Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 39 (1998), No. 2, 467-472.

Lower Levels of Euclidean Planes

Erhard Quaisser

Universität Potsdam, Institut für Mathematik
Postfach 60 15 53, D-14415 Potsdam, Germany

Abstract. This paper begins with weak assumptions and develops its conclusions from orthogonality and reflection, in conjugation with certain meta-theoretical arguments. The symmetry of orthogonality and certain reflection-geometrical representations are of particular interest.
Some lower levels of Euclidean planes are discussed.

1. Basic structure

$(\mathbf{P}, \mathbf{L}, \perp)$ is called a *pre-Euclidean plane*, if

(A) $(\mathbf{P}, \mathbf{L})$ is an affine plane, where the *points* (the elements of $\mathbf{P}$) will be denoted by $A, B, \ldots$, the *lines* (the elements of $\mathbf{L}$) by $a, b, \ldots$ $(a, b \subset \mathbf{P})$.

(O) $\perp$ is a binary relation in $\mathbf{L}$ which induces a fixed-point-free transformation in the set of directions.

The notation $a \parallel b$ means that the lines a, b are *parallel*, and AB stands for the *line connecting* the points A, B.
If A, B, C are three noncollinear points, then there will be exactly one point D so that $ABCD$ is a parallelogram. This point will be denoted by $\operatorname{Par}(ABC)$.
The relation $\perp$ is called *orthogonality*. The symmetry of orthogonality is here deliberately not required.

(OL) For every line a and every point B there is exactly one line l through B with $l \perp a$ (the *left-perpendicular line* $\operatorname{lot}_L(a, B)$) and exactly one line r through B with $a \perp r$ (the *right-perpendicular line* $\operatorname{lot}_R(a, B)$).

2. Reflections in pre-Euclidean planes

A transformation $\sigma : \mathbf{P} \to \mathbf{P} : A \mapsto A'$ is called a *right-reflection* in the line a if

(a) $P \in a \Longleftrightarrow P' = P$;

(b) If $P \notin a$, then $a \perp PP'$; and for arbitrary points $A \in a \backslash PP'$ and $Q \in PP' \backslash a$ the identity $\mathrm{Par}(PAP') = \mathrm{Par}(QAQ') \in a$ is valid;

(c) $PQ \parallel a \Longrightarrow P'Q' \parallel a$.

(See Fig. 1).

A *left-reflection* in the line a is analogously defined.

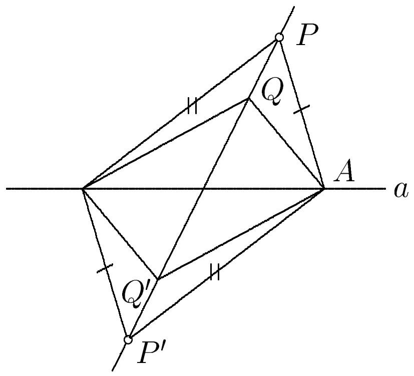

Figure 1

2.1. Lemma. *An arbitrary reflection is involutory.*

A pre-Euclidean plane is called a *pre-reflection plane*, if for any line there is exactly one right- and one left-reflection.

3. Canonical isomorphic representation, characterization-theorem

We consider the following notations of a *reflection geometry*:
Let $(S, \circ)$ be a group, I the set of all involutions of S and $\Gamma_1, \Gamma_2 \subset I$ such that $< \Gamma_1 \cup \Gamma_2 >= S$.
We associate to $(S; \Gamma_1, \Gamma_2)$ the geometric structure:
Let $\Pi := \Gamma_1 \circ \Gamma_2 \cap I$ be the set of *points*, $\Lambda := \{(\alpha) | \alpha \in \Gamma_1 \cup \Gamma_2\}$ where $(\alpha) := \{P \in \Pi | P \circ \alpha \in I\}$ the set of *lines* and for $\gamma, \delta \in \Gamma_1 \cup \Gamma_2$ let $(\gamma) \perp (\delta) :\Leftrightarrow \exists \alpha \in \Gamma_1,\ \beta \in \Gamma_2 : \alpha \circ \beta \in I$ and $(\alpha) = (\gamma),\ (\beta) = (\delta)$. $E(S) := (\Pi, \Delta, \perp)$ is called the *group plane* of $(S; \Gamma_1, \Gamma_2)$.

Now let $E = (\mathbf{P}, \mathbf{L}, \perp)$ be a pre-Euclidean plane, Γ_1 the set of right-, Γ_2 the set of left-reflections of E and let $S(E) :=< \Gamma_1 \cup \Gamma_2 >$. Then $E(S(E))$ is called *the group plane of the pre-Euclidean plane E*.

A pre-Euclidean plane E is called *reflection geometrically representable* (briefly: an *RGR-plane*) if there are bijections $\varphi : \mathbf{P} \to \Pi$ and $\lambda : \mathbf{L} \to \Lambda$ preserving incidence and orthogonality such that:

(i) If α is a right- or left-reflection in the line $g \in \mathbf{L}$ then $\lambda(g) = (\alpha)$.

Repeatedly, this canonical isomorphism is a requirement for a reflection geometric representability. (See requirement (c*) in [5] or the "1. Isomorphiesatz" in [4].)

3.1. Lemma. *Let* $E = (\mathbf{P}, \mathbf{L}, \perp)$ *be an* RGR*-plane. Then*

(1) *E is a pre-reflection plane.*

(2) *If $a, b \in \mathbf{L}$ with $a \perp b$ and $C := a \cap b$ then there exists a right- resp. left-reflection α in a resp. β in b such that $(\alpha) = \lambda(a), (\beta) = \lambda(b), \alpha \circ \beta \in I$ and $\alpha \circ \beta$ is a collineation with* $\mathrm{Fix}\,(\alpha \circ \beta) = C$*, i.e. $\alpha \circ \beta$ is a reflection in the point C.*

(3) $(\mathbf{P}, \mathbf{L})$ *is a translation plane of characteristic $\neq 2$.*

(4) *If $\alpha \in \Gamma_1$, $a :=$* $\mathrm{Fix}\,\alpha$*, $C \in a$ and σ the reflection in the point C, then $\sigma \circ \alpha$ is the left-reflection in $lot_R(a, C)$.*

Proof. Let $\alpha, \beta \in \Gamma_1 \cup \Gamma_2$ with $(\alpha) = \lambda(a), (\beta) = \lambda(b)$.
By (i) $\mathrm{Fix}\,\alpha = a$ and $\mathrm{Fix}\,\beta = b$. If $\alpha \in \Gamma_1$, we choose $a, b \in \mathbf{L}$ such that $a \perp b$. Then, $(\alpha) \perp (\beta)$, i.e. there is an $\alpha' \in \Gamma_1$ and a $\beta' \in \Gamma_2$ with $\alpha' \circ \beta' \in I$ and $(\alpha') = (\alpha), (\beta') = (\beta)$. This shows (1) and (2) up to the point $\alpha \circ \beta \in I$.
Let $g \in \mathbf{L}$ arbitrary, $d := \mathrm{lot}_R(g, C), c := \mathrm{lot}_L(d, C), \gamma \in \Gamma_1$ with $\mathrm{Fix}\,\gamma = c$, $\delta \in \Gamma_2$ with $\mathrm{Fix}\,\delta = d$ (cf. (1)). Then by $c \perp d, \gamma \circ \delta \in I \cap \Gamma_1 \circ \Gamma_2 = \Pi$ and $\varphi(C) = \varphi(c \cap d) = \lambda(c) \cap \lambda(d) = (\gamma) \cap (\delta) = \gamma \circ \delta = \varphi(a \cap b) = \alpha \circ \beta$ and $c \parallel g$. Consequently $\alpha \circ \beta(g) = \gamma \circ \delta(g) = \gamma(g)$, since $g \perp d$ and $\delta \in \Gamma_2$. The image set $\gamma(g)$ is a line, since $c \parallel g$.
Therefore, $\alpha \circ \beta(g) \in \mathbf{L}$, i.e. $\alpha \circ \beta$ is an involutory collineation and so $\mathrm{Fix}\,(\alpha \circ \beta) = C$. From (2) follows that for each point $C \in \mathbf{P}$ there exists the point reflection and so (3) is a consequence of (2). And the same is true of (4). □

Conversely, if E is a pre-Euclidean plane such that $(\mathbf{P}, \mathbf{L})$ is a translation plane with characteristic $\neq 2$, then E is an RGR-plane. (See [6].)

To sum up:

3.2. Characterization theorem. *A pre-Euclidean plane E is reflection geometrically representable if and only if the incidence structure of E is a translation plane with characteristic $\neq 2$.*

In view of this representation the following property is remarkable.

3.3. Lemma. *If E is an* RGR*-plane, $\alpha \in \Gamma_i, i \in \{1, 2\}$ and $\alpha^* : B \mapsto \alpha \circ B \circ \alpha$ for all $B \in \Pi$, then $\alpha^* \in \Gamma_i^*$ (where Γ_1^* resp. Γ_2^* denotes the right- resp. left-reflections in $E(S(E))$ and* $\mathrm{Fix}\,\alpha^* = (\alpha)$*).*

Proof. Let α be the right-reflection in the line a and $B \in \Pi(E)$, i.e., B is the reflection in the point $Q := \pi^{-1}(B) \in \mathbf{P}$. Let β be the left-reflection in the $\mathrm{lot}_R(a, Q)$. Then, $\alpha \circ \beta$, $B \circ \beta \in I$, $C := \alpha \circ \beta \in \Pi(E)$ and $\alpha \circ B \circ \alpha = C \circ B \circ C$. In a translation plane (with characteristic $\neq 2$) $C \circ B \circ C$ is a reflection in a point, i.e., $\alpha \circ B \circ \alpha \in \Pi(E)$. For the rest see Lemma 3.1.□

The property 3.3 is nearly the requirement for a concept of the reflection-geometric representation. (See requirement (d_1) of the representation in [5].)
If we assume the existence of a representation without requirement (i), then from the property 3.3 follows (i).

A lower level of Euclidean planes is given by the RGR-plane.
In every translation plane $(\mathbf{P}, \mathbf{L})$ with characteristic $\neq 2$ there is a relation $\perp$ such that $(\mathbf{P}, \mathbf{L}, \perp)$ is an RGR-plane.
Therefore, a wide spectrum of levels of Euclidean planes exists up to the Euclidean plane over fields. In the literature there are numerous relation-theoretical, algebraic, metatheoretical (representation-theoretical), or other requirements for the characterization of a specific level. (See publications by F. Bachmann, R. Baer, H. Karzel, P.E. Kustaanheimo, H. Lenz, H. Naumann, W. Rautenberg, K. Reidemeister, J. T. Schmidt, E. Schröder, K. Schütte, R. Stanik, H. Struve, among others.)

In this paper we describe only the levels at which the right- or left-reflections are collineations or special orthogonal collineations. In particular, we are interested in a condition for the symmetry of the orthogonality.

4. Pre-Euclidean planes in which the reflections in lines are collineations

Right- or left-reflections which preserve the collinearity are special axial affinities ("Schräg-spiegelungen" [5]): The reflection-direction together with the direction of the axis of reflection presents in each case a pair of orthogonal directions.

Let (dP) denote the following special case of the Desargues theorem:

(dP) Suppose that $A_1A_2A_3$ and $B_1B_2B_3$ are triangles, that g_1, g_2, g_3 are distinct lines with the common point O and that $A_i, B_i \in g_i, \notin g_k (i \neq k; i, k = 1, 2, 3)$. If $A_1A_2 \parallel B_1B_2 \parallel g_3$ and $A_2A_3 \parallel B_2B_3 \parallel g_1$, then $A_3A_1 \parallel B_3B_1$. (See Fig. 2, *parallelogram-desarguesian*)

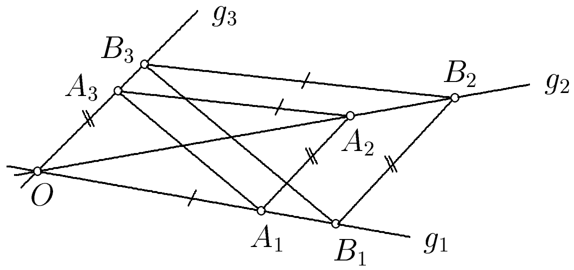

Figure 2

In every affine Moufang-plane the property (dP) is valid [3].
Conversely, every translation plane (with characteristic $\neq 2$) satisfying property (dP) is a Moufang-plane [5].

By $(\text{dP})_\perp$ we denote the special case of (dP) with the additional assumption $g_2 \perp A_1A_3$.

4.1. Theorem. *Let $(\mathbf{P}, \mathbf{L}, \perp)$ be a pre-reflection plane. Each reflection is a collineation if and only if in the plane the property $(dP)_\perp$ is valid.*

Proof. a) Suppose that every reflection is a collineation and that the premises of $(\text{dP})_\perp$ are valid, see Fig. 2. We consider the right-reflection α in the line g_2. Then $\alpha(A_1) = A_3$ and $\alpha(B_1) = B_3$, because α is a collineation. Consequently, $A_3A_1 \parallel B_3B_1$, i.e., the property $(\text{dP})_\perp$ is valid.
b) The properties $(\text{dP})_\perp$ and (c) of Section 2 imply that a reflection is a collineation. □

Another characterization is the following:

4.2. Theorem. *In an* RGR-*plane* $E = (\mathbf{P}, \mathbf{L}, \perp)$ *every reflection is a collineation if the following property is valid:*

(Inv') *For all* $\alpha, \beta \in \Gamma(E)$ *the set* $(\beta)^\alpha = \{P^\alpha : P \in \Pi, P \circ \beta \in I\}$ *is a line in* Λ.

Proof. If α is a right-reflection in E, then α^* is a right-reflection in the plane $E(S(E))$ (see Lemma 3.3). By (Inv') the reflection α^* is a collineation. The canonical isomorphism from $E(S(E))$ to E then shows that α is also a collineation in E. An analogous statement is valid for a left-reflection in E.

The converse can be proved in the same way. □

Remark. The property (Inv') is a special case of the well-known fundamental requirement in reflection geometry that the system Γ of generators is invariant under the inner automorphism:

(Inv) If $\alpha, \beta \in \Gamma$, then $\alpha^\beta (= \beta\alpha\beta) \in \Gamma$. ("Grundannahme", [1])

5. Pre-reflection planes in which the reflections in lines are orthogonal collineations

Some remarkable consequences follow from the existence of a single right- or left-reflection in pre-Euclidean planes:

5.1. Lemma. *If there exists a right-reflection* α *in the line* a *that is an orthogonal collineation and if* $a \perp b$, *then* $b \perp a$.

Proof. Let c be the right-perpendicular from the intersection S of a, b to the line b. From $\alpha(b) = b$ and $\alpha(b) \perp \alpha(c)$ follows $b \perp \alpha(c)$. Then, $\alpha(c) = c$, because there exits only one right-perpendicular from S to the line b. Consequently, $c = a$ and hence $b \perp a$ or $a \perp c$. The second case is not possible, as $a \parallel b$ contradicts the assumption $a \perp b$. □

5.2. Corollary. *If any reflection is an orthogonal collineation, then the orthogonality relation is symmetrical.*

A pre-reflection plane is called a *reflection plane* if each reflection in a line is an orthogonal collineation.

Reflection planes are investigated in [7], where there following result is shown:

5.3. Theorem.

(a) *If* $(\mathbf{P}, \mathbf{L})$ *is a Moufang-plane with characteristic* $\neq 2$ *and* a, b *are intersecting lines, then there exists an orthogonality relation* $\perp$ *in* $\mathbf{L}$ *such that* $a \perp b$ *and* $(\mathbf{P}, \mathbf{L}, \perp)$ *is a reflection plane.*

(b) *Let* $(\mathbf{P}, \mathbf{L})$ *be an affine plane. If for each pair of intersecting lines* a, b *there exists an orthogonal relation* $\perp$ *in* $\mathbf{L}$ *such that* $a \perp b$ *and* $(\mathbf{P}, \mathbf{L}, \perp)$ *is a reflection plane, then* $(\mathbf{P}, \mathbf{L})$ *is a Moufang-plane with characteristic* $\neq 2$.

The following strengthening of Theorem 4.2 is based on the canonical isomorphism in [7]:

5.4. Theorem. *Let* $E=(\mathbf{P}, \mathbf{L}, \perp)$ *be an* RGR-*plane. In* E *any reflection in a line is an orthogonal collineation, iff the "Grundannahme"* (Inv) *is valid.* (See the above remark in connection with Theorem 4.2.)

References

[1] Bachmann, F.: *Aufbau der Geometrie aus dem Spiegelungsbegriff.* Springer Verlag, 2nd ed. 1973.

[2] Folda, K.; Quaisser, E.: *Reflection geometrical aspects in some special affine planes.* Wiss. Z. PH Potsdam **34** (1990) (1), 99–104.

[3] Lingenberg, R.: *Grundlagen der Geometrie.* BI-Verlag, 2nd ed. 1976.

[4] Quaisser, E.: *Zum Aufbau der affinen Ebenen aus dem Spiegelungsbegriff.* Zeitschr. f. math. Logik und Grundl. d. Math. **27** (1981), 131–140.

[5] Quaisser, E.: *Darstellbarkeitsfragen im Rahmen der ebenen pseudoeuklidischen Geometrie.* Preprint 92/15 (1992), Univ. Potsdam.

[6] Quaisser, E.: *Zum Stufenaufbau von Translationsebenen in spiegelungsgeometrischer Darstellung.* Zeitschr. f. math. Logik und Grundl. d. Math. **34** (1988), 89–95.

[7] Quaisser, E.: *Volle gruppentheoretische Darstellung metrisch-affiner Ebenen auf der Grundlage von Orthogonalspiegelungen.* Wiss. Z. PH Potsdam **27** (1983), 43–51.

[8] Quaisser, E.: *Schwache Kongruenzebenen.* Beiträge Algebra Geom. **38** (1997) (2), 315–323.

[9] Schröder, E.: *Metric geometry.* In: Handbook of incidence geometry (Ed. F. Buekenhout), Elsevier 1995, 945–1013.

[10] Schütte, K.: *Eine Winkelmetrik in der affin-orthogonalen Ebene.* Math. Ann. **130** (1955), 183–195.

[11] Struve, H.: *Affine Ebenen mit Orthogonalitätsrelation.* Zeitschr. f. math. Logik und Grundl. d. Math. **30** (1984), 131–140.

Received August 25, 1997, revised version December 8, 1997