Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 40 (1999), No. 1, 79-87.

On Optimal Route Planning Evading Cubes in the Three Space

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Abstract. Let $\mathcal{P}$ be a finite arrangement of non-overlapping open cubes with side-lengths not exceeding 1 in the 3-dimensional Euclidean space. Let S and T be two points lying outside the open cubes. Assume one needs to find a short path emanating from S and terminating at T avoiding the cubes of $\mathcal{P}$ under the restriction that the cubes are not known prior to the search. In fact the positions and the side-lengths of the cubes become known to the searcher as the cubes are contacted. We give an algorithm to construct a path of length less than $\frac{3}{2}d + 3\sqrt{3}\log d + 5$, where $d > 3\sqrt{3}$ denotes the distance between S and T.

1. Introduction and preliminaries

Finding the shortest path is one of the most studied problems in computational geometry. We consider the following version of the problem.

Problem. *Suppose that a searcher must traverse a 3-dimensional scene with impenetrable obstacles from a start position S to a goal position T. The assumptions are the following: a) the obstacles are non-intersecting open cubes with edge-lengths not exceeding 1 such that neither S nor T are within an open cube, b) the obstacles become known only as they are contacted (that is, the searcher is navigating around the obstacles without having a map). Find a strategy which achieves a bounded ratio (the smaller the ratio the better) between the length of the path and the distance from S to T.*

The planar version of this problem. The analoguous problem in the plane, where the obstacles are squares has been considered before. That version of this shortest path

* The author was partially supported by the Hungarian National Science Foundation, grant numbers A-221/95 and T16387 and by NSF grant No. DMS-9704319.

problem where condition *b*) is replaced with the condition that all square obstacles are known prior to the search (i.e. where a map is given) was first studied by L. Fejes Tóth [4] in 1968. His study inspired several further results (see, for example the survey by G. Fejes Tóth [3]). Specifically, using a mean value argument, J. Pach [5] proved that S and T can always be connected by a path evading the cubes and having length $\leq \frac{3}{2}d + 4\sqrt{d} + 1$, where d denotes the distance between points S and T. Later, confirming a conjecture of L. Fejes Tóth, G. Fejes Tóth [2] improved the upper bound to $\frac{3}{2}d + \frac{1}{2}$. G. Fejes Tóth's bound is sharp for infinitely many values of d. To see this consider a lattice packing of open unit squares which arises from a grid of squares in horizontal rows by translating every other row of squares horizontally by $\frac{1}{2}$.

Both proofs show only the existence of short paths. Thus the question whether a strategy exists which achieves, as d grows, a ratio arbitrarily close to $\frac{3}{2}$ (asymptotically the best possible ratio) remained open. C.H. Papadimitriou and M. Yannakakis [6] had an affirmative answer to this question in the special case, when the squares are non-intersecting unit squares with sides parallel to the line ST. Later K.-F. Chan and T.W. Lam [1] settled the problem without these restrictions.

Returning to the three dimensional problem first we show that the following simple strategy achieves a ratio of 2.

Evade a cube and return to the line *ST* heuristic. *Start at S and go towards T until T is reached or a cube c blocks the way (see Figure 1). Then let X and Y be (in this order) the first and the last common points of the directed segment ST and the boundary of c. Evade c according to the following two cases:*
Case 1. If X and Y lie on neighbouring faces than go from X to the common edge of these faces on the perpendicular line and then from this edge straight to Y.
Case 2. If X and Y lie on opposite faces of the cube, then choose one edge of these faces and consider the intersection of the surface of the cube with that plane which contains X and Y and is parallel to the selected edge. Choose the shorter arc on this intersection to go from X to Y.
Continue going along the line ST until a second cube blocks the way or T is reached. Evade this cube in a similar way as the first one and then travel again towards T. Continue this process until T is reached.

In order to show that the above strategy achieves a ratio not greater than 2 it is enough to verify that the length of the path between X and Y is at most twice the distance between X and Y. In Case 1 the path between X and Y consists of two perpendicular line segments, so its length is at most $\sqrt{2}$ times the distance between X and Y. In Case 2 the intersection is a rectangle whose sides are not longer than the distance between X and Y. Therefore the shorter arc connecting X and Y on the boundary of this rectangle has length at most twice of the distance between X and Y.

The purpose of this paper is to give a strategy which achieves the ratio $\frac{3}{2}$. Note that the above mentioned planar result is a special case of our theorem.

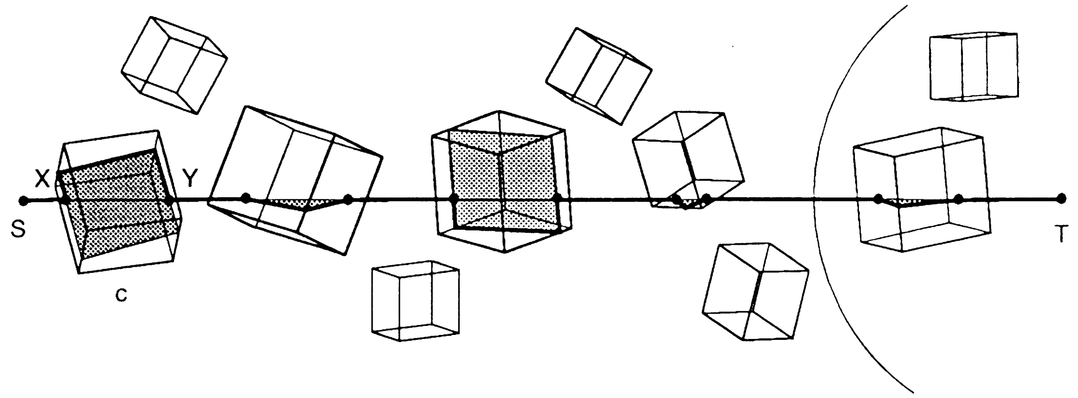

Figure 1

2. Evade a cube and go straight towards the target heuristic

Notation. Points are usually denoted by capitals. The (closed) segment with endpoints X and Y is denoted by XY. The vector from X to Y is denoted by $\overrightarrow{XY}$. The length of the segment XY is denoted by $|XY|$.

Let X and Y be two points of the space and let $\underline{d}$ be a given direction. Let $h_{\underline{d}}(X,Y) = |XY|\sin\alpha$ and let $v_{\underline{d}}(X,Y) = |XY|\cos\alpha$, where α is the angle between the vector $\overrightarrow{XY}$ and the direction $\underline{d}$. Assuming that the direction $\underline{d}$ is vertical and the searcher made a move from X to Y, we say that $v_{\underline{d}}(X,Y)$ measures the *vertical progress* and $h_{\underline{d}}(X,Y)$ measures the *horizontal detour* relative to the vertical line through X. If X and Y are boundary points of a cube obstacle, then $l(X,Y)$ denotes the length of the shortest path connecting X and Y on the boundary of the cube. We will use the following strategy:

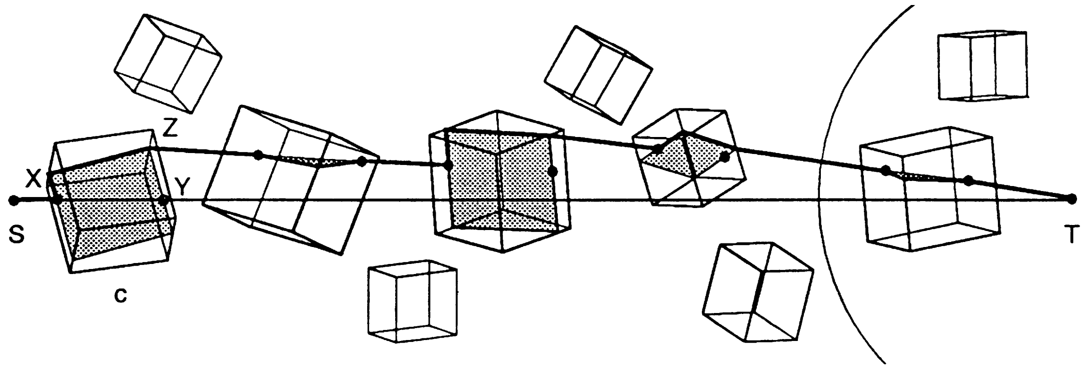

Figure 2

Evade a cube and go straight towards the target heuristic. *Initially a variable $\underline{d}$ is set to be equal to the direction of $\overrightarrow{ST}$. Start at S and*
1. Go in direction $\underline{d}$, i.e. directly towards T until the target point T is reached or a cube, say c, blocks the way (see Figure 2). In the latter case let X and Y be (in this order) the first and the last common points of the directed segment ST and the boundary of c.

2. If $|XT| < 3\sqrt{3}$, then go to T using the previously discussed simple "evade a cube and return to the line XT" strategy.
3. If $|XT| \geq 3\sqrt{3}$, then evade c according to the following two cases:
Case 1: If X and Y lie on adjacent faces of c then go from X to the common edge of these faces on the perpendicular line and then from this edge straight to Y.
Case 2: If X and Y lie on opposite faces of the cube, then choose one edge of these faces and consider the intersection of the surface of the cube with that plane which contains X and Y and is parallel to the selected edge. This intersection is a rectangle. Along the shorter arc of this rectangle's boundary go to that vertex Z, which belongs to the side of Y and satisfies the following inequalities (see Lemma 3 for the existence of such vertex Z):

$$\frac{h_{\underline{d}}(X,Z)}{v_{\underline{d}}(X,Z)} \leq 1 \quad \text{and} \quad \frac{l(X,Z)}{v_{\underline{d}}(X,Z)} \leq \frac{3}{2}.$$

4. Set the direction $\underline{d}$ to be equal to the direction pointing from the last point of the path towards the target point T. Repeat step 1.

We shall show

Theorem. *Let $\mathcal{P}$ be any arrangement of non-overlapping open cubes with edge-lengths not exceeding 1. Let S and T be two points outside the cubes at a distance $d \geq 3\sqrt{3}$. Then the "evade a cube and go straight towards the target heuristic" leads to a path evading the cubes of $\mathcal{P}$ which connects the point S with T by a path of length smaller than $\frac{3}{2}d + 3\sqrt{3}\log d + 5$.*

We need the following generalization of a simple lemma of L. Fejes Tóth [4]:

Lemma 1. *Let X be a point of the side S_1S_2 of the square $s = S_1S_2S_3S_4$ such that the ray emanating from X in the given direction $\underline{d}$ intersects the side S_3S_4. Then at least one of the vertices $S_i, i = 3, 4$ satisfies each of the following inequalities:*

$$\frac{h_{\underline{d}}(X,S_i)}{v_{\underline{d}}(X,S_i)} \leq 1 \quad \text{and} \quad \frac{l(X,S_i)}{v_{\underline{d}}(X,S_i)} \leq \frac{3}{2}.$$

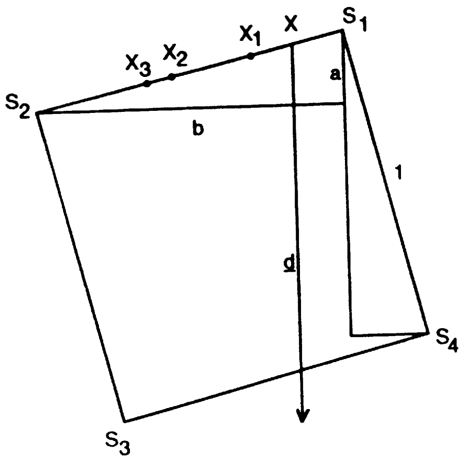

Figure 3

Proof of Lemma 1. Without loss of generality assume that s is a unit square in such a position that $\underline{d}$ is vertical and is pointing down. Assume also that the highest vertex of s is S_1. We assume that the side S_1S_2 is neither horizontal nor has a slope of 1, because otherwise Lemma 1 is obvious. For every $X \in S_1S_2$ let x denote the distance $|S_1X|$. We introduce the notation $a = v_{\underline{d}}(S_1, S_2)$ and $b = h_{\underline{d}}(S_1, S_2)$. Notice that $a < b$ and $a^2+b^2 = 1$ and thus $0 < a < \frac{1}{\sqrt{2}} < b < 1$ (see Figure 3).

Since for every $X \in S_1S_2$ we have $h_{\underline{d}}(X, S_3) < b$ and $v_{\underline{d}}(X, S_3) > b$, X satisfies $\frac{h_{\underline{d}}(X,S_3)}{v_{\underline{d}}(X,S_3)} \leq 1$, which is one of the four inequalities of Lemma 1. While X travels on the segment S_1S_2 from S_1 to S_2 the ratio $\frac{l(X,S_3)}{v_{\underline{d}}(X,S_3)}$ continuously decreases and the ratios $\frac{h_{\underline{d}}(X,S_4)}{v_{\underline{d}}(X,S_4)}$ and $\frac{l(X,S_4)}{v_{\underline{d}}(X,S_4)}$ continuously increase. Therefore to prove Lemma 1 it is enough to show that there exists a point X on the side S_1S_2 which satisfies the remaining three inequalities of Lemma 1, which can be written in the following form:

$$\text{(1)} \qquad \frac{l(X,S_3)}{v_{\underline{d}}(X,S_3)} = \frac{2-x}{a+b-ax} \leq \frac{3}{2},$$

$$\text{(2)} \qquad \frac{h_{\underline{d}}(X,S_4)}{v_{\underline{d}}(X,S_4)} = \frac{a+bx}{b-ax} \leq 1,$$

$$\text{(3)} \qquad \frac{l(X,S_4)}{v_{\underline{d}}(X,S_4)} = \frac{1+x}{b-ax} \leq \frac{3}{2}.$$

Let $X_1 \in S_1S_2$ be the point closest to S_1 which satisfies (1). As we already mentioned, the ratio $\frac{l(X,S_3)}{v_{\underline{d}}(X,S_3)}$ continuously decreases and since $\frac{l(S_2,S_3)}{v_{\underline{d}}(S_2,S_3)} = \frac{1}{b} \leq \sqrt{2} \leq \frac{3}{2}$, thus X_1 exists and it might be equal to the vertex S_1. Let $X_2 \in S_1S_2$ be the point for which in (2) equality holds. An easy computation shows that $|X_2S_1| = x_2 = \frac{b-a}{b+a} \in (0,1)$. Let $X_3 \in S_1S_2$ be the point for which in (3) equality holds. An easy computation shows that $|X_3S_1| = x_3 = \frac{3b-2}{2+3a} \in (0,1)$. Obviously (1) is the same as

$$\text{(4)} \qquad x(3a-2) \leq 3a+3b-4.$$

In view of the monotonicities mentioned above, X_1 satisfies the inequalities (1), (2) and (3) if and only if both x_2 and x_3 satisfy (4). Substituting x_2 in (4) we get an inequality equivalent to $3-2b+(3b-6)\sqrt{1-b^2}+3(1-b^2) \geq 0$, and substituting x_3 in (4) we get an inequality equivalent to $1-4b+3b^2 \leq 0$, which are, for $\frac{1}{\sqrt{2}} < b < 1$, true (see Figure 4 which shows the graph of the first function drawn by Mathematica).

Lemma 2. *Lemma 1 remains true if $s = S_1S_2S_3S_4$ is not necessarily a square but a rectangle with side lengths $|S_1S_2| \leq |S_1S_4|$.*

Proof of Lemma 2. Let S_3^* and S_4^* be points on the boundary of s such that $s^* = S_1S_2S_3^*S_4^*$ is a square. According to Lemma 1 at least one of the vertices S_i^*, $i = 3, 4$, satisfies each of the following inequalities:

$$\frac{h_{\underline{d}}(X,S_i^*)}{v_{\underline{d}}(X,S_i^*)} \leq 1 \quad \text{and} \quad \frac{l(X,S_i^*)}{v_{\underline{d}}(X,S_i^*)} \leq \frac{3}{2}.$$

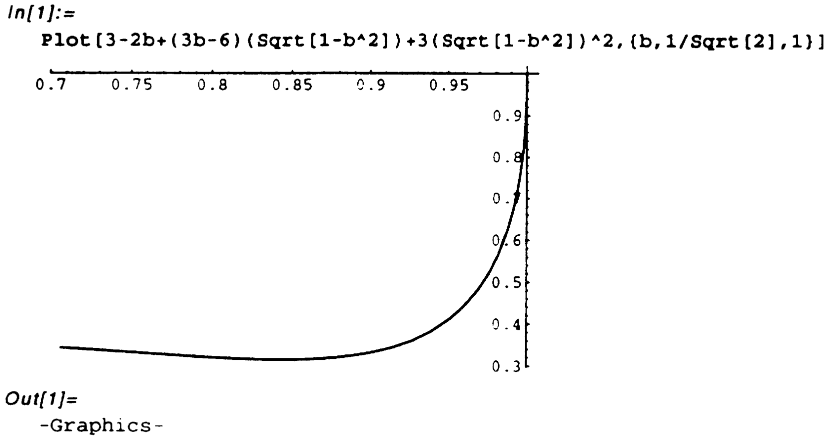

Figure 4

Since the sides containing S_3^* and S_4^* have slopes with absolute values ≥ 1, the ratios, $\frac{h_{\underline{d}}(X,S_i)}{v_{\underline{d}}(X,S_i)}$ and $\frac{l(X,S^i)}{v_{\underline{d}}(X,S^i)}$ are even smaller than their counterparts in the above inequalities.

Stair-path. A polygonal path is called *stair-path*, if the path can be partitioned into a finite number of pairs of consecutive sides which are orthogonal to each other and have equal length. We say a stair-path *aims* to the point T, if the extension of every odd numbered side passes through the point T.

Lemma 3. *Let $e > 0$ be a given number and let S and T be two points at a distance $d > 2e$ from each other. Let p be a stair-path emanating at S and aiming to T. If all sides of p have length $\leq e$ and the endpoint of p lies at a distance $\geq 2e$ from T, then the length of the path p is less than $2d + 4e \log \frac{d}{e} - 2e$.*

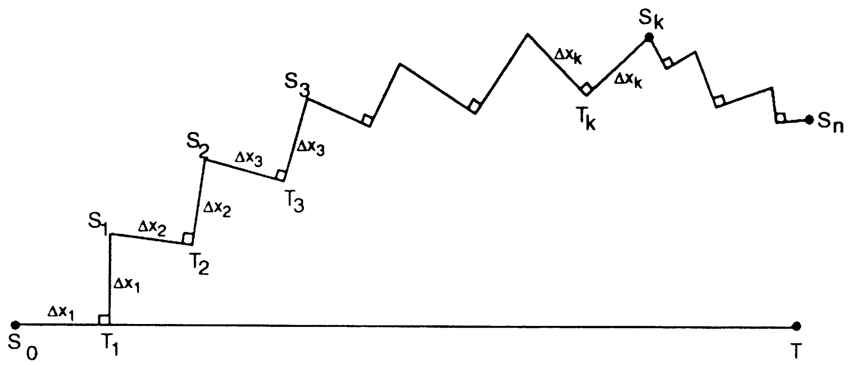

Figure 5

Proof of Lemma 3. First we restrict ourselves to the case where $e = 1$. Let $S = S_0, T_1, S_1, T_2, S_2 \ldots T_n, S_n$ be the vertices of the stair-path p. Let Δx_i be the common length of the i-th pair of equal sides, as we go along the polygonal path p starting at S_0

$(i = 1, \dots, n)$. If $i < j$ then S_j lies closer to T than S_i does, therefore $|S_nT| \geq 2$ implies that $|S_iT| > 2$ $(i = 0, \dots, n-1)$.

Then the simple side-length comparison in the triangle $S_{i-1}S_iT$ yields that

$$\frac{1}{2}\Delta x_i < |S_{i-1}T| - |S_iT| < \Delta x_i \quad (i = 1, \dots, n). \tag{5}$$

The lower bound in (5) implies that the lengths of those polygonal paths which satisfy the conditions in Lemma 3 have a supremum. Denote this supremum by L. It suffices to show the inequality in Lemma 3 for those paths whose length is close to the supremum, let say whose length is greater than $L - 2\epsilon$, where $\epsilon > 0$ is a fixed small number. Under this restriction we may assume that $2 \leq |S_nT| < 2+\epsilon$, otherwise path p could be extended with a piece of length 2ϵ without violating the properties from Lemma 3. This implies that $\frac{1}{2}l(p) = \sum_{i=1}^{n} \Delta x_i > d - (2+\epsilon)$. Let $1 \leq k \leq n$ be the smallest integer such that $\sum_{i=1}^{k} \Delta x_i \geq d - (2+\epsilon)$. First we show

$$\Sigma = \sum_{i=1}^{k} (|S_iT| - |T_iT|) < \log d. \tag{6}$$

Let $x_i = d - \sum_{j=1}^{i} \Delta x_j$, where $i = 1, \dots, k$ (see Figure 6).

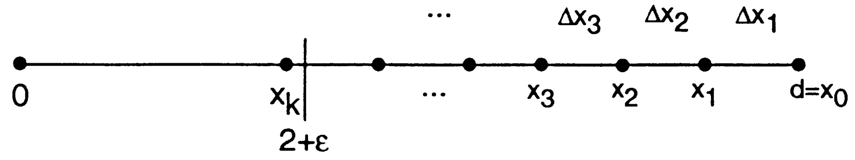

Figure 6

Clearly $x_i \leq |T_iT|$. Since $\Delta x_i \leq 1$ and $\sqrt{c^2 + x^2} - x$ is a decreasing function of x, we have that

$$\begin{aligned}
\Sigma &= \sum_{i=1}^{k} \sqrt{(\Delta x_i)^2 + |T_iT|^2} - |T_iT| \leq \sum_{i=1}^{k} \left(\sqrt{(\Delta x_i)^2 + {x_i}^2} - x_i\right) \\
&= \sum_{i=1}^{k} \frac{(\Delta x_i)^2}{\sqrt{(\Delta x_i)^2 + {x_i}^2} + x_i} < \sum_{i=1}^{k} \frac{1}{2x_i}\Delta x_i < \sum_{i=1}^{k} \frac{1}{x_{i-1}}\Delta x_i \\
&< \int_{1+\epsilon}^{d} \frac{1}{x} dx < \log d.
\end{aligned}$$

In the above chain of inequalities the step involving the definite integral is also obvious once we notice that the numbers $1+\epsilon < x_k < x_{k-1} < \ldots < x_1 < d =: x_0$ partition the interval $[1+\epsilon, d]$ so that $x_{i-1} - x_i = \Delta x_i$ $(i = 0, \dots, k)$. Since $|S_kT| = d + \Sigma - \sum_{i=1}^{k} \Delta x_i$, in view of (6) and the definition of k, we get

$$|S_kT| < d + \log d - (d - (2+\epsilon)) = \log d + 2 + \epsilon. \tag{7}$$

The point S_k divides path p into two parts. Denote the path from S to S_k by p_1 and the path from S_k to S_n by p_2. On one hand we have the simple bound

$$l(p_1) = 2\sum_{i=1}^{k} \Delta x_i < 2(d - (1+\epsilon)), \tag{8}$$

on the other hand, in view of (7) and the lower bound of (5), we have

$$l(p_2) < 4(|S_kT| - |S_nT|) < 4((\log d + 2 + \epsilon) - 2) = 4\log d + 4\epsilon. \tag{9}$$

Using (8) and (9) we get $l(p) = l(p_1) + l(p_2) < 2d + 4\log d - 2 + 2\epsilon$, as we wanted.

Finally we consider the case when e is not necessarily equal to 1. Assume for a moment that e is in fact the unit length in the plane, then the upper bound which we obtained is $2\frac{d}{e} + 4\log(\frac{d}{e}) - 2$ and that gives, using the original unit length again, $l(p) < 2d + 4e\log(\frac{d}{e}) - 2e$, as needed.

Proof of Theorem. Let X_i be the boundary point at which we first reach the i-th blocking cube and let Y_i be the last boundary point at which we have contact with this cube. Let m be the greatest index such that $|X_mT| \geq 3\sqrt{3}$. Since X_i, Y_i belong to the same cube, $|X_iY_i| \leq \sqrt{3}$ and thus $|Y_mT| \geq 2\sqrt{3}$. We divide the constructed path into two parts. Let the division point $D = Y_m$, if $|Y_mT| \leq 3\sqrt{3}$ otherwise let D be that point on the constructed path whose distance from T is $3\sqrt{3}$. Let p_1 be the path from S to D and p_2 be the path from D to T. Case 1 in our strategy clearly implies that $l(p_2) \leq 6\sqrt{3}$. Points $X_1, Y_1, X_2, Y_2, \dots X_m$ (, Y_m resp.) partition path p_1 into $2m-1$ ($2m$ resp.) pieces. Since X_i, Y_i belong to the same cube, $|X_iY_i| \leq \sqrt{3}$. We may assume that the segments Y_iX_{i+1} (which point towards T) have length $\leq \sqrt{3}$ too, otherwise we insert extra subdivision points into the segment Y_iX_{i+1} in order to satisfy this condition. Assume we get n subdivision points including both S and D. For convenience let us relabel all these subdivision points as we go along path p_1 starting at S. Specifically let they be $P_1, P_2, \dots, P_n$. Utilizing the notation of Lemma 3 consider the stair-path $p = S_0T_1S_1T_2S_2\dots T_nS_n$ emanating at S and aiming to T, such that $|S_iT_i| = v_{\overrightarrow{P_iT}}(P_iP_{i+1}) \leq \sqrt{3}$. The sides of p have lengths $\leq \sqrt{3}$, thus by Lemma 3 we have $l(p) < 2d + 4\sqrt{3}\log\frac{d}{\sqrt{3}} - 2\sqrt{3}$. It is easy to see that $l(p_1) \leq \frac{3}{2}\frac{1}{2}l(p)$, thus we have $p_1 + p_2 < \frac{3}{4}(2d + 4\sqrt{3}\log\frac{d}{\sqrt{3}} - 2\sqrt{3}) + 6\sqrt{3} < \frac{3}{2}d + 3\sqrt{2}\log d + 5$, as needed.

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Received October 16, 1997; revised version March 30, 1998