \to \overline{M}$ *of a space of constant curvature is 2-symmetric if* $f(M)$ *is locally invariant under reflections of* $\overline{M}$ *at any point* $p \in M$ *with respect to a subspace of the normal space* $\mathcal{N}_p^{odd}M$ (see Section 1 for the definition of $\mathcal{N}_p^{odd}M$. In particular, it is easy to see that symmetric submanifolds are 2-symmetric).

In [4] and [2] are given examples of 2-symmetric submanifolds of Euclidean space and the sphere. In particular one has helical geodesic immersions and standard minimal immersions into spheres.

In this note we examine 2-symmetric submanifolds of the hyperbolic space $\mathbf{H}^N$. Generalizing to $\mathbf{H}^N$ a result in [3] on submanifolds of spaces of constant non-negative curvature we give the following differential characterization of 2-symmetric submanifolds in terms of the higher fundamental forms s^k (see Section 1 and [16] for their definition; s^0 is the usual second fundamental form).

Theorem A. *A nicely curved, substantial submanifold of the hyperbolic space $f : M \to \mathbf{H}^N$ is 2-symmetric if and only if each higher fundamental form s^k is parallel in $\perp^{k+1} M$, $k = 0, \dots, m$.*

Here m denotes the degree of the immersion and $\perp^{k+1} M$ is the $(k+1)^{th}$-normal bundle (see Section 1 for the definitions).

The proof will be given in Section 2 and will use the Frenet theory for immersed curves in a Riemannian manifold and the study of certain symmetry properties of curves related to the decomposition of the normal bundle into k^{th}-normal bundles. This is the same method as the one used by W. Strübing to show that a submanifold with parallel second fundamental form is symmetric [17]. We remark that this proof actually works for submanifolds of any space of constant curvature (but the result for non-negative curvature is already known, [3]).

Using Theorem A, in Section 3 we can examine examples of 2-symmetric submanifolds of the hyperbolic space form $\mathbf{H}^N$. There are two classes of examples: the compact and the non-compact ones. The latter are to some extent trivial since they can be obtained by composition of compact 2-symmetric submanifolds of the sphere S^n with a totally umbilical embedding of S^n into $\mathbf{H}^N$ (following [20] we shall say that they are not full). Thus "new examples" can arise only in the non-compact case. In this situation, we observe that the class of full (non-compact) 2-symmetric submanifolds of $\mathbf{H}^N$ is not empty. Indeed one can see, as a consequence of Theorem A, that any homogeneous curve in $\mathbf{H}^N$ (i.e. any image in $\mathbf{H}^N$ of a one-parameter subgroup of $O(N,1)$) is 2-symmetric. (Perhaps it is worthwhile to remark this gives a difference with the case of symmetric submanifolds of $\mathbf{H}^N$. In fact there are no full irreducible symmetric submanifolds of $\mathbf{H}^N$, see [18] or [1]).

We also observe that 2-symmetric submanifolds arise as singular orbits of the representation of the group $I(M, \mathbf{H}^N)$ of the isometries of $\mathbf{H}^N$ which leave M invariant into the isometry group $O(N,1)$ of $\mathbf{H}^N$.

In Section 4 we prove a splitting theorem for 2-symmetric submanifolds of a space of constant curvature. We prove a more general result (Theorem 4.2) which in particular yields the following

Theorem B. *A 2-symmetric submanifold of a space of constant curvature with parallel mean curvature is either minimal or minimal in a hypersphere or it splits as an extrinsic product of 2-symmetric submanifolds.*

1. Normal spaces and higher fundamental forms

Let $f : M \to \overline{M}$ be an isometric immersion of a Riemannian manifold M into a space $\overline{M}$ of constant curvature. Since f is locally an embedding, we may locally identify M with its image in $\overline{M}$ and a vector field Y on M with its image $f_*(Y)$.

Denoted by $\overline{\nabla}$ the Levi-Civita connection of $\overline{M}$ and by ∇ the induced connection on M, the Gauß formula is

$$\overline{\nabla}_{X_p} Y = \nabla_{X_p} Y + s^0_p(X_p, Y_p),$$

where X_p and Y_p are in the tangent space T_pM of M at p, Y is any extension of Y_p to a vector field. The map $s^0_p : (X_p, Y_p) \mapsto s^0_p(X_p, Y_p) = (\overline{\nabla}_{X_p} Y)^{\perp}$ is the second fundamental form of f at p ($\perp$ denotes the normal projection on the orthogonal complement $\perp_p M$ of T_pM in $T_p\overline{M}$). Indeed, $s^0(X_p, Y_p)$ depends only on the value Y_p of Y at p.

The linear subspace $\perp^1_p M$ of the normal space $\perp_p M$ of M at p, spanned by the image of s^0_p, is called the *first normal space* of M at p.

The higher fundamental form s^1_p of f at p is defined by

$$s^1_p(X_p, \xi^1_p) = (\overline{\nabla}_{X_p} \xi^1)^{\perp^2}$$

for $X_p \in T_pM$, $\xi^1_p \in \perp^1_p M$ arbitrary extended to a vector field ξ^1 ($\perp^2$ is the normal projection on the orthogonal complement of $T_pM \oplus \perp^1_p M$).

We denote by $\perp^2_p M$ the linear span of the image of s^1_p and we call it the *second normal space* of M at p. Then s^1_p is a bilinear form from $T_pM \times \perp^1_p M$ into $\perp^2_p M$.

In general, if k is an integer, the *higher fundamental form* s^k_p is defined by

$$s^k_p(X_p, \xi^k_p) = (\overline{\nabla}_{X_p} \xi^k)^{\perp^{k+1}},$$

where $X_p \in T_pM$, $\xi^k_p \in \perp^k_p M$ is arbitrary extended to a vector field ξ^k and $\perp^{k+1}$ is the normal projection on the orthogonal complement of $\oplus^k_{i=0} \perp^i_p M$ (we set $\perp^0_p M = T_pM$). The $(k+1)^{th}$-*normal space* $\perp^{k+1}_p M$ is the linear span of the image of s^k_p. Clearly this process must stop at some index k since $\dim(\oplus^k_{i=0} \perp^i_p M) \leq \dim T_p\overline{M}$. The first index m such that $\perp^m_p M = \perp^{m+1}_p M = \ldots$ is called the *degree* of the immersion f at p (note that $m \geq 1$).

The submanifold M is said *nicely curved* if the dimension of each normal space $\perp^k_p M$ does not depend on $p \in M$.

If M is nicely curved, then one has a vector bundle $\perp^k M$ ($k = 0, \ldots, m$), whose fibre at p is $\perp^k_p M$. The connection $\overline{\nabla}$ induces a connection ∇^k on the vector bundle $\perp^k M$, by the formula

$$\nabla^k_{X_p} \xi^k = \perp^k (\overline{\nabla}_{X_p} \xi^k), \qquad X_p \in T_pM, \xi^k \in \Gamma(\perp^k M)$$

where $\Gamma(\perp^k M)$ is the sheaf of local sections of $\perp^k M$. Setting

$$\perp^{k-1} (\overline{\nabla}_{X_p} \xi^k) = -A^k_p(X_p, \xi^k_p),$$

we define the k^{th}-*Weingarten map*. Note that $A^1 =: A$ is the usual Weingarten map or shape operator. We have the *generalized Frenet formulas*

$$\overline{\nabla}_{X_p} \xi^k = -A^k_p(X_p, \xi^k_p) + \nabla^k_{X_p} \xi^k + s^k_p(X_p, \xi^k_p), \tag{1.1}$$

(see [16, page 250]). We set $A^0_p = 0$ and $s^m_p = 0$. Note that s^k_p and A^k_p determine tensor fields s^k and A^k.

Moreover, denoted by $\langle \, , \, \rangle$ the inner product of $\overline{M}$,

$$\langle A^k_p(X_p, \xi^k_p), \eta^{k-1}_p \rangle = \langle s^{k-1}_p(X_p, \eta^{k-1}_p), \xi^k_p \rangle. \tag{1.2}$$

The covariant derivatives of s^k and A^k are defined by the formulas

$$(\nabla s^k)(X_p, Y_p, \xi^k_p) = \nabla^{k+1}_{X_p} s^k(Y, \xi^k) - s^k(\nabla_{X_p} Y, \xi^k) - s^k(Y, \nabla^k_{X_p} \xi^k),$$
$$(\nabla A^k)(X_p, Y_p, \xi^k_p) = \nabla^{k-1}_{X_p} A^k(Y, \xi^k) - A^k(\nabla_{X_p} Y, \xi^k) - A^k(Y, \nabla^k_{X_p} \xi^k),$$

for $Y \in \Gamma(TM)$ and $\xi^k \in \Gamma(\perp^k M)$ arbitrary extensions of Y_p and ξ^k_p respectively. For any $p \in M$, ∇s^k and ∇A^k are symmetric with respect to the first two arguments. It is known (see [3]) that if M is nicely curved, then

$$\nabla s^k = 0 \text{ if and only if } \nabla A^k = 0.$$

We shall say that s^k is *parallel* in $\perp^{k+1} M$ if $\nabla s^k = 0$.

Moreover one can introduce the higher order fundamental forms S^k recursively by defining $S^0_p(v_1, v_2) = s^0(v_1, v_2)$, if $v_1, v_2 \in T_pM$ and

$$S^k_p(X_p, v_1, \ldots, v_{k+1}) = (\overline{\nabla}_{X_p} S^{k-1}(V_1, \ldots, V_{k+1}))^{\perp^{k+1}}$$

for $X_p \in T_pM$, $v_1, \ldots, v_{k+1} \in T_pM$ arbitrarily extended to vector fields $V_1, \ldots, V_{k+1}$. One can also define the covariant derivative ∇S^k of S^k and show that $\nabla S^k = 0$ if and only if $\nabla s^k = 0$.

We denote by $\mathcal{N}^{odd}M$ the bundle given by the direct sum of the odd normal bundles:

$$\mathcal{N}^{odd}M = \oplus_{i=0}^{[\frac{m}{2}]} \perp^{2i+1} M$$

and by $\mathcal{N}^{even}M$ the bundle given by the direct sum of the even normal bundles:

$$\mathcal{N}^{even}M = \oplus_{i=0}^{[\frac{m}{2}]} \perp^{2i} M.$$

Note that $\mathcal{N}^{odd}M \subset \perp M$ and $TM \subset \mathcal{N}^{even}M$. The submanifold $f(M) \subset \overline{M}$ is *substantial* if it is not contained in any totally geodesic submanifold of $\overline{M}$. For a nicely curved substantial submanifold of $\overline{M}$ we have

$$\perp M = \oplus_{i=1}^{m} \perp^i M.$$

Consider the vector bundle $\mathcal{F}$ on M whose fibre at p is $\mathcal{F}_p = \oplus_{i=0}^{m} \perp^m_p M$ and define the tensors

$$s, A : TM \otimes \mathcal{F} \to \mathcal{F}$$

by

$$s(X_p, \xi_p) = \sum_{k=0}^{m} s^k(X_p, \xi^k_p),$$
$$A(X_p, \xi_p) = \sum_{k=0}^{m} A^k(X_p, \xi^k_p),$$

with $X_p \in T_pM$ and $\xi_p = \xi_p^0 + \ldots + \xi_p^m$, $\xi_p^i \in \perp_p^i M$ $(i = 0, \ldots, m)$. We have that

$$s_{|TM\otimes\mathcal{N}^{odd}M}, A_{|TM\otimes\mathcal{N}^{odd}M} : TM \otimes \mathcal{N}^{odd} \to \mathcal{N}^{even},$$
$$s_{|TM\otimes\mathcal{N}^{even}M}, A_{|TM\otimes\mathcal{N}^{even}M} : TM \otimes \mathcal{N}^{even} \to \mathcal{N}^{odd}.$$

We can define two connections ∇^o and ∇^e on $\mathcal{N}^{odd}M$ and $\mathcal{N}^{even}M$ respectively, setting

$$\nabla^o_{X_p}\xi^o = (\overline{\nabla}_{X_p}\xi^o)^{\perp^o},$$

for $X_p \in T_pM$, $\xi^o \in \Gamma(\mathcal{N}^{odd}M)$, where $\perp^o$ is the orthogonal projection on $\mathcal{N}^{odd}M$, and

$$\nabla^e_{X_p}\xi^e = (\overline{\nabla}_{X_p}\xi^e)^{\perp^e},$$

for $X_p \in T_pM$, $\xi^e \in \Gamma(\mathcal{N}^{even}M)$. Note that we have a connection $\nabla^{\mathcal{F}}$ on the vector bundle $\mathcal{F}$ (given by the sum of the connections in each normal bundle) and, by the generalized Frenet formulas (1.1),

$$\nabla^{\mathcal{F}} = \nabla^o + \nabla^e,$$

i.e., if $\xi \in \Gamma(\mathcal{F})$ and $\xi = \xi^o + \xi^e$, where $\xi^o \in \Gamma(\mathcal{N}^oM)$ and $\xi^e \in \Gamma(\mathcal{N}^{even}M)$, we have

$$\nabla^{\mathcal{F}}_{X_p}\xi = \nabla^o_{X_p}\xi^o + \nabla^e_{X_p}\xi^e. \tag{1.3}$$

We say ξ is $\nabla^{\mathcal{F}}$-*parallel* if $\nabla^{\mathcal{F}}\xi = 0$. By (1.3), ξ is $\nabla^{\mathcal{F}}$-parallel if and only if ξ^o and ξ^e are parallel in $\mathcal{N}^{odd}M$ and $\mathcal{N}^{even}M$ respectively. By the generalized Frenet formulas (1.1) it is easy to check that $\xi^o \in \Gamma(\mathcal{N}^{odd}M)$ $(\xi^e \in \Gamma(\mathcal{N}^{even}M))$ is parallel in $\mathcal{N}^{odd}M$ $(\mathcal{N}^{even}M)$ if and only if its component in each odd (even) normal space $\perp^{2k+1} M$ $(\perp^{2k} M)$ is parallel in $\perp^{2k+1} M$ $(\perp^{2k} M)$.

Proposition 1.1. *Let $\xi \in \Gamma(\mathcal{F})$ be a $\nabla^{\mathcal{F}}$-parallel section of $\mathcal{F}$ along a curve with tangent vector X_p at p. Then*

$$\overline{\nabla}_{X_p}\xi = (s - A)(X_p, \xi_p). \tag{1.4}$$

Proof. Setting $\xi = \sum_{k=0}^m \xi^k$, $\xi^k \in \Gamma(\perp^k M)$, since (by assumption and what remarked above) $\nabla^k_{X_p}\xi^k = 0$, by the Frenet formulas we have

$$\overline{\nabla}_{X_p}\xi = \sum_{k=0}^m s^k(X_p, \xi_p^k) - A^k(X_p, \xi_p^k) = (s - A)(X_p, \xi_p).$$

Next, we prove the following

Proposition 1.2. *Let $f : M \to \overline{M}$ be a nicely curved, substantial 2-symmetric submanifold of a space $\overline{M}$ of constant curvature. For any $\xi^o \in \Gamma(\mathcal{N}^{odd}M)$ ∇^o-parallel in $\mathcal{N}^{odd}M$, $\overline{\nabla}_{X_p}\xi^o$*

is a ∇^e*-parallel section of* $\mathcal{N}^{even}M$ *and for any* $\eta^e \in \Gamma(\mathcal{N}^{even}M)$ ∇^e*-parallel in* $\mathcal{N}^{even}M$, $\overline{\nabla}_{X_p}\eta^e$ *is a* ∇^o*-parallel section of* $\mathcal{N}^{odd}M$.

Proof. Let X be any parallel extension of X_p along a curve with tangent vector Z_p at p. Then, by Theorem A and (1.4), we have

$$\begin{aligned}\nabla^e_{Z_p}(\overline{\nabla}_{X_p}\xi^o) =&\nabla^e_{Z_p}(s-A)(X,\xi^o) = \\ =&\sum_{k=1}^{[\frac{m}{2}]}(\nabla^{2k+1}_{Z_p}s^{2k+1}(X,\xi^{2k+1}) - \nabla^{2k+1}_{Z_p}A^{2k+1}(X,\xi^{2k+1})) = 0\end{aligned}$$

The other case has an analogous proof.

2. Frenet curves of osculating rank d

We recall first some notions about Frenet curves (cf. [17]). Let $\gamma : J \to \overline{M}$ be a (differentiable) curve in a Riemannian manifold $\overline{M}$, parametrized by arc length s. Denote by $\overline{\nabla}$ the Riemannian connection on $\overline{M}$, let $V_1(s) = X(s)$ be the unit tangent vector, put $k_0 = \|X\|$ and $k_1 = \|\overline{\nabla}_X X\|$, where $\| \ \|$ is the norm. If k_1 is not identically zero, then we define V_2 by $k_1 V_2 = \overline{\nabla}_X X$ on the set $I_1 = \{s \in J \mid k_1(s) \neq 0\}$. Set $k_2 = \|\overline{\nabla}_X V_2 + k_1 X\|$. If k_2 is not identically zero, we can define V_3 by $k_2 V_3 = \overline{\nabla}_X V_2 + k_1 X$ on the set $I_1 = \{s \in J \mid k_2(s) \neq 0\}$. By recursion, one then defines k_i and V_{i+1}. The well-known Frenet formulas are

$$\overline{\nabla}_X V_i + k_{i-1}V_{i-1} = k_i V_{i+1}. \tag{2.1}$$

$V_1, V_2, \ldots, V_{i+1}$ are called *Frenet vectors* and $k_0, k_1, \ldots, k_i$ are called *curvatures*.

We call $\gamma : J \to \overline{M}$ a *Frenet curve in* $\overline{M}$ *of osculating rank* d if the higher order derivatives of V_1

$$V_1(s), (\overline{\nabla}_X V_1)(s), \ldots, (\overline{\nabla}^{d-1}_X V_1)(s)$$

are linearly independent and

$$V_1(s), (\overline{\nabla}_X V_1)(s), \ldots, (\overline{\nabla}^{d}_X V_1)(s)$$

are linearly dependent in $T_{\gamma(s)}M$.

Note that in this case $\{V_1, V_2, \ldots, V_d\}$ is an orthonormal frame along γ, called the *Frenet frame* of γ.

It is easy to see that γ is a Frenet curve of osculating rank d if and only if the curvature functions $k_1, k_2, \ldots, k_{d-1}$ are nowhere zero and k_d is identically zero [16].

Strübing [17] proved the following

Proposition 2.1. *Given a Frenet curve* $\gamma : (-\epsilon, \epsilon) \to \overline{M}$ *of osculating rank* d *whose curvature functions are even functions on* $(-\epsilon, \epsilon)$, *then for any isometry* $\sigma : \overline{M} \to \overline{M}$ *with*

(a) $\sigma(\gamma(0)) = \gamma(0)$,

(b) $\sigma_*(V_i(0)) = (-1)^i V_i(0)$ *for all* $i = 1, \ldots, d,$ *the equation*

$$\sigma \circ \gamma(t) = \gamma(-t)$$

holds for all $t \in (-\epsilon, \epsilon)$.

Next we prove

Theorem 2.2. *Let* $f : M \to \overline{M}$ *be an isometric immersion of a Riemannian manifold* M *into a space of constant curvature* c*. Denote by* $\overline{\nabla}$ *the Levi-Civita connection on* $\overline{M}$ *and by* ∇ *the induced connection on* M*. Let* $\gamma : J \to M$ *be a geodesic of* M *and* $\overline{\gamma} = f \circ \gamma : J \to \overline{M}$ *be the image of* γ *in* $\overline{M}$*. Suppose that each higher fundamental form* s^k *is* ∇^{k+1}*-parallel in* $\perp^{k+1} M$ *along* γ *for all vector fields* X ∇*-parallel along* γ *and all* $\xi^k \in \Gamma(\perp^k M)$*,* ∇^k*-parallel in* $\perp^k M$ *along* γ*, for* $k = 0, \ldots, m$*. (Obviously the above holds if* $\nabla s^k = 0$*,* $k = 0, \ldots, m$*.) Then*

i) *The curve* $\overline{\gamma}$ *is a curve of osculating rank* m *and all curvature functions* $k_i(s)$ $(i = 0, \ldots, m-1)$ *are constant along* γ*.*

ii) *The Frenet m-frame* $\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_m$ *satisfies the following conditions*

(a) $\overline{V}_{2i+1}$ *are* ∇^e*-parallel sections in* $\mathcal{N}^{even}M$ *along* γ*,*

(b) $\overline{V}_{2i+2}$ *are* ∇^o*-parallel sections in* $\mathcal{N}^{odd}M$ *along* γ

for $i = 0, \ldots, [\frac{m}{2}]$*.*

Proof. We set $\overline{V}_1 = \dot{\gamma}(s) = X(s)$, thus $\overline{V}_1$ is ∇-parallel in TM and $k_0 = \|X\| = 1$ is constant along γ.

We proceed by induction. Suppose that (a) and (b) hold for all i such that $2i+1, 2i+2 \leq \rho$. Setting

$$\overline{\nabla}_X \overline{V}_\rho + k_{\rho-1}\overline{V}_{\rho-1} = k_\rho \overline{V}_{\rho+1},$$

by Proposition 1.1 we have

$$(s - A)(X, \overline{V}_\rho) + k_{\rho-1}\overline{V}_{\rho-1} = k_\rho \overline{V}_{\rho+1}. \tag{2.2}$$

The first member of (2.2) is a ∇^e-parallel vector field of $\mathcal{N}^e M$ if ρ is even and a ∇^o-parallel vector field of $\mathcal{N}^o M$ if ρ is odd, since it is the sum of either two ∇^e-parallel vector fields or two ∇^o-parallel vector fields. Thus k_ρ is constant along γ and $\overline{V}_{\rho+1}$ is either a ∇^e-parallel vector field in $\mathcal{N}^{even}M$ or a ∇^o-parallel vector field in $\mathcal{N}^{odd}M$.

Next we are able to give the

Proof of Theorem A. If $f : M \to H^N$ is 2-symmetric (substantial, nicely curved), then it is proved in [3] (in the general case of a submanifold of a space of constant curvature) that M has parallel higher fundamental forms.

For the converse we still deal with the general case of a submanifold of a space $\overline{M}$ of constant curvature. Let U and $\overline{U}$ be neighbourhoods of p in M and of $f(p)$ in $\overline{M}$ respectively. Denote by $\sigma : \overline{U} \to \overline{U}$ the local isometry of $\overline{M}$ such that $\sigma(f(p)) = f(p)$, $\sigma_{*p}(\xi^e) = -\xi^e$, for any $\xi^e \in \mathcal{N}^{even}_p M$, $\sigma_{*p}(\eta^o) = \eta^o$, for any $\eta^o \in \mathcal{N}^{odd}_p M$. Let $X_p \in T_p M$

be a unit vector and consider the geodesic $\gamma : (-\epsilon, \epsilon) \to M$ defined by $\gamma(t) = \exp_p^M(tX_p)$, for all $t \in (-\epsilon, \epsilon)$. Setting $\overline{\gamma} = f \circ \gamma$, according to Theorem 2.2, the Frenet vectors of $\overline{\gamma}$ are such that $\overline{V}_{2i}(0) \in \mathcal{N}_p^{odd}M$, $\overline{V}_{2i+1}(0) \in \mathcal{N}_p^{even}M$ and the curvatures are constant along γ. Thus $\sigma(\overline{\gamma}(0)) = \sigma(f(p) = f(p) = \overline{\gamma}(0)$ and $\sigma_{*p}\overline{V}_i(0) = (-1)^i\overline{V}_i(0)$. By Proposition 2.1 we have $\sigma(\overline{\gamma}(t)) = \overline{\gamma}(-t)$. Then σ maps $f(U)$ into itself and it is the reflection with respect to the affine subspace at p parallel to $\mathcal{N}_p^{odd}M$.

3. Examples of 2-symmetric submanifolds of the hyperbolic space

Let $\mathbf{R}^{N,1}$ be $\mathbf{R}^{N+1}$ endowed with the Lorentz pseudometric $ds^2 = -dx_1^2 + dx_2^2 + ... + dx_{N+1}^2$. In the sequence we identify $\mathbf{H}^N$ with the N-sphere of radius -1 of $\mathbf{R}^{N,1}$

$$\{X \in \mathbf{R}^{N,1} \mid \langle X, X \rangle \ = -1\},$$

where $\langle\, , \rangle$ is the inner product induced by the Lorentz pseudometric.

Let $M \to \mathbf{H}^N$ be a submanifold of the hyperbolic space $\mathbf{H}^N$. Like in [20] we define M to be *full* if it is not contained in a proper totally umbilical submanifold of $\mathbf{H}^N$. It is well-known that the latter are the intersections of hyperplanes of $\mathbf{R}^{N,1}$ with $\mathbf{H}^N$.

Note that, by Cartan's Theorem, any compact homogeneous submanifold of $\mathbf{H}^N$ is contained in a sphere (hence it is not full). Since a 2-symmetric submanifold is homogeneous, we deduce that any compact 2-symmetric submanifold of $\mathbf{H}^N$ is not full. Therefore any compact 2-symmetric submanifold of $\mathbf{H}^N$ is the composition of a 2-symmetric submanifold of the sphere S^n with the totally umbilical embedding of S^n into $\mathbf{H}^N$.

Examples of 2-symmetric submanifolds of the sphere are given in [2]. They include helical geodesic immersions and standard minimal immersions into spheres.

Let us turn now to the non-compact case. First, as an application of Theorem A, we show that any homogeneous curve in $\mathbf{H}^N$ is 2-symmetric. Indeed, let $\gamma : s \mapsto g_s \cdot p$ be a homogeneous curve in $\mathbf{H}^N$, where $g_s \in O(N, 1)$ and γ is parametrized by arc length. If $V_1 := X \cdot p$ is the (unit) tangent vector at γ at 0 (with $X = \frac{d}{ds}_{s=0} g_s$), and V_i and k_i ($k > 1$) are defined as in Section 2, then the Frenet equations

$$\overline{\nabla}_X V_i = k_i V_{i+1} - k_{i-1} V_{i-1},$$

show that $S^{i-1}(V_1, \ldots, V_1) = k_1 \cdot \ldots \cdot k_i V_{i+1}$ and $\nabla_{V_1} S^i = 0$.

This result implies that it is not difficult to find reducible non-compact 2-symmetric submanifolds of $\mathbf{H}^N$. Recall that a submanifold M of $\mathbf{H}^N \subseteq \mathbf{R}^{N,1}$ is reducible if it is an extrinsic product $(M_1 \times M_2, i_1 \times i_2)$ of submanifolds $i_1 : M_1 \to \mathbf{R}^{n_1}$, $i_2 : M_2 \to \mathbf{R}^{n_2,1}$, $n_1 + n_2 = N$. Hence to produce a full non-compact (reducible) 2-symmetric submanifold of $\mathbf{H}^N$ it suffices to consider the extrinsic product of a 2-symmetric submanifold of $\mathbf{R}^{n_1}$ and a homogeneous curve in $\mathbf{H}^{n_2}$.

To find a full irreducible non-compact 2-symmetric submanifold of $\mathbf{H}^N$ (of dimension greater than one) seems to be involved.

In general a 2-symmetric submanifold M of a space of constant curvature $\overline{M}$ is an orbit of a representation $\tau : G \to I(\overline{M})$ (where $I(\overline{M})$ is the isometry group of $\overline{M}$). To see this one can claim that, by Theorem A, a canonical connection on $TM \oplus \perp M$ ([13], [6]) is given by $\nabla \oplus \nabla^{\mathcal{F}}$ (where ∇ is the Levi-Civita connection on M and $\nabla^{\mathcal{F}}$ was defined by (1.3) in Section 1). Thus by [6, Theorem A], M is an orbit. One can see this more directly, describing also a representation having M as an orbit. To this aim, recall first that a *regular 2-structure* on M is a family of isometries of $\overline{M}$, $\{\sigma_x\}_{x\in M}$, such that:

(1) $\sigma_x(M) \subseteq M$.
(2) x is an isolated fixed point of $\sigma_{x|M}$.
(3) for any $x, y \in M$, $\sigma_x \cdot \sigma_y = \sigma_z \cdot \sigma_x$, $z = \sigma_x(y)$.

If $\{\sigma_x\}_{x\in M}$ is of finite order 2 (i.e. $\sigma^2 = id$), M is a 2-symmetric submanifold.

Let $Tr(M, \{\sigma_{x|M}\})$ be the group of the transvections, i.e. the group generated by the isometries $\sigma_{x|M}\sigma_{y|M}^{-1}$ of M. Then one can define the representation

$$\begin{aligned} \tau : Tr(M, \{\sigma_{x|M}\}) &\to I(\overline{M}) \\ \sigma_{x|M}\sigma_{y|M}^{-1} &\mapsto \sigma_x\sigma_y^{-1}. \end{aligned}$$

M is an orbit of $Tr(M, \{\sigma_{x|M}\})$ in the representation τ.

Let G be the group $I(M, \overline{M})$ of isometries of the ambient space $\overline{M}$ which leave M invariant. Clearly M is an orbit of the representation $\rho : G \to O(N, 1)$ (given by inclusion).

For 2-symmetric submanifolds of the hyperbolic space we have the following result

Lemma 3.1. *Let M be a full and irreducible 2-symmetric submanifold M of $\mathbf{H}^N$. Then M is a singular orbit of the representation $\rho : G \to O(N, 1)$.*

Proof. Suppose that $M = G/K$ is a principal orbit of ρ. Let S be the (extrinsic) homogeneous structure associated to a reductive decomposition of the Lie algebra of G (cf. [13], [6]). Let k be an element of the isotropy subgroup K of M at $p \in M$. The differential of k (which we shall still denote by k) acts on T_pM as isotropy representation and on $\perp_p M$ as slice representation. The slice representation is trivial, because M is a principal orbit (see e.g. [15, Proposition 5.4.7, page 86]). Since k induces an isometry $\mathbf{H}^N$ leaving M invariant, we have for any $X \in T_pM$, $\xi \in \perp_p M$

$$S_{kX}k\xi = kS_X\xi.$$

Now, using the fact that the slice representation is trivial we get

$$S_{kX}\xi = S_X\xi,$$

or

$$S_{(k-id)X}\xi = 0. \tag{3.1}$$

Taking $k = \sigma_p$ we get $(k - id)X = -2X$. So

$$S_X\xi = 0 \quad \text{for any } X \in T_pM,$$

which by the main result in [14] implies that M is a full irreducible isoparametric submanifold of $\mathbf{H}^N$. But this is absurd by [20].

4. A splitting theorem

In this section we prove a splitting theorem for 2-symmetric submanifolds of a space of constant curvature $\overline{M}$.

More in general we consider submanifolds of spaces of constant curvature for which s^0 is parallel in the first normal space. This means explicitly that

$$(4.1) \qquad \langle (\nabla^{\perp}_X s^0)(Y,Z), \xi\rangle = 0 \quad \text{for any } X, Y, Z \in T_pM,\ \xi \in \perp^1_p M,$$

where $\nabla^{\perp}_X s^0$ is the usual covariant derivative of s^0.

The splitting theorem we will prove gives a generalization of the results of Ferus in the case of symmetric submanifolds [8]. Our main tool will be the following homogeneous version of the Lemma of Moore [12]:

Lemma 4.1. *Let M be a homogeneous submanifold of $\overline{M}$. Then M is reducible if and only if there exists a non-trivial parallel distribution $\mathcal{H}$ on some open neighbourhood of q in M such that $s^0(\mathcal{H}, \mathcal{H}^{\perp}) = 0$.*

Theorem 4.2. *Let M be a submanifold M of a space of constant curvature $\overline{M}$ for which s^0 is parallel in the first normal space. Let ξ be a parallel normal vector field which lies in the first normal bundle $\perp^1 M$. If ξ is not umbilical, then M splits as an extrinsic product of submanifolds. In particular if the mean curvature H is parallel in the normal connection, then either M is minimal or minimal in a hypersphere or it splits.*

Proof. Suppose ξ is a parallel (locally defined) normal section which lies in the first normal bundle $\perp^1 M$. Consider then the shape operator $A^1_\xi = A_\xi$ and let $\lambda_1, \dots, \lambda_g$ be its (different) eigenvalues. If ξ is not umbilical, then $g > 1$.

Lemma 4.3. *The eigenvalues of A_ξ are constant.*

Proof. Let $Y \in \Gamma TM$ and let (e_{i_1}) be an orthonormal frame of the eigenspace E_1 of λ_1, (e_{j_2}) an orthonormal frame of the eigenspace E_2 of λ_2 and so on. Then by (4.1), since ∇e_{j_k} is orthogonal to e_{j_k} we have (for any e_{j_i} in the frame of E_i)

$$\begin{aligned} Y\lambda_i =& Y\langle s^0(e_{j_i}, e_{j_i}), \xi\rangle = \langle \nabla^{\perp}_Y s^0(e_{j_i}, e_{j_i}), \xi\rangle + \langle s^0(e_{j_i}, e_{j_i}), \nabla^{\perp}_Y \xi\rangle = \\ =& \langle (\nabla^{\perp}_Y s^0)(e_{j_i}, e_{j_i}), \xi\rangle - 2\langle s^0(\nabla_Y e_{j_i}, e_{j_i}), \xi\rangle = 0. \end{aligned}$$

Note that Lemma 4.3 holds also in the case that ξ is parallel just in the first normal space.

Lemma 4.3 implies that the eigendistributions E_i are autoparallel. Let $\eta \in \Gamma(\perp M)$. Then, since $\langle \nabla^{\perp}_X \xi, \eta\rangle = 0$ implies that $\langle R^{\perp}_{XY}\xi, \eta\rangle = 0$, by the Ricci equation we have

$$(4.2) \qquad [A_\xi, A_\eta] = 0.$$

This implies that the E_i are common eigendistributions of the family $\{A_\eta\}$ of the shape operators. Thus

$$s^0(E_i, E_j) = 0, \qquad \text{for } i \neq j. \tag{4.3}$$

Lemma 4.4. *The eigendistributions E_i are parallel.*

Proof. To prove that E_i is parallel it suffices to show that $\nabla_X Y \in \Gamma(E_i)$ for any $X \in \Gamma(E_j)$, $Y \in \Gamma(E_i)$. By (4.1) we get $(\nabla A)_\xi = 0$ and, since ξ is parallel and λ_i is constant,

$$A_\xi \nabla_X Y = \nabla_X A_\xi Y = \nabla_X(\lambda_i Y) = \lambda_i \nabla_X Y.$$

Thus $\nabla_X Y \in E_i$.

We can now apply Lemma 4.1 inductively to E_i and $E_i^\perp$, getting the first part of Theorem 4.2.

Next we consider the mean curvature vector field H (which is a section of the first normal bundle). If H is zero, M is minimal. If H (which we now assume to be nonzero) is umbilical, then M is pseudoumbilical with parallel mean curvature and by [5, Proposition 4.2, Chapter 4] M is a minimal submanifold of a hypersphere of $\overline{M}$. If H is not umbilical we can apply the above procedure to the mean curvature vector field and we get a splitting of M as $M_1 \times \ldots \times M_g \to \overline{M}_1 \times \ldots \times \overline{M}_g$ (where $\overline{M}_i$ is a space of constant curvature). Observe that any M_i is pseudoumbilical in $\overline{M}_i$.

Remark. If M_i has parallel mean curvature then by [5, Proposition 4.2] M_i is a minimal submanifold of a hypersphere of $\overline{M}_i$.

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