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Classification of Dodecahedral Space Forms

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1. Introduction

Considering the regular dodecahedron tilings in the spaces of constant curvature we look for the fixed point free discrete transformation groups that act simply transitively on the tiles. The isometries of a group, mapping a distinguished tile onto its neighbours, identify the faces and determine a dodecahedral space form, moreover, these transformations generate this fundamental group. In order to find all these groups, in this paper we apply a polyhedron algorithm, which was developed theoretically by E. Molnár [5, 7] and implemented to computer by the author [6, 7, 9, 10]. Because of the large number of cases we modify the general algorithm and apply it for finding all the non-equivariant face identifications of a regular polyhedron, and for choosing those ones that generate fixed point free groups, if the number of polyhedra around the edges is given. In this way we obtain the fundamental groups of dodecahedral space forms given by generators and relations. We classify these non-equivariant forms according to the obvious orientability and to the first homology groups. When the vertices are ideal points we shall take into consideration the so-called cusp structure refining our classification. The fundamental groups in different classes will be non-isomorphic, but we shall leave the general problem of isomorphism within the classes open for later papers. The isomorphism problem has already been solved for hyperbolic octahedron spaces in [9], recently. So this paper gives a complete list of regular dodecahedral space forms, but in some small classes the difference of the manifolds (up to isometry) cannot be guaranteed yet. From the results we emphasize that we improved the list of compact dodecahedron spaces by L. A. Best in [1], and we extended the investigations for non-orientable space forms, too.

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2. Regular dodecahedron tilings

In the spherical space $\mathbf{S}^3$ and in the hyperbolic space $\mathbf{H}^3$ we consider the regular face-to-face dodecahedron tilings $\mathcal{D}^{(n)}$ in which there are n dodecahedra around each edge. The dihedral angles of a dodecahedron $D^{(n)}$ in $\mathcal{D}^{(n)}$ are $\frac{2\pi}{n}$. Let $k^{(n)}$ denote the number of dodecahedra around the vertices. There are four tilings of this kind. These are as follows.

The tiling $\mathcal{D}^{(3)}$ is realized in $\mathbf{S}^3$ (as the projection of the regular 120 cell onto the surface of the 4-dimensional sphere from its centre). The symmetry group $\operatorname{Sym}\mathcal{D}^{(3)}$ is the Coxeter group of diagram

$$\bigcirc\overset{5}{\text{———}}\bigcirc\overset{3}{\text{———}}\bigcirc\overset{3}{\text{———}}\bigcirc.$$

The number of dodecahedra around the vertices is $k^{(3)} = 4$; the neigbourhood of any vertex is tetrahedronlike.

The tiling $\mathcal{D}^{(4)}$ appears in $\mathbf{H}^3$. The diagram of its symmetry group $\operatorname{Sym}\mathcal{D}^{(4)}$ is

$$\bigcirc\overset{5}{\text{———}}\bigcirc\overset{3}{\text{———}}\bigcirc\overset{4}{\text{———}}\bigcirc,$$

and the neighbourhood of the vertices is octahedronlike, $k^{(4)} = 8$.

The tiling $\mathcal{D}^{(5)}$ also appears in $\mathbf{H}^3$. The diagram of its symmetry group $\operatorname{Sym}\mathcal{D}^{(5)}$ is

$$\bigcirc\overset{5}{\text{———}}\bigcirc\overset{3}{\text{———}}\bigcirc\overset{5}{\text{———}}\bigcirc,$$

and the neighbourhood of the vertices is icosahedronlike, $k^{(5)} = 20$.

Finally, the tiling $\mathcal{D}^{(6)}$ of $\mathbf{H}^3$ contains noncompact dodecahedra whose vertices are ideal points lying on the absolute, but the tiles have finite volume. (See [8] for hexagonal bipyramids and [11] for cubes.) The tiling has the symmetry group $\operatorname{Sym}\mathcal{D}^{(6)}$ of Coxeter diagram

$$\bigcirc\overset{5}{\text{———}}\bigcirc\overset{3}{\text{———}}\bigcirc\overset{6}{\text{———}}\bigcirc.$$

Since the vertices are ideal points (so called parabolic vertices or ends), all its edges concurrent to such an ideal vertex belong to a parabolic bundle. A horosphere centred at this vertex is orthogonal to this bundle and intersects each tile around the vertex in a horospherical triangle. Since a horosphere is isometric to the Euclidean plane and the angles are congruent to the dihedral angles of $D^{(6)}$ (which is $\frac{2\pi}{6}$), the triangles are regular. Thus, the neighbourhood of any vertex corresponds to the regular triangle tiling $\{3, 6\}$ of the Euclidean plane, $k^{(6)} = \infty$.

3. How to generate fixed point free groups acting on a tiling

Considering a tiling $\mathcal{D}^{(n)}$ we look for the fixed point free discrete isometry groups that act transitively on the tiles. Let $\mathcal{G}^{(n)}$ be a group of this kind, and let us fix a dodecahedron $D^{(n)}$ in $\mathcal{D}^{(n)}$. Since fixed points cannot be inside $D^{(n)}$, the action of $\mathcal{G}^{(n)}$ on $\mathcal{D}^{(n)}$ must be simply transitive, so $D^{(n)}$ is a fundamental domain of $\mathcal{G}^{(n)}$. Similarly, because fixed points cannot be inside the faces of $D^{(n)}$ either, each face must be identified with another face by the transformation and its inverse that map $D^{(n)}$ onto its neighbours along these faces. In this way $\mathcal{G}^{(n)}$ determines a pairing of the 12 faces of $D^{(n)}$ so that each of the 6 pairs contains different faces. Moreover, an isometry and its inverse that identify the paired faces are attached to each pair. Conversely, the face pairing with the identifying transformations generate the group $\mathcal{G}^{(n)}$ geometrically (in general, this generator system is not independent as the following relations may yield).

4. Defining relations by the Poincaré algorithm

We consider a dodecahedron tiling $\mathcal{D} := \mathcal{D}^{(n)}$ and fix a tile $D := D^{(n)}$ in it. We give a face pairing of D and attach an isometry $\mathbf{g}$ and its inverse $\mathbf{g}^{-1}$ to each pair (g, g^{-1}) (where $g \neq g^{-1}$), so that $\mathbf{g}$ maps g^{-1} onto g and D onto its neighbour $D^{\mathbf{g}}$ along g. Then $\mathbf{g}^{-1}$ maps g onto g^{-1} and D onto $D^{\mathbf{g}^{-1}}$ along g^{-1}. Let $\mathcal{F}$ denote the set of these identifying transformations (together with their inverses).

We look for the defining relations of the group $\mathcal{G}$ generated by $\mathcal{F}$ and apply the so-called Poincaré algorithm. (We do not know yet whether $\mathcal{G}$ is a fixed point free space group.) We choose a directed edge e_1 of D and try to form the transformations (as products of the generators) that map D onto its neigbours along e_1 one after the other, surrounding the edge.

There are two faces along e_1 in D, let g_1 be one of them. The isometry $\mathbf{g}_1 \in \mathcal{F}$ maps $D = D^{\mathbf{1}}$ onto $D^{\mathbf{g}_1}$ along g_1 and maps the edge $e_1^{\mathbf{g}_1^{-1}} =: e_2$ of the face g_1^{-1} onto e_1.

Now we consider the face $g_2^{\mathbf{g}_1}$ of $D^{\mathbf{g}_1}$ along the directed edge $e_2^{\mathbf{g}_1} = e_1$, where $g_2^{\mathbf{g}_1} \neq (g_1^{-1})^{\mathbf{g}_1} \equiv g_1$. The isometry $\mathbf{g}_1^{-1}\mathbf{g}_2\mathbf{g}_1$ $(\mathbf{g}_2 \in \mathcal{F})$ maps $D^{\mathbf{g}_1}$ onto $(D^{\mathbf{g}_1})^{\mathbf{g}_1^{-1}\mathbf{g}_2\mathbf{g}_1} = D^{\mathbf{g}_2\mathbf{g}_1}$ along $g_2^{\mathbf{g}_1}$. So the transformation $\mathbf{g}_2\mathbf{g}_1$ maps $D^{\mathbf{1}}$ onto its neighbour $D^{\mathbf{g}_2\mathbf{g}_1}$ along the edge e_1 and it maps the edge $e_1^{\mathbf{g}_1^{-1}\mathbf{g}_2^{-1}} =: e_3$ of $D^{\mathbf{1}}$ onto e_1.

We continue this procedure considering the face $g_3^{\mathbf{g}_2\mathbf{g}_1}$ along the directed edge $e_3^{\mathbf{g}_2\mathbf{g}_1} = e_1$ in $D^{\mathbf{g}_2\mathbf{g}_1}$, where $g_3^{\mathbf{g}_2\mathbf{g}_1} \neq (g_2^{-1})^{\mathbf{g}_2\mathbf{g}_1} \equiv g_2^{\mathbf{g}_1}$. We apply the isometry $\mathbf{g}_1^{-1}\mathbf{g}_2^{-1}\mathbf{g}_3\mathbf{g}_2\mathbf{g}_1$ $(\mathbf{g}_3 \in \mathcal{F})$, which maps $D^{\mathbf{g}_2\mathbf{g}_1}$ onto its neighbour $D^{\mathbf{g}_3\mathbf{g}_2\mathbf{g}_1}$ along $g_3^{\mathbf{g}_2\mathbf{g}_1}$. Mapping $D^{\mathbf{1}}$ onto its neighbour $D^{\mathbf{g}_3\mathbf{g}_2\mathbf{g}_1}$ along e_1, the transformation $\mathbf{g}_3\mathbf{g}_2\mathbf{g}_1$ maps the edge $e_1^{\mathbf{g}_1^{-1}\mathbf{g}_2^{-1}\mathbf{g}_3^{-1}} =: e_4$ of $D^{\mathbf{1}}$ onto e_1.

Finally, the isometry

$$\mathbf{g}_1^{-1}\mathbf{g}_2^{-1}\cdots\mathbf{g}_{l-1}^{-1}\mathbf{g}_l\mathbf{g}_{l-1}\cdots\mathbf{g}_2\mathbf{g}_1 \quad (\mathbf{g}_1, \ldots, \mathbf{g}_l \in \mathcal{F})$$

maps $D^{\mathbf{g}_{l-1}\cdots\mathbf{g}_2\mathbf{g}_1}$ onto $D^{\mathbf{g}_l\mathbf{g}_{l-1}\cdots\mathbf{g}_2\mathbf{g}_1}$ along $g_l^{\mathbf{g}_{l-1}\cdots\mathbf{g}_2\mathbf{g}_1}$. The transformation $\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1$ maps $D^{\mathbf{1}}$ onto its neighbour $D^{\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1}$ along e_1, mapping $e_1^{\mathbf{g}_1^{-1}\mathbf{g}_2^{-1}\cdots\mathbf{g}_l^{-1}} =: e_{l+1}$ of $D^{\mathbf{1}}$ onto e_1. Thus, the directed edge e_{l+1} is identical with e_1, moreover the face $g_{l+1}^{\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1}$, which is different from $(g_l^{-1})^{\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1} \equiv g_l^{\mathbf{g}_{l-1}\cdots\mathbf{g}_2\mathbf{g}_1}$, along $e_{l+1}^{\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1} = e_1^{\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1}$ is identical with $g_1^{\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1}$ in $D^{\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1}$.

During this algorithm we put together a collection of l dodecahedra $D = D^{\mathbf{1}}$, $D^{\mathbf{g}_1}$, $\ldots$, $D^{\mathbf{g}_{l-1}\cdots\mathbf{g}_2\mathbf{g}_1}$, and we get a cycle $\{e_1, e_2, \ldots, e_l\}$ of directed edges, moreover, a product (a so-called *cycle transformation*) $\mathbf{r} := \mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1$ of the generators, which is metrically a rotation about e_1 as $g_1 \mapsto g_1^{\mathbf{g}_l\cdots\mathbf{g}_2\mathbf{g}_1}$ indicates (see [5] for more details).

Let us take into consideration that the generated group $\mathcal{G}$ should be a fixed point free space group. Then a proper rotation axis, whose all points are fixed, cannot appear. Moreover, oppositely directed edges cannot occur in the cycle, because if an edge and its image occur in opposite directions (then each edge does likewise), its midpoint would be fix (as inversion centre). So in order that $\mathcal{G}$ be a fixed point free space group, it is necessary that l should be equal to n (the number of dodecahedra around the edges in the tiling), and the cycle $\{e_1, e_2, \ldots, e_n\}$ should contain different edges. Thus, we get the defining relation $\mathbf{r} = \mathbf{g}_l \cdots \mathbf{g}_2 \mathbf{g}_1 = \mathbf{1}$.

In order to get the other edge cycles defining new relations, we choose a next edge $e \notin \{e_1, e_2, \ldots, e_l\}$, and starting with e we apply the previous method until each edge has occurred in a cycle. Since each cycle has to contain exactly $l = n$ different edges, n has to be a divisor of 30 (the number of edges of a dodecahedron), and the number of cycles is $30/n$. This criterion excludes the existence of fixed point free $\mathcal{G}$ acting on $\mathcal{D}^{(4)}$ (i.e. n cannot be equal to 4).

5. Fixed points at proper vertices

Similarly to the previous method we could describe also the tilings around the vertices determining the vertex stabilizer subgroups of $\mathcal{G}$ (see Section 8, moreover [5]). Since we look for only fixed point free groups, these stabilizers must be trivial. This means that the transformations, mapping D onto its neighbour around a vertex in $\mathcal{D} = \mathcal{D}^{(n)}$, cannot map this vertex onto itself, namely, exactly $k^{(n)}$ different vertex domains form the local tiling around each vertex. If we add this necessary condition to the previous ones from Section 4, we can exclude the existence of fixed points.

Let the faces be given by their vertex cycles:

$$\begin{array}{rcl} g & = & V_1\, V_2\, V_3\, V_4\, V_5 \\ g^{-1} & = & W_1 W_2 W_3 W_4 W_5 \\ & \vdots & \end{array}$$

Now we consider a face pairing $(g,\, g^{-1})$, ... with the identifying isometries $(\mathbf{g},\, \mathbf{g}^{-1})$, ..., and describe how the isometries map the vertices:

$$\begin{array}{rcccc} \mathbf{g} & : & W_1 W_2 W_3 W_4 W_5 & \mapsto & V_{i_1}\, V_{i_2}\, V_{i_3}\, V_{i_4}\, V_{i_5} \\ \mathbf{g}^{-1} & : & V_1\, V_2\, V_3\, V_4\, V_5 & \mapsto & W_{j_1} W_{j_2} W_{j_3} W_{j_4} W_{j_5} \\ & & & \vdots & \end{array}$$

Starting with the set $\mathcal{V} = \{W_1\}$ we look for W_1 on the left side of the previous mappings, and add the corresponding vertex from the right side to $\mathcal{V}$. So we get $\mathcal{V} = \{W_1, V_{i_1}, \ldots\}$ at the end of this step. Then, starting with the second element V_{i_1} of $\mathcal{V}$, we add its images to $\mathcal{V}$, and so on while we can find a new vertex as the image of any element of $\mathcal{V}$. In this way, finally, $\mathcal{V}$ contains the class of vertices whose image domains surround W_1 combinatorially. Taking the metric conditions into consideration, the cardinality of $\mathcal{V}$ must be equal to $k^{(n)}$. Then we consider a vertex $V \notin \mathcal{V}$ and look for its equivalent vertices, and so on. We assign to each vertex its class. Since each class has to contain $k^{(n)}$ vertices, $k^{(n)}$ must be a divisor of 20 (the number of vertices of the dodecahedron). This criterion excludes the tiling $\mathcal{D}^{(4)}$ again.

Remarks. In the case of $\mathcal{D}^{(6)}$ we do not have to examine the problem of fixed points at the vertices because they are ideal points lying on the absolute. Moreover, it would be enough to guarantee the correct number in each edge class, if we did not look for the defining relations, too.

6. Finding non-equivariant generator systems

First we look for all the non-equivariant pairings of the 12 faces. A face pairing π is a permutation of the faces formed by 6 disjoint transpositions. We consider the elements $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_{120}$ of $\operatorname{Sym} D$ (the symmetry group of the dodecahedron) and create the face permutations $\sigma_1, \sigma_2, \ldots, \sigma_{120}$ determined by them. (These permutations can be computed independently of the space in which the regular dodecahedron D is given.) We say that two pairings π_1 and π_2 are equivariant if a symmetry $\mathbf{s}_i$ transforms π_1 onto π_2, namely $\pi_2 = \sigma_i^{-1}\pi_1\sigma_i$.

Enumerating all the face pairings, we collect the non-equivariant ones into a set Π, which is the empty set in the beginning. We consider the next pairing π in turn and check whether the set of conjugacies

$$\{\sigma_1^{-1}\pi\sigma_1, \quad \sigma_2^{-1}\pi\sigma_2, \quad \ldots, \quad \sigma_{120}^{-1}\pi\sigma_{120}\}$$

and Π are disjoint. If so, then we put π into Π $(\Pi := \Pi \cup \{\pi\})$. Having run the programme of this algorithm on our computer, we obtained 147 non-equivariant face pairings among the 10 395 ones.

Now we choose a tiling $\mathcal{D}^{(n)}$ from $\mathcal{D}^{(3)}$, $\mathcal{D}^{(5)}$ and $\mathcal{D}^{(6)}$ (fixing also the space). We consider the face pairings $\pi_1, \pi_2, \ldots, \pi_{147} \in \Pi$ one after the other and equip each pairing π_j $(j \in \{1, 2, \ldots, 147\})$ with identifying transformations, attaching an isometry and its inverse to each pair (as we have seen above), in all possible manner. We enumerate these systems of transformations. Since two pentagons can be identified in 10 different ways, the 6 face pairs involve a million cases.

We collect the non-equivariant systems that generate fixed point free discrete transformation groups acting simply transitively on $\mathcal{D}^{(n)}$ into a set $\mathcal{F}_j^{(n)}$, which is the empty set in the beginning. We consider the next system $\mathcal{F} = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{12}\}$ (containing the identifying isometries) in turn. First we form the edge cycles and the equivalence classes of vertices, to decide whether the generated group is suitable, as we have seen in Section 4 and 5. If so, then we create the set

$$\{\mathbf{s}_1^{-1}\mathcal{F}\mathbf{s}_1, \quad \mathbf{s}_2^{-1}\mathcal{F}\mathbf{s}_2, \quad \ldots, \quad \mathbf{s}_{120}^{-1}\mathcal{F}\mathbf{s}_{120}\}$$

of conjugacies by the symmetries, and check whether this set and $\mathcal{F}_j^{(n)}$ are disjoint, in other words we examine if $\mathcal{F}$ is equivariant with an earlier generator system in $\mathcal{F}_j^{(n)}$. If not, then we put $\mathcal{F}$ into $\mathcal{F}_j^{(n)}$ $(\mathcal{F}_j^{(n)} := \mathcal{F}_j^{(n)} \cup \{\mathcal{F}\})$.

Enumerating each pairing π_j of Π, we get the sets $\mathcal{F}_1^{(n)}$, $\mathcal{F}_2^{(n)}$, $\ldots$, $\mathcal{F}_{147}^{(n)}$, and their union

$$\mathcal{F}^{(n)} := \bigcup_{j=1}^{147} \mathcal{F}_j^{(n)}$$

contains the generator systems of the desired groups. Having examined the 147 million cases by computer, we got these systems. As a result we obtained the numbers: $|\mathcal{F}^{(3)}| = 2$, $|\mathcal{F}^{(5)}| = 12$ and $|\mathcal{F}^{(6)}| = 78$.

Remark. We mention that our algorithm works combinatorially [10]. We substitute the corresponding automorphisms of the incidence structure for the symmetries and the corresponding incidence preserving isomorphisms between faces for the identifying isometries. Since we have assumed that the dodecahedron is regular, the correspondence between the combinatorial mappings and the isometries is one-to-one.

7. Classification of the groups

We enumerate the generator systems contained by $\mathcal{F}^{(n)}$. Applying the Poincaré algorithm we obtain the defining relations in each case. Let us consider a system from $\mathcal{F}^{(n)}$, its identifying isometries generate a fixed point free space group $\mathcal{G}^{(n)}$. Thus, the identification of faces determines a space form $\mathbf{S}^3/\mathcal{G}^{(n)}$ $(n = 3)$ or $\mathbf{H}^3/\mathcal{G}^{(n)}$ $(n = 5, 6)$. We can compute its first homology group $\mathcal{H}_1^{(n)}$, which is the Abelianization (the commutator factor group) of the fundamental group $\mathcal{G}^{(n)}$:

$$\mathcal{H}_1^{(n)} = \mathcal{G}^{(n)}/[\mathcal{G}^{(n)}, \mathcal{G}^{(n)}].$$

If the first homology groups of two manifolds are different, then the fundamental groups cannot be isomorphic. First, we include two identifications (or generator systems) from $\mathcal{F}^{(n)}$ into the same class in the course of our classification if their first homology groups are isomorphic. In this way we obtain the following results.

Theorem 1. *There are 2 non-equivariant face identifications of $D^{(3)}$ in $\mathcal{F}^{(n)}$ yielding fixed point free fundamental dodecahedron tilings in $\mathbf{S}^3$ (Section 9). Both of them determine orientable spherical manifolds. Since their first homology groups are different, the fundamental groups are non-isomorphic, the manifolds are not isometric.*

Remarks. $\mathbf{S}^3/\mathcal{G}_1^{(3)}$ is the spherical dodecahedral space of H. Poincaré with the first homology group 0. The second dodecahedral space $\mathbf{S}^3/\mathcal{G}_2^{(3)}$ was mentioned by P. J. Lorimer in [4] and B. Everitt in [3]. Our result also shows that non-orientable manifolds (and other orientable ones) do not exist.

Theorem 2. *There are 12 non-equivariant face identifications of $D^{(5)}$ yielding fixed point free fundamental dodecahedron tilings in $\mathbf{H}^3$ (Section 10). Each of them determines compact orientable hyperbolic manifolds. They can be classified into 7 classes according to their first homology groups, so they determine at least 7 different manifolds.*

Remarks. The first compact hyperbolic dodecahedral space $\mathcal{G}_6^{(5)}$ was discovered by C. Weber and H. Seifert [12]. Then L. A. Best listed 7 more manifolds in [1]. Unfortunately, his cases 4, 5 and 6 are equivariant, in Section 10 the corresponding group is $\mathcal{G}_2^{(5)}$. His other cases 1, 2, 3 and 7 appear in that section table at $\mathcal{G}_5^{(5)}$, $\mathcal{G}_4^{(5)}$, $\mathcal{G}_8^{(5)}$, $\mathcal{G}_{12}^{(5)}$, respectively. Among our other identifications there are at least 3 new manifolds according to their homology groups. These are $\mathcal{G}_7^{(5)}$, $\mathcal{G}_9^{(5)}$, moreover, $\mathcal{G}_{10}^{(5)}$ and $\mathcal{G}_{11}^{(5)}$ ($\mathcal{G}_{10}^{(5)}$ and $\mathcal{G}_{11}^{(5)}$ may be isomorphic). Similarly to the spherical cases our result also shows that non-orientable manifolds cannot appear.

In the course of the classification of non-compact hyperbolic space forms we can apply a new approach to the classes determined by the first homology group. In the tiling $\mathcal{D}^{(6)}$ the vertices are ideal points and their domains, intersecting by a horosphere, form an infinite regular triangle tiling around the ends. A fixed point free space group $\mathcal{G}^{(6)}$ glues together these ends defining a so-called cusp. Gluing together the corresponding triangles, we can form a fundamental domain (with side pairing generators) of the vertex stabilizer subgroup, which is a planar Euclidean crystallographic group. Since $\mathcal{G}^{(6)}$ is fixed point free, the stabilizer must also be a fixed point free discontinuous group of the horosphere, i.e. of the Euclidean plane, and each stabilizer is either **p1** or **pg**.

We compute the cusps for each identification of $\mathcal{F}^{(6)}$ and determine their stabilizer group. Let us consider two identifications $\mathcal{F}_i$ and $\mathcal{F}_j$ from $\mathcal{F}^{(6)}$ and let x_i and x_j denote the number of cusps with stabilizer group **p1**, moreover y_i and y_j denote the number of cusps with **pg** determined by $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively. The numbers x_i, y_i and x_j, y_j are characteristic to the manifolds described by $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively, namely if the ordered pairs (x_i, y_i) and (x_j, y_j) are not equal then $\mathcal{F}_i$ and $\mathcal{F}_j$ generate non-isomorphic fundamental groups. Finally, we put the identifications $\mathcal{F}_i$ and $\mathcal{F}_j$ into the same class iff both of them are orientable or both of them are non-orientable, furthermore their first homology groups $\mathcal{H}_{1,i}$ and $\mathcal{H}_{1,j}$ are isomorphic and $(x_i, y_i) = (x_j, y_j)$.

Theorem 3. *There are 78 non-equivariant face identifications of $D^{(6)}$ yielding fixed point free fundamental dodecahedron tilings in $\mathbf{H}^3$. They determine 10 orientable (Section 11) and 68 non-orientable (Section 12) non-compact hyperbolic manifolds with finite volume. The 10 orientable manifolds (each of them has an enantiomorphic pair) can be classified into 6 classes according to their first homology groups and we get 7 classes taking the cusp structures into consideration as an additional aspect. The 68 non-orientable manifolds can be classified into 22 homology classes and we obtain 31 classes according to their cusp structures. In this way the identifications determine at least 7 different (non-isometric) orientable and 31 different non-orientable manifolds.*

Remarks. In an algebraic way, looking for the decompositions of the symmetry group of the tiling $\mathrm{Sym}\,\mathcal{D}^{(6)}$ as a product of the symmetry group of the dodecahedron $\mathrm{Sym}\,D^{(6)}$ and a space group $\mathcal{G}^{(6)}$, B. Everitt found the 10 orientable cases about a year earlier [3]. (He distinguished the enantiomorphic pairs, thus he obtained 20 cases.) He used the group theory computer program LOWINDEX. In this manner we get the same result by applying different methods. Our results related with the orientable tetrahedral cubic and octahedral manifolds are also identical [6, 11, 9, resp.]. Intentionally, he did not examine the non-orientable cases.

8. An example for non-compact dodecahedral manifold

The table of Section 12 contains the group $\mathcal{G}^{(6)}_{74}$ in Class 36 in the following way:

$$\mathcal{G}^{(6)}_{74} = \Big\{\mathbf{a}\Big[{ABCDE \atop LRKAB}\Big],\ \mathbf{u}\Big[{BLQMC \atop EDNTO}\Big],\ \mathbf{b}\Big[{CMPND \atop PFGQM}\Big],\ \mathbf{v}\Big[{EOSKA \atop IJFGH}\Big],\ \mathbf{c}\Big[{GQLRH \atop KRHIS}\Big],\ \mathbf{d}\Big[{ISOTJ \atop NTJFP}\Big]$$

$$\text{—}\ \mathbf{1} = \mathbf{aua^2c\bar{v}} = \mathbf{uvdbc\bar{a}} = \mathbf{avcv\bar{b}^2} = \mathbf{ud\bar{v}d^2\bar{b}} = \mathbf{c^2d\bar{u}\bar{b}\bar{u}}\Big\}.$$

In the first row we can see the face identifying generators (Fig. 1). In our tables (Sections 9, 10, 11 and 12) **a**, **b**, **c**, **d**, **e** and **f** will denote the orientation preserving transformations (screw motions), moreover **u**, **v**, **w**, **x**, **y** and **z** will denote the orientation reversing transformations (glide reflections). Now, the face identifying generators are four screw motions **a**, **b**, **c** and **d**, moreover two glide reflections **u** and **v**:

$$\begin{array}{cclcccl}
\mathbf{a} & : & a^{-1} & = & ABCDE & \longmapsto & a = LRKAB, \\
\mathbf{b} & : & b^{-1} & = & CMPND & \longmapsto & b = PFGQM, \\
\mathbf{c} & : & c^{-1} & = & GQLRH & \longmapsto & c = KRHIS, \\
\mathbf{d} & : & d^{-1} & = & ISOTJ & \longmapsto & d = NTJFP, \\
\mathbf{u} & : & u^{-1} & = & BLQMC & \longmapsto & u = EDNTO, \\
\mathbf{v} & : & v^{-1} & = & EOSKA & \longmapsto & v = IJFGH.
\end{array}$$

These transformations map the dodecahedron $D^{(6)}$ onto its neighbours along the faces a, b, c, d, u and v, respectively. In Figure 1 the face identifications are denoted by corresponding "small flags" ⚑ inside the faces. The letters A, B, C, ..., T for vertices are valid for each identifications in the tables of Section 9, 10, 11 and 12.

In the second row we can see the defining relations, which are given by the Poincaré algorithm in equivalent inverse form to that of Section 4 (see also [5]). For example starting with the directed edge $\overrightarrow{AE}$ and the face $a^{-1} = \overrightarrow{AE}DCB$ along $\overrightarrow{AE}$ we get the following cycle:

$$\begin{array}{cclclcl}
\mathbf{a} & : & \overrightarrow{AE}DCB & \longmapsto & \overrightarrow{LB}AKR & []\!\uparrow\![] & \overrightarrow{LB}CMQ \\
\mathbf{u} & : & \overrightarrow{LB}CMQ & \longmapsto & \overrightarrow{DE}OTN & []\!\uparrow\![] & \overrightarrow{DE}ABC \\
\mathbf{a} & : & \overrightarrow{DE}ABC & \longmapsto & \overrightarrow{AB}LRK & []\!\uparrow\![] & \overrightarrow{AB}CDE \\
\mathbf{a} & : & \overrightarrow{AB}CDE & \longmapsto & \overrightarrow{LR}KAB & []\!\uparrow\![] & \overrightarrow{LR}HGQ \\
\mathbf{c} & : & \overrightarrow{LR}HGQ & \longmapsto & \overrightarrow{HI}SKR & []\!\uparrow\![] & \overrightarrow{HI}JFG \\
\mathbf{v}^{-1} & : & \overrightarrow{HI}JFG & \longmapsto & \overrightarrow{AE}OSK & []\!\uparrow\![] & \overrightarrow{AE}DCB
\end{array}$$

and the edge cycle $\{\overrightarrow{AE}, \overrightarrow{LB}, \overrightarrow{DE}, \overrightarrow{AB}, \overrightarrow{LR}, \overrightarrow{HI}\}$, moreover the relation $\mathbf{aua^2cv^{-1}} = \mathbf{1}$. In our tables we use a bar for the inverse transformations, e.g. the notation $\bar{\mathbf{g}} := \mathbf{g}^{-1}$ for the sake of brevity. In similar way we obtain the other edge cycles and relations. In Figure 1 the edges in different cycles are denoted by different types of arrows and we can see the corresponding relations. Now we can compute the first homology group:

$\mathcal{H}^{(6)}_{1,74} = \mathcal{Z}_6 \times \mathcal{Z}_{16} \times \mathcal{Z}$, since $\mathbf{v}$ resp. $\mathbf{b}$ are expressible from the first and second relations, and the others provide the existence of the cyclic group components $\mathcal{Z}_6$ (of order 6) and $\mathcal{Z}_{16}$ by commutativity.

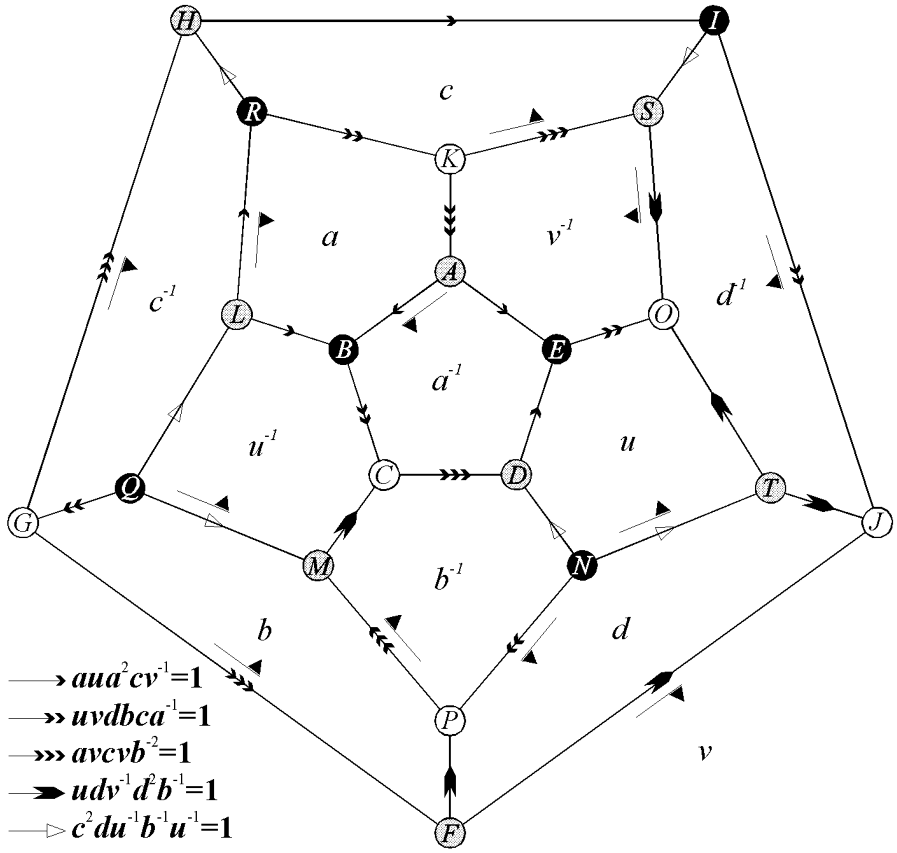

Figure 1

Then we look for the fundamental domains of the vertex (end) stabilizer subgroups (the cusps). For example we start with the end A. Intersecting the dodecahedron $\tilde{D} := D^{(6)}$ with a horosphere centred at the ideal point A we get a regular horospherical triangle. Its sides are the intersecting lines with the faces v^{-1}, a, and a^{-1}, moreover its vertices are the intersecting points with the edges denoted by "one", "three" and "one" arrows (Figure 2). The "upwards" ("one") arrows show that the directed edges point out from A and the "downwards" ("three") arrow denotes that the corresponding edge points into A.

Now we choose a side of the triangle, for example a, and apply the corresponding transformation $\mathbf{a}$. This transformation maps $\tilde{D}^{\mathbf{1}} = \tilde{D}$ onto its neighbour $\tilde{D}^{\mathbf{a}}$ along the face $a = ABLRK$ mapping $a^{-1} = DEABC$ onto a. Then the horospherical triangle around the end $D^{\mathbf{a}} \equiv A$ in $\tilde{D}^{\mathbf{a}}$ is glued to the previous one, uniting the sides a of $\tilde{D}^{\mathbf{1}}$ and $(a^{-1})^{\mathbf{a}}$ of $\tilde{D}^{\mathbf{a}}$. The endpoints of this side is derived from the edges of "one" and "three" arrows. Thus we obtain a horospherical rhomb from the previous triangles. Its sides are a^{-1}, v^{-1}, $(b^{-1})^{\mathbf{a}}$ and $u^{\mathbf{a}}$. (In Figure 2 we use the brief notations D, b^{-1} and u for $D^{\mathbf{a}}$, $(b^{-1})^{\mathbf{a}}$ and $u^{\mathbf{a}}$, respectively, moreover we discard the notations of the transformed images at the other ends and their sides, too.)

We choose a side of the rhomb, for example $(b^{-1})^{\mathbf{a}}$ of $\tilde{D}^{\mathbf{a}}$, and apply the corresponding transformation (determined by the face identification of $\tilde{D}^{\mathbf{1}}$) $\mathbf{a}^{-1}\mathbf{b}^{-1}\mathbf{a}$, which maps $\tilde{D}^{\mathbf{a}}$ onto its neighbour $\tilde{D}^{\mathbf{b}^{-1}\mathbf{a}}$ along the face $(b^{-1})^{\mathbf{a}} = (DCMPN)^{\mathbf{a}}$ of $\tilde{D}^{\mathbf{a}}$ mapping $b^{\mathbf{a}} = (MPFGQ)^{\mathbf{a}}$ onto $(b^{-1})^{\mathbf{a}}$. Then the horospherical triangle around $M^{\mathbf{b}^{-1}\mathbf{a}} \equiv D^{\mathbf{a}} \equiv A$ in $\tilde{D}^{\mathbf{b}^{-1}\mathbf{a}}$ is glued to the rhomb forming a trapezium.

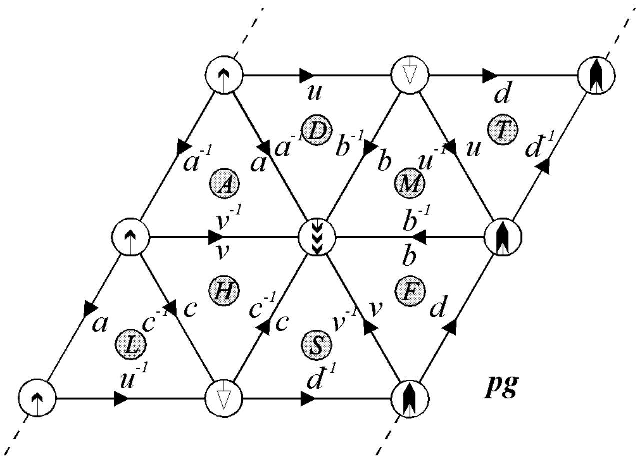

Figure 2

We can follow the previous method, by choosing a free side of the actual plane figure and gluing the corresponding horospherical triangle along it if its center end did not occur earlier. If we cannot find such a triangle, then the fundamental domain is ready with the identification of its sides. During this method we usually have some freedom in choosing the next side to glue a triangle along it. In our example we have obtained a "big" horospherical rhomb gluing together the triangles around the ends

$$A \in \tilde{D}^{\mathbf{1}}, \qquad D^{\mathbf{a}} \in \tilde{D}^{\mathbf{a}}, \qquad M^{\mathbf{b}^{-1}\mathbf{a}} \in \tilde{D}^{\mathbf{b}^{-1}\mathbf{a}}, \qquad T^{\mathbf{u}^{-1}\mathbf{b}^{-1}\mathbf{a}} \in \tilde{D}^{\mathbf{u}^{-1}\mathbf{b}^{-1}\mathbf{a}},$$
$$L^{\mathbf{c}\mathbf{v}^{-1}} \in \tilde{D}^{\mathbf{c}\mathbf{v}^{-1}}, \quad H^{\mathbf{v}^{-1}} \in \tilde{D}^{\mathbf{v}^{-1}}, \quad S^{\mathbf{c}^{-1}\mathbf{v}^{-1}} \in \tilde{D}^{\mathbf{c}^{-1}\mathbf{v}^{-1}}, \quad F^{\mathbf{v}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}} \in \tilde{D}^{\mathbf{v}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}}.$$

These ends, which are denoted by "grey" circles in our figures, form an equivalence class under the group $\mathcal{G}^{(6)}_{74}$. (In this way, surrounding the edges of "three" arrows we also get the defining relation $\mathbf{avcvb}^{-2} = \mathbf{1}$ again.)

Considering the triangle sides and the symbols of edge classes along the boundary of the domain we can recognize the side identifications and the generators of the stabilizer subgroup. The transformation $\mathbf{g}_1$ maps the side a^{-1} of $\tilde{D}^{\mathbf{1}}$ (at the end A) onto $a^{\mathbf{c}\mathbf{v}^{-1}}$ of $\tilde{D}^{\mathbf{c}\mathbf{v}^{-1}}$ (at the image end L) and maps the domain onto its neighbour along $a^{\mathbf{c}\mathbf{v}^{-1}}$. In this way we get the glide reflection $\mathbf{g}_1 = \mathbf{acv}^{-1}$. Similarly, the transformation $\mathbf{g}_2$ maps the side $(d^{-1})^{\mathbf{u}^{-1}\mathbf{b}^{-1}\mathbf{a}}$ of $\tilde{D}^{\mathbf{u}^{-1}\mathbf{b}^{-1}\mathbf{a}}$ (at the image end T between the "black" arrows) onto $d^{\mathbf{v}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}}$ of $\tilde{D}^{\mathbf{v}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}}$ (at the image end F between the "black" arrows again) and maps the domain onto its neighbour along $d^{\mathbf{v}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}}$. So we obtain the glide reflection $\mathbf{g}_2 = \mathbf{a}^{-1}\mathbf{budv}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}$, (i.e. we map back $\tilde{D}^{\mathbf{u}^{-1}\mathbf{b}^{-1}\mathbf{a}}$ onto $\tilde{D}^{\mathbf{1}}$ and map $\tilde{D}^{\mathbf{1}}$ onto the neighbouring dodecahedron of $\tilde{D}^{\mathbf{v}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}}$ along $d^{\mathbf{v}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}}$). Finally, the identifications of $u^{\mathbf{a}}$ and $(u^{-1})^{\mathbf{c}\mathbf{v}^{-1}}$ (at the image ends D and L, resp.), moreover $d^{\mathbf{u}^{-1}\mathbf{b}^{-1}\mathbf{a}}$ and $(d^{-1})^{\mathbf{c}^{-1}\mathbf{v}^{-1}}$ (at the image ends T and S, resp.) involve the translations $\mathbf{t}_1 = \mathbf{a}^{-1}\mathbf{u}^{-1}\mathbf{c}\mathbf{v}^{-1}$ and $\mathbf{t}_2 = \mathbf{a}^{-1}\mathbf{b}\mathbf{u}\mathbf{d}^{-1}\mathbf{c}^{-1}\mathbf{v}^{-1}$, respectively. We can observe that $\mathbf{t}_1 = \mathbf{t}_2 = \mathbf{g}_1^2 = \mathbf{g}_2^2$, and the sides $u^{\mathbf{a}}$ and $(u^{-1})^{\mathbf{c}\mathbf{v}^{-1}}$ can be extended with $d^{\mathbf{u}^{-1}\mathbf{b}^{-1}\mathbf{a}}$ and $(d^{-1})^{\mathbf{c}^{-1}\mathbf{v}^{-1}}$, respectively, determining new longer sides. Topologically, this fundamental domain is a hexagon. The glide reflections $\mathbf{g}_1$, $\mathbf{g}_2$ and the translation $\mathbf{t} := \mathbf{t}_1 = \mathbf{t}_2$ generate the planar crystallographic group **pg**.

We know that the stabilizer groups should be either **p1** or **pg**, and among them only **pg** contains orientation reversing transformations (horospherical glide reflections). Therefore, in general, we only have to examine whether orientation reversing side identification occurs. If so, then the stabilizer is **pg** else **p1**.

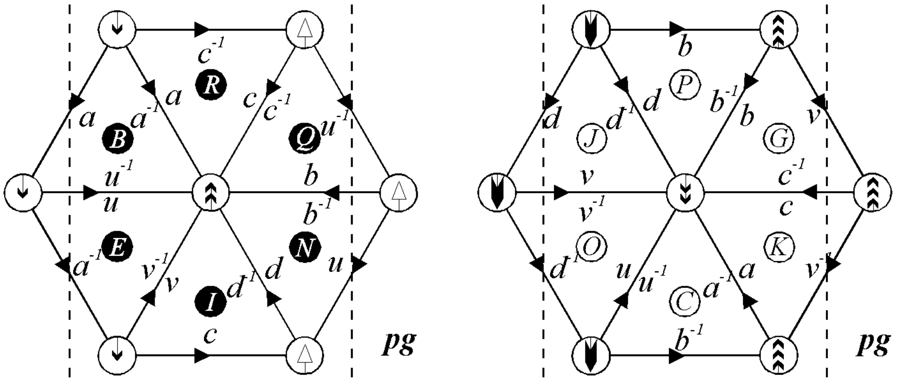

Figure 3

Starting with a vertex (end) that does not belong to any previous class, we can find the other classes. In our case there are two more classes (Figure 3). The vertices (ends) belonging to them are denoted by "black" and "white" circles, respectively. The side identifications of the fundamental domains determine the plane group **pg** again. So we get the cusp structure: "**pg**: 3 cusps".

9. Table of the spherical dodecahedral space forms

Class 1 ○ $\mathcal{H}_{1,1}^{(3)} = 0$ ....

- $\mathcal{G}_1^{(3)} = \big\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\HGFJI\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\TJFPN\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\SIJTO\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}CMPND\\RHISK\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}DNTOE\\QGHRL\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}EOSKA\\PFGQM\end{smallmatrix}\right]$ — $\mathbf{1} = \mathbf{be\bar{a}} = \mathbf{cf\bar{a}} = \mathbf{db\bar{a}} = \mathbf{ec\bar{a}} = \mathbf{fd\bar{a}} = \mathbf{fc\bar{b}} = \mathbf{bd\bar{c}} = \mathbf{ce\bar{d}} = \mathbf{df\bar{e}} = \mathbf{eb\bar{f}}\big\}$

Class 2 ○ $\mathcal{H}_{1,2}^{(3)} = \mathcal{Z}_3 \times \mathcal{Z}_5$ ....

- $\mathcal{G}_2^{(3)} = \big\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\BCMQL\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\CDNPM\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}DNTOE\\HGFJI\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}EOSKA\\PFGQM\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}GQLRH\\RHISK\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}ISOTJ\\PNTJF\end{smallmatrix}\right]$ — $\mathbf{1} = \mathbf{a^2\bar{b}} = \mathbf{ba\bar{d}} = \mathbf{ae\bar{c}} = \mathbf{ab\bar{d}} = \mathbf{ceb} = \mathbf{cf\bar{d}} = \mathbf{fc\bar{d}} = \mathbf{f^2\bar{c}} = \mathbf{e^2d} = \mathbf{ef\bar{b}}\big\}$

10. Table of the compact hyperbolic dodecahedral space forms

Class 1 $\circ$ $\mathcal{H}_{1,1}^{(5)} = \mathcal{H}_{1,2}^{(5)} = \mathcal{Z}_3^2$ . . . .

- $\mathcal{G}_1^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\QMPFG\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\HGFJI\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\NDEOT\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}CMPND\\KSOEA\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}GQLRH\\IJTOS\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}HRKSI\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{bfd\bar{c}\bar{a}} = \mathbf{ad\bar{e}f\bar{c}} = \mathbf{dbef\bar{a}} = \mathbf{ca\bar{b}f\bar{e}} = \mathbf{cdae\bar{b}} = \mathbf{df\bar{b}e\bar{c}}\}$
- $\mathcal{G}_2^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\HRLQG\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\IHGFJ\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\DEOTN\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}CMPND\\IJTOS\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}EOSKA\\PFGQM\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}HRKSI\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{af\bar{c}d\bar{b}} = \mathbf{cde\bar{b}\bar{a}} = \mathbf{ace\bar{f}\bar{d}} = \mathbf{ca\bar{e}f\bar{b}} = \mathbf{ed\bar{f}b\bar{a}} = \mathbf{bfd\bar{c}\bar{e}}\}$

Class 2 $\circ$ $\mathcal{H}_{1,3}^{(5)} = \ldots = \mathcal{H}_{1,6}^{(5)} = \mathcal{Z}_5^3$ . . . .

- $\mathcal{G}_3^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\MQLBC\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\LQGHR\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}CMPND\\NDEOT\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}EOSKA\\FGQMP\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}FGHIJ\\ISKRH\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}ISOTJ\\TJFPN\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{b^2ed\bar{a}} = \mathbf{dca^2\bar{b}} = \mathbf{ab\bar{e}f\bar{c}} = \mathbf{ac^2f\bar{d}} = \mathbf{def^2c} = \mathbf{fe^2b\bar{d}}\}$
- $\mathcal{G}_4^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\QGHRL\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\HGFJI\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\TNDEO\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}CMPND\\JTOSI\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}EOSKA\\FGQMP\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}HRKSI\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{bfde\bar{a}} = \mathbf{ed\bar{c}a\bar{b}} = \mathbf{af\bar{c}b\bar{d}} = \mathbf{ecab\bar{f}} = \mathbf{acdf\bar{e}} = \mathbf{ce\bar{b}f\bar{d}}\}$
- $\mathcal{G}_5^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\HRLQG\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\FJIHG\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\DEOTN\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}CMPND\\OSIJT\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}EOSKA\\QMPFG\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}HRKSI\\JFPNT\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{af\bar{b}e\bar{b}} = \mathbf{abf\bar{d}\bar{c}} = \mathbf{acec\bar{d}} = \mathbf{ca\bar{e}a\bar{b}} = \mathbf{dedf\bar{c}} = \mathbf{fd\bar{b}f\bar{e}}\}$
- $\mathcal{G}_6^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\JIHGF\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\PNTJF\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\TOSIJ\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}CMPND\\SKRHI\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}DNTOE\\RLQGH\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}EOSKA\\QMPFG\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{df\bar{b}a\bar{c}} = \mathbf{eb\bar{c}a\bar{d}} = \mathbf{fc\bar{d}a\bar{e}} = \mathbf{bd\bar{e}a\bar{f}} = \mathbf{ce\bar{f}a\bar{b}} = \mathbf{becfd}\}$

Class 3 $\circ$ $\mathcal{H}_{1,7}^{(5)} = \mathcal{Z}_{29}$ . . . .

- $\mathcal{G}_7^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\NDEOT\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\MCDNP\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\JIHGF\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}EOSKA\\GHRLQ\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}FPMQG\\ISKRH\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}ISOTJ\\PNTJF\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{bedb\bar{a}} = \mathbf{a^2f^2\bar{c}} = \mathbf{ebad\bar{c}} = \mathbf{de\bar{d}f\bar{a}} = \mathbf{bce\bar{c}\bar{d}} = \mathbf{cfef\bar{b}}\}$

Class 4 $\circ$ $\mathcal{H}_{1,8}^{(5)} = \mathcal{Z}_3 \times \mathcal{Z}_5^2$ . . . .

- $\mathcal{G}_8^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\GFJIH\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\NPMCD\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\TNDEO\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}EOSKA\\PFGQM\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}GQLRH\\TOSIJ\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}HRKSI\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{ed\bar{a}b\bar{c}} = \mathbf{de\bar{c}a\bar{f}} = \mathbf{ba\bar{e}f\bar{c}} = \mathbf{dcaf\bar{b}} = \mathbf{ae\bar{f}b\bar{d}} = \mathbf{bcd\bar{f}\bar{e}}\}$

Class 5 $\circ$ $\mathcal{H}_{1,9}^{(5)} = \mathcal{Z}_3^2 \times \mathcal{Z}_5^2$ . . . .

- $\mathcal{G}_9^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\QGHRL\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\NPMCD\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\TNDEO\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}EOSKA\\QMPFG\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}FGHIJ\\ISKRH\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}ISOTJ\\TJFPN\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{ac\bar{b}a\bar{d}} = \mathbf{aed\bar{f}\bar{c}} = \mathbf{ba\bar{e}f\bar{c}} = \mathbf{abcdc} = \mathbf{def^2\bar{b}} = \mathbf{fe^2b\bar{d}}\}$

Class 6 $\circ$ $\mathcal{H}_{1,10}^{(5)} = \mathcal{H}_{1,11}^{(5)} = \mathcal{Z}_3 \times \mathcal{Z}_{16}$ . . . .

- $\mathcal{G}_{10}^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\IHGFJ\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\DEOTN\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\EAKSO\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}CMPND\\FGQMP\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}GQLRH\\HISKR\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}ISOTJ\\JFPNT\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{bd^2e\bar{a}} = \mathbf{ae^2b\bar{c}} = \mathbf{cf\bar{d}a\bar{d}} = \mathbf{cba\bar{f}\bar{e}} = \mathbf{caf^2\bar{b}} = \mathbf{bfdc\bar{e}}\}$
- $\mathcal{G}_{11}^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\GFJIH\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\PMCDN\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\PFGQM\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}DNTOE\\HRLQG\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}EOSKA\\KRHIS\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}ISOTJ\\JFPNT\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{dc\bar{a}b\bar{f}} = \mathbf{ef\bar{a}c\bar{b}} = \mathbf{af^2db} = \mathbf{e^2\bar{a}d\bar{a}} = \mathbf{bd\bar{e}f\bar{c}} = \mathbf{ebc^2\bar{d}}\}$

Class 7 $\circ$ $\mathcal{H}_{1,12}^{(5)} = \mathcal{Z}_5 \times \mathcal{Z}_7$ . . . .

- $\mathcal{G}_{12}^{(5)} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\MQLBC\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\HGFJI\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}CMPND\\DEOTN\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}EOSKA\\QGHRL\end{smallmatrix}\right], \mathbf{e}\left[\begin{smallmatrix}FPMQG\\ISKRH\end{smallmatrix}\right], \mathbf{f}\left[\begin{smallmatrix}ISOTJ\\TJFPN\end{smallmatrix}\right]$
  — $\mathbf{1} = \mathbf{aebe\bar{b}} = \mathbf{aca^2\bar{d}} = \mathbf{abf\bar{c}^2} = \mathbf{db\bar{f}d\bar{b}} = \mathbf{cde\bar{d}\bar{e}} = \mathbf{ef^2cf}\}$

11. Table of the non-compact orientable hyperbolic dodecahedral space forms

Class 1 ∘ $\mathcal{H}^{(6)}_{1,1} = \mathcal{Z}_2 \times \mathcal{Z}$ ∘ **p1**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_1 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{RKABL}, \mathbf{b}\genfrac{[}{]}{0pt}{}{BLQMC}{QGHRL}, \mathbf{c}\genfrac{[}{]}{0pt}{}{CMPND}{OTNDE}, \mathbf{d}\genfrac{[}{]}{0pt}{}{EOSKA}{JIHGF}, \mathbf{e}\genfrac{[}{]}{0pt}{}{FPMQG}{KRHIS}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{PNTJF}$
  — $\mathbf{1} = \mathbf{cdfe\bar{a}^2} = \mathbf{aded\bar{b}^2} = \mathbf{abefc^2} = \mathbf{cf^2\bar{d}a\bar{b}} = \mathbf{b\bar{e}c\bar{f}d\bar{e}}\}$

Class 2 ∘ $\mathcal{H}^{(6)}_{1,2} = \mathcal{H}^{(6)}_{1,3} = \mathcal{H}^{(6)}_{1,4} = \mathcal{Z}_2 \times \mathcal{Z}^2$ ∘ **p1**: 2 cusps . . . .

- $\mathcal{G}^{(6)}_2 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{GHRLQ}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{LBCMQ}, \mathbf{c}\genfrac{[}{]}{0pt}{}{CMPND}{OEAKS}, \mathbf{d}\genfrac{[}{]}{0pt}{}{DNTOE}{IHGFJ}, \mathbf{e}\genfrac{[}{]}{0pt}{}{FPMQG}{RHISK}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{TJFPN}$
  — $\mathbf{1} = \mathbf{dfd\bar{a}b\bar{a}} = \mathbf{deba\bar{e}\bar{f}} = \mathbf{abcd\bar{f}\bar{c}} = \mathbf{cae\bar{c}d\bar{e}} = \mathbf{b^2ef^2c}\}$
- $\mathcal{G}^{(6)}_3 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{NPMCD}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{BCMQL}, \mathbf{c}\genfrac{[}{]}{0pt}{}{DNTOE}{GFJIH}, \mathbf{d}\genfrac{[}{]}{0pt}{}{EOSKA}{MPFGQ}, \mathbf{e}\genfrac{[}{]}{0pt}{}{GQLRH}{HISKR}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{TJFPN}$
  — $\mathbf{1} = \mathbf{b^2ef^2\bar{a}} = \mathbf{de\bar{c}d\bar{a}\bar{b}} = \mathbf{ce^2b\bar{a}^2} = \mathbf{ed\bar{c}\bar{a}d\bar{b}} = \mathbf{cfc\bar{f}d\bar{f}}\}$
- $\mathcal{G}^{(6)}_4 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{RLQGH}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{QLBCM}, \mathbf{c}\genfrac{[}{]}{0pt}{}{CMPND}{FJIHG}, \mathbf{d}\genfrac{[}{]}{0pt}{}{DNTOE}{ISKRH}, \mathbf{e}\genfrac{[}{]}{0pt}{}{EOSKA}{PFGQM}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{PNTJF}$
  — $\mathbf{1} = \mathbf{aba\bar{b}e\bar{b}} = \mathbf{fde\bar{a}c\bar{e}} = \mathbf{dfc\bar{d}a\bar{c}} = \mathbf{ecf\bar{e}d\bar{a}} = \mathbf{db^2c\bar{f}^2}\}$

Class 3 ∘ $\mathcal{H}^{(6)}_{1,5} = \mathcal{H}^{(6)}_{1,6} = \mathcal{Z}_2^2 \times \mathcal{Z}$ ∘ **p1**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_5 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{CMQLB}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{DNPMC}, \mathbf{c}\genfrac{[}{]}{0pt}{}{DNTOE}{QGHRL}, \mathbf{d}\genfrac{[}{]}{0pt}{}{EOSKA}{IHGFJ}, \mathbf{e}\genfrac{[}{]}{0pt}{}{FPMQG}{ISKRH}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{NTJFP}$
  — $\mathbf{1} = \mathbf{ba\bar{c}ab\bar{a}} = \mathbf{a^2eb\bar{f}\bar{d}} = \mathbf{bce\bar{c}f\bar{d}} = \mathbf{cbede\bar{d}} = \mathbf{f^2efc\bar{d}}\}$
- $\mathcal{G}^{(6)}_6 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{PFGQM}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{TNDEO}, \mathbf{c}\genfrac{[}{]}{0pt}{}{BLQMC}{IHGFJ}, \mathbf{d}\genfrac{[}{]}{0pt}{}{CMPND}{SOEAK}, \mathbf{e}\genfrac{[}{]}{0pt}{}{GQLRH}{OSIJT}, \mathbf{f}\genfrac{[}{]}{0pt}{}{HRKSI}{NTJFP}$
  — $\mathbf{1} = \mathbf{ef\bar{a}b\bar{e}\bar{c}} = \mathbf{bac\bar{a}c\bar{e}} = \mathbf{df\bar{c}d\bar{e}\bar{a}} = \mathbf{cfdad\bar{b}} = \mathbf{bdb\bar{f}e\bar{f}}\}$

Class 4 ∘ $\mathcal{H}^{(6)}_{1,7} = \mathcal{Z}_2^2 \times \mathcal{Z}^2$ ∘ **p1**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_7 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{HGFJI}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{SOEAK}, \mathbf{c}\genfrac{[}{]}{0pt}{}{BLQMC}{QMPFG}, \mathbf{d}\genfrac{[}{]}{0pt}{}{CMPND}{LQGHR}, \mathbf{e}\genfrac{[}{]}{0pt}{}{DNTOE}{KRHIS}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{NTJFP}$
  — $\mathbf{1} = \mathbf{afd\bar{a}b\bar{e}} = \mathbf{dcd\bar{c}a\bar{c}} = \mathbf{dba\bar{e}f\bar{a}} = \mathbf{b^2f^2\bar{c}^2} = \mathbf{ebefe\bar{d}}\}$

Class 5 ∘ $\mathcal{H}^{(6)}_{1,8} = \mathcal{Z}_2^2 \times \mathcal{Z}^2$ ∘ **p1**: 2 cusps . . . .

- $\mathcal{G}^{(6)}_8 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{CMQLB}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{DNPMC}, \mathbf{c}\genfrac{[}{]}{0pt}{}{DNTOE}{QGHRL}, \mathbf{d}\genfrac{[}{]}{0pt}{}{EOSKA}{FJIHG}, \mathbf{e}\genfrac{[}{]}{0pt}{}{FPMQG}{KRHIS}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{NTJFP}$
  — $\mathbf{1} = \mathbf{ba\bar{c}ab\bar{a}} = \mathbf{ded\bar{e}\bar{a}^2} = \mathbf{bcefc\bar{d}} = \mathbf{cbe\bar{c}f\bar{d}} = \mathbf{f^2eb\bar{f}\bar{d}}\}$

Class 6 ∘ $\mathcal{H}^{(6)}_{1,9} = \mathcal{Z}_2 \times \mathcal{Z}_3 \times \mathcal{Z}^2$ ∘ **p1**: 2 cusps . . . .

- $\mathcal{G}^{(6)}_9 = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{DEOTN}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{LBCMQ}, \mathbf{c}\genfrac{[}{]}{0pt}{}{CMPND}{OEAKS}, \mathbf{d}\genfrac{[}{]}{0pt}{}{FGHIJ}{RLQGH}, \mathbf{e}\genfrac{[}{]}{0pt}{}{FPMQG}{RHISK}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{TJFPN}$
  — $\mathbf{1} = \mathbf{a^2\bar{f}d^2\bar{b}} = \mathbf{dbc\bar{a}\bar{b}\bar{e}} = \mathbf{cfd\bar{e}\bar{f}\bar{a}} = \mathbf{ede\bar{c}\bar{a}\bar{c}} = \mathbf{b^2ef^2c}\}$

Class 7 ∘ $\mathcal{H}^{(6)}_{1,10} = \mathcal{Z}_2^2 \times \mathcal{Z}_3 \times \mathcal{Z}^2$ ∘ **p1**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_{10} = \{\mathbf{a}\genfrac{[}{]}{0pt}{}{ABCDE}{HRLQG}, \mathbf{b}\genfrac{[}{]}{0pt}{}{AKRLB}{PFGQM}, \mathbf{c}\genfrac{[}{]}{0pt}{}{BLQMC}{AKSOE}, \mathbf{d}\genfrac{[}{]}{0pt}{}{CMPND}{RHISK}, \mathbf{e}\genfrac{[}{]}{0pt}{}{DNTOE}{GFJIH}, \mathbf{f}\genfrac{[}{]}{0pt}{}{ISOTJ}{NTJFP}$
  — $\mathbf{1} = \mathbf{bd\bar{e}\bar{c}d\bar{a}} = \mathbf{ca\bar{e}a\bar{b}\bar{a}} = \mathbf{db\bar{e}d\bar{c}\bar{a}} = \mathbf{bcf^2\bar{b}\bar{c}} = \mathbf{efdfe\bar{f}}\}$

12. Table of the non-compact non-orientable hyperbolic dodecahedral space forms

Class 8 ∘ $\mathcal{H}^{(6)}_{1,11} = \ldots = \mathcal{H}^{(6)}_{1,14} = \mathcal{Z}$ ∘ **p1**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_{11} = \{\mathbf{u}\genfrac{[}{]}{0pt}{}{ABCDE}{HGQLR}, \mathbf{v}\genfrac{[}{]}{0pt}{}{AKRLB}{BLQMC}, \mathbf{w}\genfrac{[}{]}{0pt}{}{CMPND}{SIHRK}, \mathbf{x}\genfrac{[}{]}{0pt}{}{DNTOE}{FGHIJ}, \mathbf{y}\genfrac{[}{]}{0pt}{}{EOSKA}{MQGFP}, \mathbf{z}\genfrac{[}{]}{0pt}{}{ISOTJ}{JTNPF}$
  — $\mathbf{1} = \mathbf{vu\bar{y}zx\bar{u}} = \mathbf{wy\bar{x}wv\bar{u}} = \mathbf{uv\bar{y}xz\bar{x}} = \mathbf{yw\bar{x}zw\bar{u}} = \mathbf{v^2wz^2\bar{y}}\}$

- $\mathcal{G}_{12}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop LRHGQ}\right], \mathbf{v}\left[{AKRLB \atop LQMCB}\right], \mathbf{w}\left[{CMPND \atop HRKSI}\right], \mathbf{x}\left[{DNTOE \atop GHIJF}\right], \mathbf{y}\left[{EOSKA \atop FPMQG}\right], \mathbf{z}\left[{ISOTJ \atop TNPFJ}\right]$
— $\mathbf{1} = \mathbf{v^2u\bar{w}\bar{v}\bar{u}} = \mathbf{wzx\bar{w}u\bar{x}} = \mathbf{x\bar{y}u\bar{v}y\bar{u}} = \mathbf{xz^2\bar{x}y\bar{z}} = \mathbf{zwy\bar{v}\bar{w}\bar{y}}\}$
- $\mathcal{G}_{13}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop GQLRH}\right], \mathbf{v}\left[{AKRLB \atop LQMCB}\right], \mathbf{w}\left[{CMPND \atop KSIHR}\right], \mathbf{x}\left[{DNTOE \atop GHIJF}\right], \mathbf{y}\left[{EOSKA \atop FPMQG}\right], \mathbf{z}\left[{ISOTJ \atop TNPFJ}\right]$
— $\mathbf{1} = \mathbf{v^2u\bar{v}y\bar{u}} = \mathbf{uvwy\bar{v}\bar{w}} = \mathbf{x\bar{y}u\bar{x}w\bar{u}} = \mathbf{xz^2\bar{x}y\bar{z}} = \mathbf{ywzx\bar{w}\bar{z}}\}$
- $\mathcal{G}_{14}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop GQLRH}\right], \mathbf{v}\left[{AKRLB \atop MCBLQ}\right], \mathbf{w}\left[{CMPND \atop KSIHR}\right], \mathbf{x}\left[{DNTOE \atop GHIJF}\right], \mathbf{y}\left[{EOSKA \atop FPMQG}\right], \mathbf{z}\left[{ISOTJ \atop TNPFJ}\right]$
— $\mathbf{1} = \mathbf{vwy\bar{v}u\bar{y}} = \mathbf{uv^2\bar{u}\bar{v}\bar{w}} = \mathbf{x\bar{y}u\bar{x}w\bar{u}} = \mathbf{xz^2\bar{x}y\bar{z}} = \mathbf{ywzx\bar{w}\bar{z}}\}$

Class 9 ∘ $\mathcal{H}_{1,15}^{(6)} = \mathcal{Z}^2$ ∘ **p1**: 1 cusp, **pg**: 1 cusp ....

- $\mathcal{G}_{15}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop HIJFG}\right], \mathbf{v}\left[{AKRLB \atop QGFPM}\right], \mathbf{w}\left[{BLQMC \atop OSKAE}\right], \mathbf{x}\left[{CMPND \atop RKSIH}\right], \mathbf{y}\left[{DNTOE \atop GQLRH}\right], \mathbf{z}\left[{ISOTJ \atop JTNPF}\right]$
— $\mathbf{1} = \mathbf{vwv\bar{y}x\bar{u}} = \mathbf{wy\bar{x}u\bar{z}\bar{u}} = \mathbf{xv\bar{u}y\bar{u}\bar{w}} = \mathbf{wzyw\bar{x}\bar{v}} = \mathbf{zxz^2\bar{v}\bar{y}}\}$

Class 10 ∘ $\mathcal{H}_{1,16}^{(6)} = \mathcal{H}_{1,17}^{(6)} = \mathcal{Z}_2 \times \mathcal{Z}^2$ ∘ **p1**: 1 cusp, **pg**: 1 cusp ....

- $\mathcal{G}_{16}^{(6)} = \{\mathbf{a}\left[{ABCDE \atop EOTND}\right], \mathbf{u}\left[{AKRLB \atop CMPND}\right], \mathbf{b}\left[{BLQMC \atop LQGHR}\right], \mathbf{v}\left[{EOSKA \atop GFPMQ}\right], \mathbf{w}\left[{FGHIJ \atop SIHRK}\right], \mathbf{x}\left[{ISOTJ \atop TNPFJ}\right]$
— $\mathbf{1} = \mathbf{avwx\bar{a}\bar{u}} = \mathbf{bu\bar{x}v\bar{x}\bar{a}} = \mathbf{b^2\bar{v}a^2\bar{u}} = \mathbf{vbw^2\bar{b}\bar{u}} = \mathbf{x^2wv\bar{u}\bar{w}}\}$
- $\mathcal{G}_{17}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop QGFPM}\right], \mathbf{v}\left[{AKRLB \atop GHIJF}\right], \mathbf{w}\left[{BLQMC \atop DNTOE}\right], \mathbf{x}\left[{CMPND \atop HGQLR}\right], \mathbf{y}\left[{EOSKA \atop RKSIH}\right], \mathbf{z}\left[{ISOTJ \atop TNPFJ}\right]$
— $\mathbf{1} = \mathbf{wux\bar{u}v\bar{u}} = \mathbf{uwz\bar{u}x\bar{y}} = \mathbf{wyv\bar{y}v\bar{x}} = \mathbf{wxvz^2\bar{v}} = \mathbf{y^2z\bar{w}\bar{x}\bar{z}}\}$

Class 11 ∘ $\mathcal{H}_{1,18}^{(6)} = \mathcal{H}_{1,19}^{(6)} = \mathcal{Z}_2^2 \times \mathcal{Z}$ ∘ **pg**: 1 cusp ....

- $\mathcal{G}_{18}^{(6)} = \{\mathbf{a}\left[{ABCDE \atop HGFJI}\right], \mathbf{b}\left[{AKRLB \atop MPFGQ}\right], \mathbf{u}\left[{BLQMC \atop DNTOE}\right], \mathbf{v}\left[{CMPND \atop GQLRH}\right], \mathbf{c}\left[{EOSKA \atop RHISK}\right], \mathbf{d}\left[{ISOTJ \atop NTJFP}\right]$
— $\mathbf{1} = \mathbf{bud\bar{a}v\bar{a}} = \mathbf{uadvb\bar{a}} = \mathbf{cb\bar{d}^2c\bar{a}} = \mathbf{c^2d\bar{u}\bar{v}\bar{b}} = \mathbf{uv\bar{c}\bar{u}v\bar{b}}\}$
- $\mathcal{G}_{19}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop FGHIJ}\right], \mathbf{a}\left[{AKRLB \atop MPFGQ}\right], \mathbf{b}\left[{BLQMC \atop EOTND}\right], \mathbf{c}\left[{CMPND \atop KRHIS}\right], \mathbf{d}\left[{EOSKA \atop HRLQG}\right], \mathbf{v}\left[{ISOTJ \atop JTNPF}\right]$
— $\mathbf{1} = \mathbf{ab\bar{v}da\bar{u}} = \mathbf{buv\bar{u}d\bar{u}} = \mathbf{cdbvc\bar{u}} = \mathbf{bd\bar{c}\bar{a}d\bar{a}} = \mathbf{bcv^2\bar{a}\bar{c}}\}$

Class 12 ∘ $\mathcal{H}_{1,20}^{(6)} = \mathcal{H}_{1,21}^{(6)} = \mathcal{Z}_2^2 \times \mathcal{Z}$ ∘ **pg**: 2 cusps ....

- $\mathcal{G}_{20}^{(6)} = \{\mathbf{a}\left[{ABCDE \atop LQGHR}\right], \mathbf{b}\left[{AKRLB \atop NPMCD}\right], \mathbf{c}\left[{BLQMC \atop DEOTN}\right], \mathbf{d}\left[{EOSKA \atop GQMPF}\right], \mathbf{u}\left[{FGHIJ \atop SIHRK}\right], \mathbf{v}\left[{ISOTJ \atop TNPFJ}\right]$
— $\mathbf{1} = \mathbf{acd\bar{a}c\bar{b}} = \mathbf{bau^2\bar{a}\bar{c}} = \mathbf{duv\bar{c}\bar{b}\bar{a}} = \mathbf{dcv\bar{d}b\bar{v}} = \mathbf{v^2ud\bar{b}\bar{u}}\}$
- $\mathcal{G}_{21}^{(6)} = \{\mathbf{a}\left[{ABCDE \atop FJIHG}\right], \mathbf{b}\left[{AKRLB \atop FGQMP}\right], \mathbf{c}\left[{BLQMC \atop OTNDE}\right], \mathbf{d}\left[{CMPND \atop RLQGH}\right], \mathbf{u}\left[{EOSKA \atop RKSIH}\right], \mathbf{v}\left[{ISOTJ \atop PFJTN}\right]$
— $\mathbf{1} = \mathbf{u^2v\bar{b}a\bar{v}} = \mathbf{cub\bar{d}\bar{v}\bar{a}} = \mathbf{d\bar{u}a\bar{b}u\bar{a}} = \mathbf{dbcd\bar{a}\bar{c}} = \mathbf{cv^2\bar{c}\bar{d}\bar{b}}\}$

Class 13 ∘ $\mathcal{H}_{1,22}^{(6)} = \mathcal{H}_{1,23}^{(6)} = \mathcal{Z}_2^2 \times \mathcal{Z}$ ∘ **pg**: 3 cusps ....

- $\mathcal{G}_{22}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop MQGFP}\right], \mathbf{v}\left[{AKRLB \atop FGHIJ}\right], \mathbf{w}\left[{BLQMC \atop OEDNT}\right], \mathbf{x}\left[{CMPND \atop GQLRH}\right], \mathbf{y}\left[{EOSKA \atop IHRKS}\right], \mathbf{z}\left[{ISOTJ \atop PFJTN}\right]$
— $\mathbf{1} = \mathbf{uwx\bar{y}z\bar{v}} = \mathbf{wz^2\bar{w}x\bar{u}} = \mathbf{y^2v\bar{x}u\bar{v}} = \mathbf{uxwu\bar{z}\bar{y}} = \mathbf{vzxv\bar{y}\bar{w}}\}$
- $\mathcal{G}_{23}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop MQGFP}\right], \mathbf{v}\left[{AKRLB \atop FGHIJ}\right], \mathbf{w}\left[{BLQMC \atop MPNDC}\right], \mathbf{x}\left[{DNTOE \atop HRKSI}\right], \mathbf{y}\left[{EOSKA \atop LRHGQ}\right], \mathbf{z}\left[{ISOTJ \atop PFJTN}\right]$
— $\mathbf{1} = \mathbf{uwx\bar{y}z\bar{v}} = \mathbf{w^2u\bar{v}y\bar{u}} = \mathbf{xz\bar{u}x\bar{v}\bar{y}} = \mathbf{yw\bar{z}\bar{v}w\bar{u}} = \mathbf{z^2xv\bar{y}\bar{x}}\}$

Class 14 ∘ $\mathcal{H}_{1,24}^{(6)} = \mathcal{H}_{1,25}^{(6)} = \mathcal{Z}_2^2 \times \mathcal{Z}^2$ ∘ **p1**: 1 cusp, **pg**: 1 cusp ....

- $\mathcal{G}_{24}^{(6)} = \{\mathbf{u}\left[{ABCDE \atop IJFGH}\right], \mathbf{v}\left[{AKRLB \atop MQGFP}\right], \mathbf{a}\left[{BLQMC \atop NPMCD}\right], \mathbf{w}\left[{DNTOE \atop RKSIH}\right], \mathbf{x}\left[{EOSKA \atop HGQLR}\right], \mathbf{b}\left[{ISOTJ \atop TJFPN}\right]$
— $\mathbf{1} = \mathbf{ubwxa\bar{v}} = \mathbf{awv\bar{x}b\bar{u}} = \mathbf{va^2u\bar{v}\bar{x}} = \mathbf{w\bar{x}u\bar{w}x\bar{u}} = \mathbf{wb^2\bar{a}v\bar{b}}\}$
- $\mathcal{G}_{25}^{(6)} = \{\mathbf{a}\left[{ABCDE \atop RKABL}\right], \mathbf{b}\left[{BLQMC \atop JIHGF}\right], \mathbf{c}\left[{CMPND \atop EAKSO}\right], \mathbf{u}\left[{DNTOE \atop QLRHG}\right], \mathbf{v}\left[{FPMQG \atop IHRKS}\right], \mathbf{d}\left[{ISOTJ \atop NTJFP}\right]$
— $\mathbf{1} = \mathbf{cu\bar{b}v\bar{a}^2} = \mathbf{b\bar{d}u\bar{v}c\bar{a}} = \mathbf{abdc\bar{v}\bar{u}} = \mathbf{bvdu\bar{a}\bar{c}} = \mathbf{cd^2v\bar{b}\bar{u}}\}$

Class 15 $\circ$ $\mathcal{H}^{(6)}_{1,26} = \mathcal{Z}_2^2 \times \mathcal{Z}^2$ $\circ$ **p1**: 1 cusp, **pg**: 2 cusps . . . .

- $\mathcal{G}^{(6)}_{26} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\JFGHI\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}AKRLB\\SOEAK\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}BLQMC\\QMPFG\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}CMPND\\RKSIH\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}DNTOE\\HRLQG\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1 = cbv\bar{a}u\bar{d} = vac\bar{b}u\bar{b} = v\bar{c}v\bar{u}c\bar{u} = udvdca = a^2d^2\bar{b}^2}\big\}$

Class 16 $\circ$ $\mathcal{H}^{(6)}_{1,27} = \ldots = \mathcal{H}^{(6)}_{1,38} = \mathcal{Z}_2^2 \times \mathcal{Z}^2$ $\circ$ **p1**: 1 cusp, **pg**: 3 cusps . . . .

- $\mathcal{G}^{(6)}_{27} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\HGQLR\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\DCMPN\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\GHIJF\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}DNTOE\\OSKAE\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}FPMQG\\IHRKS\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = vxz\bar{v}w\bar{u} = wyzx\bar{y}\bar{u} = xvuw\bar{y}\bar{z} = x^2u\bar{y}\bar{v}\bar{u} = wz^2\bar{w}\bar{v}\bar{y}}\big\}$
- $\mathcal{G}^{(6)}_{28} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\MQGFP\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\TOEDN\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\OSKAE\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}CMPND\\RHGQL\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}FGHIJ\\SIHRK\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1 = uwvz^2\bar{v} = zx\bar{u}w\bar{v}\bar{y} = xvu\bar{z}\bar{y}\bar{u} = wuxy^2\bar{x} = wzy\bar{w}\bar{x}\bar{v}}\big\}$
- $\mathcal{G}^{(6)}_{29} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\GHIJF\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\MQGFP\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\OEDNT\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}CMPND\\LRHGQ\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}EOSKA\\IHRKS\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = vx\bar{y}zx\bar{u} = wz\bar{v}wy\bar{u} = xwu\bar{z}^2\bar{u} = yz\bar{w}xv\bar{u} = y^2v\bar{x}\bar{w}\bar{v}}\big\}$
- $\mathcal{G}^{(6)}_{30} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\LBAKR\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}BLQMC\\NDCMP\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}DNTOE\\OSKAE\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}FGHIJ\\QGFPM\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\SIHRK\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = u^2vwz\bar{v} = yx\bar{z}w\bar{u}\bar{v} = w^2uy^2\bar{u} = z^2xv^2\bar{x} = x^2yzw\bar{y}}\big\}$
- $\mathcal{G}^{(6)}_{31} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\TOEDN\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\SKAEO\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\HGQLR\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}CMPND\\GFPMQ\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}FGHIJ\\SIHRK\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = vzx^2\bar{z}\bar{u} = wy^2\bar{w}u\bar{v} = u^2xw^2\bar{x} = xyz\bar{u}\bar{v}\bar{w} = z^2y\bar{v}^2\bar{y}}\big\}$
- $\mathcal{G}^{(6)}_{32} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\TOEDN\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\AEOSK\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\NDCMP\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}FGHIJ\\FPMQG\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\RKSIH\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1 = v^2u\bar{z}^2\bar{u} = xyv\bar{u}w\bar{z} = wu^2\bar{w}v\bar{y} = y^2xw^2\bar{x} = vzx^2\bar{z}\bar{y}}\big\}$
- $\mathcal{G}^{(6)}_{33} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\AKRLB\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}BLQMC\\NDCMP\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}DNTOE\\FGHIJ\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\QGFPM\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\RKSIH\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1 = xv^2\bar{x}\bar{u}^2 = xy\bar{u}v\bar{z}\bar{w} = vuyz\bar{x}\bar{y} = uvw\bar{x}z\bar{w} = z^2wy^2\bar{w}}\big\}$
- $\mathcal{G}^{(6)}_{34} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\HIJFG\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\AEOSK\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\EDNTO\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}CMPND\\PMQGF\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\RKSIH\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1 = v^2uy^2\bar{u} = uzxyv\bar{w} = vz\bar{u}x\bar{z}\bar{y} = wx\bar{u}\bar{w}v\bar{y} = x^2w\bar{z}^2\bar{w}}\big\}$
- $\mathcal{G}^{(6)}_{35} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\QGFPM\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\HIJFG\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\NDCMP\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}DNTOE\\OSKAE\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\SIHRK\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = uyzx\bar{y}\bar{v} = wxz\bar{w}u\bar{v} = xv\bar{y}wu\bar{z} = x^2uw^2\bar{u} = y^2vz^2\bar{v}}\big\}$
- $\mathcal{G}^{(6)}_{36} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\TOEDN\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\SKAEO\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\HGQLR\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}CMPND\\FGHIJ\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}FPMQG\\IHRKS\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = vzx\bar{y}\bar{z}\bar{u} = w\bar{y}x\bar{w}v\bar{u} = u^2xz^2\bar{x} = xyz\bar{u}\bar{v}\bar{w} = w^2y\bar{v}^2\bar{y}}\big\}$
- $\mathcal{G}^{(6)}_{37} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\MPNDC\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}DNTOE\\JFGHI\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\FPMQG\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\RKSIH\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = v^2u\bar{z}x\bar{u} = wz^2\bar{w}\bar{u}^2 = yzw\bar{x}u\bar{v} = xyv\bar{x}\bar{y}\bar{v} = wy^2\bar{w}x\bar{z}}\big\}$
- $\mathcal{G}^{(6)}_{38} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\TOEDN\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}CMPND\\FGHIJ\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\FPMQG\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\RKSIH\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = v^2ux\bar{z}\bar{u} = u^2wz^2\bar{w} = yz\bar{u}x\bar{w}\bar{v} = xyv\bar{x}\bar{y}\bar{v} = xwy^2\bar{w}\bar{z}}\big\}$

Class 17 $\circ$ $\mathcal{H}^{(6)}_{1,39} = \mathcal{Z}_2^4 \times \mathcal{Z}$ $\circ$ **pg**: 4 cusps . . . .

- $\mathcal{G}^{(6)}_{39} = \big\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\FJIHG\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}CMPND\\FGQMP\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}DNTOE\\OEAKS\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}GQLRH\\RHISK\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = u^2av^2\bar{a} = cud\bar{a}b\bar{v} = cvb\bar{u}\bar{d}\bar{a} = dvca\bar{b}\bar{u} = b^2d^2\bar{c}^2}\big\}$

Class 18 $\circ$ $\mathcal{H}^{(6)}_{1,40} = \mathcal{Z}_4 \times \mathcal{Z}$ $\circ$ **pg**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_{40} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\PMQGF\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}AKRLB\\TNDEO\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}BLQMC\\GHIJF\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}CMPND\\SOEAK\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}GQLRH\\OSIJT\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}HRKSI\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1 = ub\bar{a}vc\bar{a} = uv\bar{c}au\bar{v} = bd\bar{v}b\bar{c}\bar{u} = vdbu\bar{d}\bar{c} = aba\bar{d}c\bar{d}}\big\}$

Class 19 $\circ$ $\mathcal{H}^{(6)}_{1,41}=\mathcal{H}^{(6)}_{1,42}=\mathcal{Z}_4\times\mathcal{Z}$ $\circ$ **pg**: 2 cusps ....

- $\mathcal{G}^{(6)}_{41}=\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\GHIJF\end{smallmatrix}\right],\ \mathbf{a}\left[\begin{smallmatrix}AKRLB\\NDEOT\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}BLQMC\\PMQGF\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}CMPND\\RHISK\end{smallmatrix}\right],\ \mathbf{w}\left[\begin{smallmatrix}EOSKA\\RHGQL\end{smallmatrix}\right],\ \mathbf{x}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{xbx\bar{a}u\bar{w}}=\mathbf{vb\bar{u}v\bar{x}\bar{a}}=\mathbf{bau\bar{x}^2\bar{u}}=\mathbf{waw\bar{b}v\bar{u}}=\mathbf{wv^2\bar{w}\bar{b}\bar{a}}\}$
- $\mathcal{G}^{(6)}_{42}=\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\MPFGQ\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}AKRLB\\HGFJI\end{smallmatrix}\right],\ \mathbf{c}\left[\begin{smallmatrix}BLQMC\\TNDEO\end{smallmatrix}\right],\ \mathbf{u}\left[\begin{smallmatrix}CMPND\\RKSIH\end{smallmatrix}\right],\ \mathbf{d}\left[\begin{smallmatrix}EOSKA\\HRLQG\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}ISOTJ\\JTNPF\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{audcu\bar{b}}=\mathbf{cvuv^2\bar{a}}=\mathbf{ub\bar{a}u\bar{d}\bar{c}}=\mathbf{aca\bar{d}b\bar{d}}=\mathbf{db\bar{v}\bar{b}c\bar{v}}\}$

Class 20 $\circ$ $\mathcal{H}^{(6)}_{1,43}=\mathcal{H}^{(6)}_{1,44}=\mathcal{Z}_4\times\mathcal{Z}^2$ $\circ$ **p1**: 1 cusp, **pg**: 1 cusp ....

- $\mathcal{G}^{(6)}_{43}=\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\LRKAB\end{smallmatrix}\right],\ \mathbf{u}\left[\begin{smallmatrix}BLQMC\\OSKAE\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}CMPND\\PFGQM\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}DNTOE\\FGHIJ\end{smallmatrix}\right],\ \mathbf{c}\left[\begin{smallmatrix}GQLRH\\KRHIS\end{smallmatrix}\right],\ \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{a^2c\bar{v}d\bar{v}}=\mathbf{uvdbc\bar{a}}=\mathbf{b^2\bar{v}bu\bar{a}}=\mathbf{uaud^2\bar{b}}=\mathbf{c^2dvc\bar{u}}\}$
- $\mathcal{G}^{(6)}_{44}=\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\QMPFG\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}AKRLB\\DEOTN\end{smallmatrix}\right],\ \mathbf{c}\left[\begin{smallmatrix}BLQMC\\HGFJI\end{smallmatrix}\right],\ \mathbf{u}\left[\begin{smallmatrix}CMPND\\KAEOS\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}GQLRH\\TJISO\end{smallmatrix}\right],\ \mathbf{d}\left[\begin{smallmatrix}HRKSI\\JFPNT\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{bu\bar{v}d\bar{c}\bar{a}}=\mathbf{auav\bar{d}\bar{c}}=\mathbf{udu\bar{b}d\bar{a}}=\mathbf{uba\bar{c}v\bar{c}}=\mathbf{cv\bar{b}vd\bar{b}}\}$

Class 21 $\circ$ $\mathcal{H}^{(6)}_{1,45}=\mathcal{Z}_2\times\mathcal{Z}_4\times\mathcal{Z}$ $\circ$ **pg**: 2 cusps ....

- $\mathcal{G}^{(6)}_{45}=\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\CMQLB\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}AKRLB\\DNPMC\end{smallmatrix}\right],\ \mathbf{c}\left[\begin{smallmatrix}DNTOE\\QGHRL\end{smallmatrix}\right],\ \mathbf{u}\left[\begin{smallmatrix}EOSKA\\IJFGH\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}FPMQG\\HRKSI\end{smallmatrix}\right],\ \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{ba\bar{c}ab\bar{a}}=\mathbf{a^2vuv\bar{u}}=\mathbf{bcvdc\bar{u}}=\mathbf{cbvb\bar{d}\bar{u}}=\mathbf{d^2v\bar{c}d\bar{u}}\}$

Class 22 $\circ$ $\mathcal{H}^{(6)}_{1,46}=\mathcal{H}^{(6)}_{1,47}=\mathcal{H}^{(6)}_{1,48}=\mathcal{Z}_2\times\mathcal{Z}_4\times\mathcal{Z}$ $\circ$ **pg**: 3 cusps ....

- $\mathcal{G}^{(6)}_{46}=\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\MPFGQ\end{smallmatrix}\right],\ \mathbf{u}\left[\begin{smallmatrix}AKRLB\\MCBLQ\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}CMPND\\EAKSO\end{smallmatrix}\right],\ \mathbf{c}\left[\begin{smallmatrix}DNTOE\\IHGFJ\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}GQLRH\\RKSIH\end{smallmatrix}\right],\ \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\PNTJF\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{abuba\bar{u}}=\mathbf{avua\bar{d}\bar{c}}=\mathbf{bc\bar{d}^2c\bar{a}}=\mathbf{vdb\bar{v}\bar{u}^2}=\mathbf{bdcv^2\bar{c}}\}$
- $\mathcal{G}^{(6)}_{47}=\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\TOEDN\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}AKRLB\\AEOSK\end{smallmatrix}\right],\ \mathbf{a}\left[\begin{smallmatrix}BLQMC\\IHGFJ\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}CMPND\\PFGQM\end{smallmatrix}\right],\ \mathbf{w}\left[\begin{smallmatrix}GQLRH\\RKSIH\end{smallmatrix}\right],\ \mathbf{x}\left[\begin{smallmatrix}ISOTJ\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{v^2u\bar{x}^2\bar{u}}=\mathbf{axbwv\bar{u}}=\mathbf{u^2ba\bar{b}^2}=\mathbf{aw^2\bar{a}v\bar{w}}=\mathbf{vx\bar{a}b\bar{x}\bar{w}}\}$
- $\mathcal{G}^{(6)}_{48}=\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\TOEDN\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right],\ \mathbf{w}\left[\begin{smallmatrix}CMPND\\GFPMQ\end{smallmatrix}\right],\ \mathbf{a}\left[\begin{smallmatrix}EOSKA\\FJIHG\end{smallmatrix}\right],\ \mathbf{x}\left[\begin{smallmatrix}GQLRH\\RKSIH\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{v^2ua\bar{b}\bar{u}}=\mathbf{wxv\bar{w}\bar{u}^2}=\mathbf{xb\bar{u}a\bar{w}\bar{v}}=\mathbf{ax^2\bar{a}\bar{x}\bar{v}}=\mathbf{abw^2\bar{b}^2}\}$

Class 23 $\circ$ $\mathcal{H}^{(6)}_{1,49}=\mathcal{H}^{(6)}_{1,50}=\mathcal{Z}_2^2\times\mathcal{Z}_4\times\mathcal{Z}$ $\circ$ **pg**: 3 cusps ....

- $\mathcal{G}^{(6)}_{49}=\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\FJIHG\end{smallmatrix}\right],\ \mathbf{u}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}CMPND\\LQGHR\end{smallmatrix}\right],\ \mathbf{c}\left[\begin{smallmatrix}DNTOE\\MPFGQ\end{smallmatrix}\right],\ \mathbf{d}\left[\begin{smallmatrix}EOSKA\\SKRHI\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{u^2av^2\bar{a}}=\mathbf{bub\bar{u}d\bar{a}}=\mathbf{vb\bar{a}c\bar{u}\bar{d}}=\mathbf{dvc\bar{v}c\bar{a}}=\mathbf{cb\bar{c}d^2\bar{b}}\}$
- $\mathcal{G}^{(6)}_{50}=\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\JFGHI\end{smallmatrix}\right],\ \mathbf{a}\left[\begin{smallmatrix}AKRLB\\PFGQM\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}BLQMC\\AKSOE\end{smallmatrix}\right],\ \mathbf{c}\left[\begin{smallmatrix}CMPND\\ISKRH\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}DNTOE\\HGQLR\end{smallmatrix}\right],\ \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{ac\bar{b}\bar{v}d\bar{u}}=\mathbf{budca\bar{u}}=\mathbf{c\bar{u}v\bar{c}v\bar{u}}=\mathbf{abd^2\bar{a}\bar{b}}=\mathbf{cdv\bar{a}\bar{v}\bar{b}}\}$

Class 24 $\circ$ $\mathcal{H}^{(6)}_{1,51}=\mathcal{Z}_2\times\mathcal{Z}_8\times\mathcal{Z}$ $\circ$ **pg**: 2 cusps ....

- $\mathcal{G}^{(6)}_{51}=\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\QMPFG\end{smallmatrix}\right],\ \mathbf{u}\left[\begin{smallmatrix}AKRLB\\HIJFG\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}BLQMC\\EOTND\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}CMPND\\AEOSK\end{smallmatrix}\right],\ \mathbf{c}\left[\begin{smallmatrix}GQLRH\\RHISK\end{smallmatrix}\right],\ \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\TJFPN\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{ucud\bar{b}\bar{a}}=\mathbf{ba\bar{u}b\bar{v}\bar{a}}=\mathbf{vu\bar{c}bd\bar{a}}=\mathbf{vac^2\bar{v}\bar{b}}=\mathbf{cd^2vd\bar{u}}\}$

Class 25 $\circ$ $\mathcal{H}^{(6)}_{1,52}=\mathcal{H}^{(6)}_{1,53}=\mathcal{H}^{(6)}_{1,54}=\mathcal{Z}_2\times\mathcal{Z}_8\times\mathcal{Z}$ $\circ$ **pg**: 3 cusps ....

- $\mathcal{G}^{(6)}_{52}=\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\RLBAK\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}BLQMC\\OEDNT\end{smallmatrix}\right],\ \mathbf{w}\left[\begin{smallmatrix}CMPND\\EOSKA\end{smallmatrix}\right],\ \mathbf{x}\left[\begin{smallmatrix}FGHIJ\\FPMQG\end{smallmatrix}\right],\ \mathbf{y}\left[\begin{smallmatrix}GQLRH\\SIHRK\end{smallmatrix}\right],\ \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\FJTNP\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{u^2y^2\bar{u}\bar{w}}=\mathbf{vz\bar{v}w\bar{v}\bar{u}}=\mathbf{yxvw\bar{u}\bar{v}}=\mathbf{xyzx^2\bar{z}}=\mathbf{xwz^2w\bar{y}}\}$
- $\mathcal{G}^{(6)}_{53}=\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\BLQMC\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}AKRLB\\TOEDN\end{smallmatrix}\right],\ \mathbf{w}\left[\begin{smallmatrix}CMPND\\EOSKA\end{smallmatrix}\right],\ \mathbf{x}\left[\begin{smallmatrix}FGHIJ\\FPMQG\end{smallmatrix}\right],\ \mathbf{y}\left[\begin{smallmatrix}GQLRH\\SIHRK\end{smallmatrix}\right],\ \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\FJTNP\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{uvwvz\bar{v}}=\mathbf{wu^2yx\bar{u}}=\mathbf{y^2v\bar{w}\bar{u}\bar{v}}=\mathbf{xyzx^2\bar{z}}=\mathbf{xwz^2w\bar{y}}\}$
- $\mathcal{G}^{(6)}_{54}=\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\IJFGH\end{smallmatrix}\right],\ \mathbf{a}\left[\begin{smallmatrix}AKRLB\\OTNDE\end{smallmatrix}\right],\ \mathbf{v}\left[\begin{smallmatrix}BLQMC\\PMQGF\end{smallmatrix}\right],\ \mathbf{w}\left[\begin{smallmatrix}CMPND\\QLRHG\end{smallmatrix}\right],\ \mathbf{x}\left[\begin{smallmatrix}EOSKA\\RKSIH\end{smallmatrix}\right],\ \mathbf{b}\left[\begin{smallmatrix}ISOTJ\\JFPNT\end{smallmatrix}\right]$
  — $\mathbf{1}=\mathbf{axa\bar{b}^2\bar{u}}=\mathbf{x^2b\bar{u}v\bar{b}}=\mathbf{wv^2\bar{w}u\bar{v}}=\mathbf{vwaw\bar{u}\bar{a}}=\mathbf{abw\bar{x}u\bar{x}}\}$

Class 26 $\circ$ $\mathcal{H}^{(6)}_{1,55} = \mathcal{Z}_2^2 \times \mathcal{Z}_8 \times \mathcal{Z}$ $\circ$ **pg**: 3 cusps ....

- $\mathcal{G}^{(6)}_{55} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\CBLQM\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}AKRLB\\NDEOT\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}CMPND\\HGQLR\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}EOSKA\\QGFPM\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}FGHIJ\\SKRHI\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}ISOTJ\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1 = u^2ax^2\bar{a} = xv\bar{u}v\bar{b}^2 = uvbau\bar{w} = avaw\bar{v}\bar{w} = xbx\bar{w}\bar{b}\bar{w}}\big\}$

Class 27 $\circ$ $\mathcal{H}^{(6)}_{1,56} = \mathcal{Z}_2 \times \mathcal{Z}_{16} \times \mathcal{Z}$ $\circ$ **pg**: 3 cusps ....

- $\mathcal{G}^{(6)}_{56} = \big\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\FGQMP\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}AKRLB\\NTOED\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}BLQMC\\GHIJF\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}CMPND\\IJTOS\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}EOSKA\\LQGHR\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}HRKSI\\PNTJF\end{smallmatrix}\right]$
  — $\mathbf{1 = ubc\bar{a}v\bar{a} = bd\bar{v}b\bar{v}\bar{a} = uab\bar{d}c\bar{v} = cucvd\bar{a} = cdb\bar{u}d\bar{u}}\big\}$

Class 28 $\circ$ $\mathcal{H}^{(6)}_{1,57} = \mathcal{H}^{(6)}_{1,58} = \mathcal{Z}_4 \times \mathcal{Z}_4 \times \mathcal{Z}$ $\circ$ **pg**: 3 cusps ....

- $\mathcal{G}^{(6)}_{57} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\CBLQM\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\DCMPN\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}DNTOE\\OSKAE\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}FGHIJ\\HGQLR\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}FPMQG\\IHRKS\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = vwz\bar{v}\bar{u}^2 = zyx\bar{u}\bar{v}\bar{w} = w^2u\bar{v}\bar{y}\bar{u} = yzw\bar{y}\bar{x}^2 = z^2x\bar{y}\bar{v}\bar{x}}\big\}$
- $\mathcal{G}^{(6)}_{58} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\LBAKR\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}BLQMC\\HGQLR\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}CMPND\\FGHIJ\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}DNTOE\\KAEOS\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}FPMQG\\IHRKS\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = u^2v\bar{w}y\bar{v} = wz^2\bar{w}u\bar{x} = v^2y\bar{x}u\bar{y} = wyzxu\bar{v} = x^2zw\bar{y}\bar{z}}\big\}$

Class 29 $\circ$ $\mathcal{H}^{(6)}_{1,59} = \mathcal{Z}_3 \times \mathcal{Z}^2$ $\circ$ **p1**: 1 cusp, **pg**: 1 cusp ....

- $\mathcal{G}^{(6)}_{59} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\JFGHI\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\BLQMC\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}CMPND\\EDNTO\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\FPMQG\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\IHRKS\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\JTNPF\end{smallmatrix}\right]$
  — $\mathbf{1 = vu\bar{x}uz\bar{u} = uyz^2\bar{x}\bar{w} = v^2wu\bar{y}\bar{x} = xw^2zw\bar{z} = yvy^2x\bar{v}}\big\}$

Class 30 $\circ$ $\mathcal{H}^{(6)}_{1,60} = \mathcal{Z}_2^2 \times \mathcal{Z}_3 \times \mathcal{Z}$ $\circ$ **pg**: 1 cusp ....

- $\mathcal{G}^{(6)}_{60} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\FGHIJ\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\CMPND\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\TOEDN\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\FPMQG\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\OTJIS\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}HRKSI\\JTNPF\end{smallmatrix}\right]$
  — $\mathbf{1 = vuz\bar{u}x\bar{u} = uyx\bar{v}z\bar{w} = xwu\bar{y}v\bar{z} = xy\bar{w}v\bar{w}\bar{v} = wx\bar{z}\bar{y}z\bar{y}}\big\}$

Class 31 $\circ$ $\mathcal{H}^{(6)}_{1,61} = \mathcal{Z}_2^2 \times \mathcal{Z}_3 \times \mathcal{Z}$ $\circ$ **pg**: 2 cusps ....

- $\mathcal{G}^{(6)}_{61} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\IJFGH\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\CBLQM\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}CMPND\\HRLQG\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}DNTOE\\ISKRH\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}EOSKA\\FPMQG\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}ISOTJ\\FJTNP\end{smallmatrix}\right]$
  — $\mathbf{1 = va\bar{b}w\bar{x}\bar{u} = bx\bar{u}\bar{v}w\bar{a} = a\bar{u}b\bar{u}w\bar{u} = xbw\bar{v}^2\bar{b} = wav\bar{a}\bar{x}^2}\big\}$

Class 32 $\circ$ $\mathcal{H}^{(6)}_{1,62} = \ldots = \mathcal{H}^{(6)}_{1,65} = \mathcal{Z}_2^2 \times \mathcal{Z}_3 \times \mathcal{Z}$ $\circ$ **pg**: 3 cusps ....

- $\mathcal{G}^{(6)}_{62} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\FGHIJ\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}CMPND\\OEDNT\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\MQGFP\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\SIHRK\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = v^2uyx\bar{u} = wz\bar{x}vy\bar{u} = xwuz^2\bar{u} = vwx\bar{v}\bar{y}^2 = xyz\bar{w}^2\bar{z}}\big\}$
- $\mathcal{G}^{(6)}_{63} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\FGHIJ\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}CMPND\\PMQGF\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}DNTOE\\HGQLR\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}EOSKA\\RKSIH\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  — $\mathbf{1 = v^2u\bar{x}w\bar{u} = w\bar{z}x\bar{v}y\bar{u} = uz^2\bar{u}y\bar{x} = xvw^2\bar{v}\bar{y} = y^2zx\bar{w}\bar{z}}\big\}$
- $\mathcal{G}^{(6)}_{64} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\IJFGH\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\NDCMP\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\EDNTO\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\GFPMQ\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}GQLRH\\JISOT\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}HRKSI\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1 = xy\bar{u}v\bar{z}\bar{u} = wx\bar{u}\bar{v}z\bar{u} = wuy\bar{z}x\bar{v} = xw\bar{z}\bar{y}w\bar{v} = yx\bar{z}y\bar{w}\bar{v}}\big\}$
- $\mathcal{G}^{(6)}_{65} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\IJFGH\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\NDCMP\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\TJISO\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}DNTOE\\MQGFP\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}EOSKA\\LRHGQ\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}HRKSI\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1 = yw\bar{u}v\bar{z}\bar{u} = wx\bar{u}\bar{v}z\bar{u} = x\bar{v}w\bar{z}y\bar{u} = vxwzx\bar{y} = yvwyz\bar{x}}\big\}$

Class 33 $\circ$ $\mathcal{H}^{(6)}_{1,66} = \ldots = \mathcal{H}^{(6)}_{1,71} = \mathcal{Z}_3 \times \mathcal{Z}_4 \times \mathcal{Z}$ $\circ$ **pg**: 2 cusps ....

- $\mathcal{G}^{(6)}_{66} = \big\{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\HIJFG\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}AKRLB\\LBCMQ\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}CMPND\\RHGQL\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}DNTOE\\SIHRK\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\QGFPM\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  — $\mathbf{1 = wbw\bar{u}a\bar{v} = waubv\bar{x} = vav\bar{w}b\bar{u} = wx\bar{b}^2x\bar{u} = xv\bar{u}x\bar{a}^2}\big\}$
- $\mathcal{G}^{(6)}_{67} = \big\{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\FJIHG\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}AKRLB\\MCBLQ\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}CMPND\\RLQGH\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}DNTOE\\SIHRK\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}EOSKA\\QGFPM\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}ISOTJ\\PFJTN\end{smallmatrix}\right]$
  — $\mathbf{1 = u\bar{w}a\bar{w}x\bar{a} = vuaxb\bar{w} = x^2v\bar{a}b\bar{v} = vw\bar{x}\bar{v}b\bar{a} = ubu^2\bar{b}\bar{w}}\big\}$

- $\mathcal{G}^{(6)}_{68} = \{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\HIJFG\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\MCBLQ\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}CMPND\\RLQGH\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}DNTOE\\HISKR\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}EOSKA\\QGFPM\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}ISOTJ\\PFJTN\end{smallmatrix}\right]$
  $—\ \mathbf{1 = v\bar{w}u\bar{a}b\bar{u} = bvuxa\bar{w} = a\bar{b}u\bar{w}x\bar{u} = vav^2\bar{a}\bar{w} = x^2bx\bar{w}\bar{b}}\}$
- $\mathcal{G}^{(6)}_{69} = \{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\RHGQL\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}AKRLB\\QLBCM\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}CMPND\\KSIHR\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}DNTOE\\JFGHI\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\PMQGF\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}ISOTJ\\TJFPN\end{smallmatrix}\right]$
  $—\ \mathbf{1 = a\bar{x}b\bar{w}v\bar{u} = uau\bar{w}b\bar{x} = va^2vx\bar{u} = wbw\bar{x}a\bar{u} = xvb^2v\bar{w}}\}$
- $\mathcal{G}^{(6)}_{70} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\LQGHR\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}CMPND\\SIHRK\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}DNTOE\\HGFJI\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}EOSKA\\FPMQG\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  $—\ \mathbf{1 = u^2a\bar{w}u\bar{a} = vw\bar{u}\bar{v}b\bar{a} = xv\bar{a}b\bar{v}\bar{w} = auvxb\bar{w} = bx^2\bar{b}x\bar{w}}\}$
- $\mathcal{G}^{(6)}_{71} = \{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\RHGQL\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\LQMCB\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}CMPND\\ISKRH\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}DNTOE\\HGFJI\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}EOSKA\\FPMQG\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}ISOTJ\\TNPFJ\end{smallmatrix}\right]$
  $—\ \mathbf{1 = v^2u\bar{b}a\bar{u} = a\bar{b}u\bar{v}w\bar{u} = uvaxb\bar{w} = bx^2\bar{b}x\bar{w} = waw\bar{v}\bar{a}\bar{x}}\}$

Class 34 ◦ $\mathcal{H}^{(6)}_{1,72} = \mathcal{Z}_2\times\mathcal{Z}_3\times\mathcal{Z}_4\times\mathcal{Z}$ ◦ **pg**: 3 cusps . . . .

- $\mathcal{G}^{(6)}_{72} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\QMPFG\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\DEOTN\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}BLQMC\\HGFJI\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}CMPND\\KSIHR\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}EOSKA\\LRHGQ\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\TJFPN\end{smallmatrix}\right]$
  $—\ \mathbf{1 = bu\bar{v}d\bar{c}\bar{a} = aud^2u\bar{c} = ubvbd\bar{a} = vc\bar{a}\bar{b}v\bar{a} = cd\bar{b}c\bar{v}\bar{u}}\}$

Class 35 ◦ $\mathcal{H}^{(6)}_{1,73} = \mathcal{Z}_3\times\mathcal{Z}_8\times\mathcal{Z}$ ◦ **pg**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_{73} = \{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\GHIJF\end{smallmatrix}\right], \mathbf{a}\left[\begin{smallmatrix}AKRLB\\MQLBC\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}CMPND\\QLRHG\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}DNTOE\\RKSIH\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\PMQGF\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}ISOTJ\\JFPNT\end{smallmatrix}\right]$
  $—\ \mathbf{1 = av\bar{a}\bar{w}v\bar{u} = xva^2u\bar{w} = ub^2wx\bar{v} = wb\bar{u}w\bar{v}\bar{b} = x\bar{b}x\bar{a}x\bar{u}}\}$

Class 36 ◦ $\mathcal{H}^{(6)}_{1,74} = \mathcal{Z}_2\times\mathcal{Z}_3\times\mathcal{Z}_{16}\times\mathcal{Z}$ ◦ **pg**: 3 cusps . . . .

- $\mathcal{G}^{(6)}_{74} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\LRKAB\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}BLQMC\\EDNTO\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}CMPND\\PFGQM\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}EOSKA\\IJFGH\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}GQLRH\\KRHIS\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  $—\ \mathbf{1 = aua^2c\bar{v} = uvdbc\bar{a} = avcv\bar{b}^2 = ud\bar{v}d^2\bar{b} = c^2d\bar{u}\bar{b}\bar{u}}\}$

Class 37 ◦ $\mathcal{H}^{(6)}_{1,75} = \mathcal{H}^{(6)}_{1,76} = \mathcal{H}^{(6)}_{1,77} = \mathcal{Z}_2^2\times\mathcal{Z}_5\times\mathcal{Z}$ ◦ **pg**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_{75} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\CMQLB\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\DNPMC\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}DNTOE\\QGHRL\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}EOSKA\\FGHIJ\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}FPMQG\\SIHRK\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  $—\ \mathbf{1 = ba\bar{c}ab\bar{a} = a^2v\bar{c}d\bar{u} = bcvb\bar{d}\bar{u} = uvu\bar{v}\bar{b}\bar{c} = d^2vdc\bar{u}}\}$
- $\mathcal{G}^{(6)}_{76} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\CMQLB\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\DNPMC\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}DNTOE\\QGHRL\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}EOSKA\\GFPMQ\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}FGHIJ\\HISKR\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}ISOTJ\\JTNPF\end{smallmatrix}\right]$
  $—\ \mathbf{1 = ba\bar{c}ab\bar{a} = bc\bar{u}a^2\bar{u} = ud^2u\bar{b}\bar{c} = vcdv^2\bar{u} = vd\bar{c}v\bar{b}\bar{d}}\}$
- $\mathcal{G}^{(6)}_{77} = \{\mathbf{a}\left[\begin{smallmatrix}ABCDE\\CMQLB\end{smallmatrix}\right], \mathbf{b}\left[\begin{smallmatrix}AKRLB\\DNPMC\end{smallmatrix}\right], \mathbf{c}\left[\begin{smallmatrix}DNTOE\\QGHRL\end{smallmatrix}\right], \mathbf{u}\left[\begin{smallmatrix}EOSKA\\MQGFP\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}FGHIJ\\IHRKS\end{smallmatrix}\right], \mathbf{d}\left[\begin{smallmatrix}ISOTJ\\NTJFP\end{smallmatrix}\right]$
  $—\ \mathbf{1 = ba\bar{c}ab\bar{a} = cb\bar{u}a^2\bar{u} = bc\bar{u}d^2\bar{u} = vuv^2b\bar{d} = dvdcv\bar{c}}\}$

Class 38 ◦ $\mathcal{H}^{(6)}_{1,78} = \mathcal{Z}_3\times\mathcal{Z}_5\times\mathcal{Z}$ ◦ **pg**: 1 cusp . . . .

- $\mathcal{G}^{(6)}_{78} = \{\mathbf{u}\left[\begin{smallmatrix}ABCDE\\HGQLR\end{smallmatrix}\right], \mathbf{v}\left[\begin{smallmatrix}AKRLB\\DCMPN\end{smallmatrix}\right], \mathbf{w}\left[\begin{smallmatrix}BLQMC\\OEDNT\end{smallmatrix}\right], \mathbf{x}\left[\begin{smallmatrix}EOSKA\\QGFPM\end{smallmatrix}\right], \mathbf{y}\left[\begin{smallmatrix}FGHIJ\\IHRKS\end{smallmatrix}\right], \mathbf{z}\left[\begin{smallmatrix}ISOTJ\\JTNPF\end{smallmatrix}\right]$
  $—\ \mathbf{1 = uy\bar{u}xw\bar{v} = wz\bar{v}wx\bar{u} = vuwuv\bar{x} = xy^2vw\bar{z} = zyz^2\bar{x}\bar{y}}\}$

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