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Pappus' Theorem for Ring-geometries

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Abstract. A famous result, due to Hilbert, states for a projective plane over a division ring K the equivalence of Pappus' theorem and the commutativity of K. In this paper we extend this result to the projective plane over an arbitrary ring.

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1. Introduction

Let $\mathcal{P} = \mathcal{P}(K)$ a projective plane over a field K. In [1] E. Artin writes: "One of the simple but fascinating results in foundations of geometry is the fact that one can find a simple geometric configuration which is equivalent with the commutative law for multiplication of our field K."

Let us take a closer look at this configuration: Given distinct lines l and m of the projective plane which intersect in a unique point, say Z. With $i = 1, 2, 3$ let L_i and M_i be pairwise distinct points on l and m, respectively, which are all distinct from Z as well. With $1 \leq i < j \leq 3$ consider the three distinct points $P_{i,j} = (L_i + M_j) \cap (L_j + M_i)$. We say *Pappus' theorem* holds iff $P_{1,2}$, $P_{1,3}$, and $P_{2,3}$ are collinear (see Fig. 1).

The crucial point of the proof of the above stated equivalence is identifying the group of multiplications of K with the subgroup $\Gamma(Z, P_{1,2} + P_{1,3})$ of the group of collineations of $\mathcal{P}$, i.e. $\sigma \in \Gamma(Z, P_{1,2} + P_{1,3})$ iff σ has *centre* Z (i.e. every line through Z remains fix under σ) and *axis* $P_{1,2} + P_{1,3}$ (i.e. every point of $P_{1,2} + P_{1,3}$ is a fix point of σ).

The challenge in extending this well-known equivalence to ring-geometries is finding a representation of the ring's additive and multiplicative operation in terms of appropriate geometric mappings. But let us not get ahead of our story. First, let us review our concept of ring-geometry.

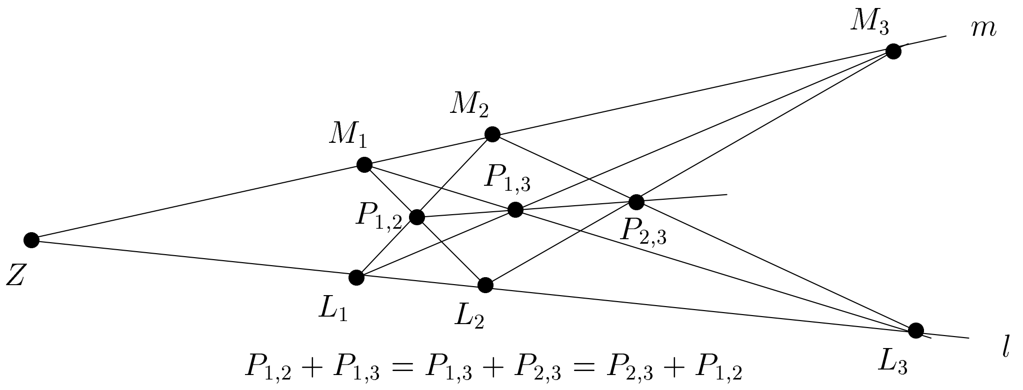

Fig. 1

2. Planar geometry over a ring

Preliminaries. Our approach to ring-geometries is a lattice-theoretic one. A given arbitrary ring[1] R induces the *planar projective lattice geometry* $PG(_RR^3) := (L, P)$, where $L := L(_RR^3)$ denotes the *lattice of all submodules* of $_RR^3$ (partially ordered by set-inclusion) and $P := P(_RR^3)$ denotes the *set of freely one generated submodules* of $_RR^3$. The elements of P are called *points*, i.e. a point is a cyclic submodule of $_RR^3$ which is (module-)isomorphic to the underlying ring R. The elements of L are called *geometric objects*.

By abuse of notation we use "$\leq$" for the order of L and "1_L" ("0_L") for the top (bottom) element of L, respectively, i.e. $1_L =_R R^3$ and $0_L = \{0\}$. The *meet* of an arbitrary family of submodules is given by their set-theoretic intersection and the meet of two geometric objects x, y is denoted by xy. The *join* of an arbitrary family of submodules U_i, $i \in I$ in L is given by $\sum_{i \in I} U_i$, i.e. the smallest submodule which contains all moduls U_i. For geometric objects $x, y \in L$ we write $x + y := \sum\{x, y\}$ and $x \oplus y$ if x and y are *disjoint*, i.e. $xy = 0_L$. A *line* l is the *span* of two disjoint points, i.e. $l = x \oplus y$ for points x, y. Three geometric objects x, y, z are *directly balanced* iff $x \oplus y = y \oplus z = z \oplus x$. A geometric object y is called *complement* of x if $x \oplus y = 1_L$ and x, y are *perspective via* z if $x \oplus z = y \oplus z$ for some $z \in L$.

For any geometric object x we obtain the *subgeometry* of x as $PG(x) := (L(x), P(x))$ where $L(x)$ denotes the sublattice $\{t \in L \mid t \leq x\}$ and $P(x) = \{p \in P \mid p \leq x\}$. Note that $P(x) = \emptyset$ is possible for some $x \in L$.

It is well-known that L is a *modular* lattice, i.e. $(x + y)z = x + yz$ holds for all $x, y, z \in L$ with $x \leq z$ (where $x + yz := x + (yz)$).

A *regular triangle* is a family of points p_i, $i \in \{1, 2, 3\}$, such that $p_3\,(p_1 \oplus p_2) = 0_L$. A *3-frame* is a set of points p_i, $i \in I = \{1, 2, 3, 4\}$, such that for every $J \subseteq I$ with $|J| = 3$ the family p_j, $j \in J$, is a regular triangle and $\sum\{p_j \mid j \in J\} = \sum\{p_{j'} \mid j' \in J'\}$ for any $J' \subseteq I$ with $|J'| = 3$. Due to modularity of L, it is easily to be shown that every regular triangle can be extended to a frame.

[1] A ring is associative with $1 \neq 0$

Projections and perspectivities. Our goal is the representation of the ring's addition and multiplication with appropriate mappings. So let us now turn to our concept of mappings of a planar geometry $G = PG(_RR^3)$ and its subgeometries:

A *mapping* between geometries $G = (L, P)$ and $G' = (L', P')$ is a mapping $\varphi : L \to L'$ of the underlying lattices L and L'. An *isomorphism* between geometries G and G' is an order-isomorphism $\varphi : L \to L'$ (i.e. φ is bijective and $x \leq y$ iff $\varphi(x) \leq \varphi(y)$ for all $x, y \in L$) with $\varphi(P) = P'$. Note that every order-isomorphism is *join-preserving*, i.e. $\varphi(\sum X) = \sum \varphi(X)$ for all $X \subseteq L$.

An important class of join-preserving mappings is given by projections:

Example 2.1. A *projection* of a geometry $G = (L, P)$ is defined as a mapping of the form $\Pi_{x,y} : L \to L$, $t \mapsto (t + x)y$ where $x, y \in L$ satisfy $x \oplus y = 1_L$.

The *centre* of a projection $\Pi_{x,y}$ is the geometric object x, its *image* the geometric object y.

What about the points of the geometry G under a projection? It is easy to check that a projection maps the points which are disjoint to its kernel onto the points which are less or equal to its image, i.e. a projection $\Pi_{x,y}$ is *point-preserving* in the following sense:

Given any geometric object z then $\Pi_{x,y}(\{p \in P(z) \mid px = 0\}) = P(\Pi_{x,y}(z))$, due to modularity of L. Furthermore, projections are *idempotent* and $\Pi_{x,y}(t) = \Pi_{x,y}(t')$ iff $t + x = t' + x$ for all $t, t' \in L$.

For disjoint points p, q and $h \in L$ such that $q \in P(p \oplus h)$ the points $p, q, \Pi_{p,h}(q)$ are directly balanced iff $q \oplus h = p \oplus h$.

The notion of projection gives rise to an important class of isomorphisms, namely *perspectivities*: Given two geometric objects x, y which are perspective via z, then the restriction $\Pi_{z,y} \mid G(x) \to G(y)$, $t \mapsto (t + z)y$ is an isomorphism of $G(x)$ and $G(y)$, called *perspectivity* of x onto y with centre z, abbreviated $x \xrightarrow{z} y$.

The ring's operations in geometric terms. We will now use projections and perspectivities for a geometric description of the ring's operations. The most important paper for us is [2] where the authors A. Day and D. Pickering construct a ternary ring to find a coordinatization of a given arguesian lattice. One of the most important features of [2] is the introduction of geometric mappings which can be adapted to our framework as follows:

Construction 2.2. Let $G = PG(_RR^3)$ a planar lattice geometry and (f_0, f_1, f_2) a regular triangle of G which we extend with f_4 to a 3-frame of the subgeometry $G(f_0 \oplus f_1 \oplus f_2)$. Define $f_3 = (f_1 \oplus f_2)(f_0 \oplus f_4)$. Since $f_3 = \Pi_{f_2, f_0 \oplus f_4}(f_1) = \Pi_{f_0, f_1 \oplus f_2}(f_4)$ holds, the triples of points (f_1, f_2, f_3) and (f_0, f_3, f_4) are directly balanced.

Define a set $D = \{f \in P(f_0 \oplus f_4) \mid f \oplus f_3 = f_0 \oplus f_3\}$. We can write $f_i = Rx_i$ for $x_i \in {}_RR^3$, $i \in \{0, 1, 2\}$, $f_3 = R(x_1 + x_2)$, and $f_4 = R(x_0 + x_1 + x_2)$ and obtain $D = \{R(x_0 + r(x_1 + x_2)) \mid r \in R\}$. Our goal is introducing an addition and multiplication on D. We therefore define the following perspectivities:

$$\varphi : f_0 \oplus f_4 \xrightarrow{f_2} f_0 \oplus f_1,\ t \mapsto \Pi_{f_2, f_0 \oplus f_1}(t);$$

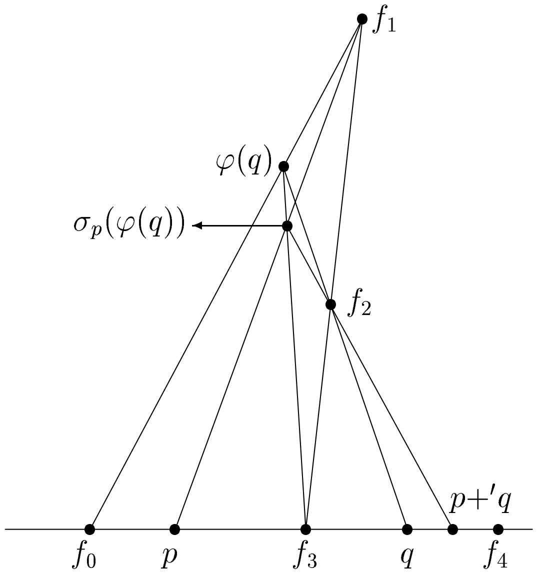

Fig. 2

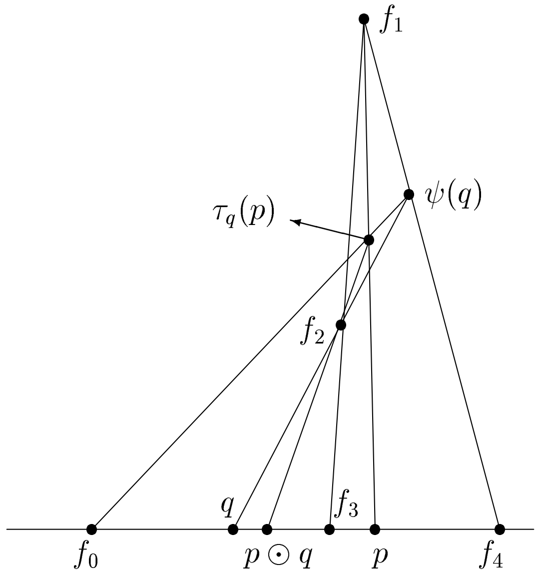

Fig. 3

$\psi : f_0 \oplus f_4 \overset{f_2}{\to} f_1 \oplus f_4,\ t \mapsto \Pi_{f_2, f_1 \oplus f_4}(t);$

For any $p \in D$ define the following perspectivities:

$\sigma_p : f_1 \oplus f_0 \overset{f_3}{\to} f_1 \oplus p,\ t \mapsto \Pi_{f_3, f_1 \oplus p}(t)$ and

$\tau_p : f_0 \oplus f_4 \overset{f_1}{\to} f_0 \oplus \psi(p),\ t \mapsto \Pi_{f_1, f_0 \oplus \psi(p)}(t);$

Using these perspectivities we introduce the following operations on D:

$p+'q := \Pi_{f_2, f_0 \oplus f_4}(\sigma_p(\varphi(q)))$ and

$p \odot q = \Pi_{f_2, f_0 \oplus f_4}(\tau_q(p)).$

Claim. *Due to these operations D becomes a ring $(D, +', \odot, f_0, f_4)$, which is isomorphic to $(R, +, \cdot, 0, 1)$ via $R \to D$, $r \mapsto R(x_0 + r(x_1 + x_2))$* (Fig. 2, 3).

Especially, R is commutative iff $p \odot q = q \odot p$ holds for all $p, q \in D$ iff $\tau_p(q) + f_2 = \tau_q(p) + f_2$ holds for all $p, q \in D$.

Proof. The first part is given by straightforward calculations. The second part follows since $\Pi_{f_2, f_0 \oplus f_4}$ is a projection with kernel f_2. □

Note that in the above construction we do not claim that the regular triangle (f_0, f_1, f_2) *spans* the planar geometry, i.e. it is not necessary to claim that $1_L = f_0 \oplus f_1 \oplus f_2$.

3. The extended Pappus' theorem

We are almost finished. We only have to find an appropriate translation for the "classical" theorem. One of the major difficulties of the more general ring-case is that the intersection of two distinct lines need not be a point. In order to adapt the configuration to our ring-case, we define:

Definition 3.1. *Let (p, q', q) be a regular triangle of a planar projective geometry $G = PG({}_RR^3)$, which we extend to a 3-frame (p, q, q', r) of the subgeometry $G(p \oplus q \oplus q')$. For points $u, t \leq p \oplus q$ such that conditions*

(1) $t \oplus q = u \oplus q = p \oplus q$ *and*

(2) p, t, u *are pairwise distinct*

hold, we set $t' = \Pi_{r,p\oplus q'}(t)$ and $u' = \Pi_{r,p\oplus q'}(u)$.

Since $q' \oplus (q \oplus t) = p \oplus q \oplus q'$, the geometric object $f_{t,u'} := \Pi_{q,q'\oplus t}(u') = (q' \oplus t)(q \oplus u')$ is a point. Similarily, $q' \oplus (u \oplus q) = p \oplus q \oplus q'$, and $f_{u,t'} := \Pi_{q,q'\oplus u}(t') = (q' \oplus u)(q \oplus t')$ is a point as well.

We say the 3-frame (p, q, q', r) satisfies the extended Pappus' theorem (EP)*, if $f_{u,t'} + r = f_{t,u'} + r$ holds for all points u, t with condition* (1) *and* (2) (cf. Fig. 4).

We say the regular triangle (p, q, q') (in this order) satisfies the extended Pappus' theorem (EP)*, if (p, q, q', r) satisfies the extended Pappus' theorem for every r where (p, q, q', r) is a 3-frame of the subgeometry $G(p \oplus q \oplus q')$.*

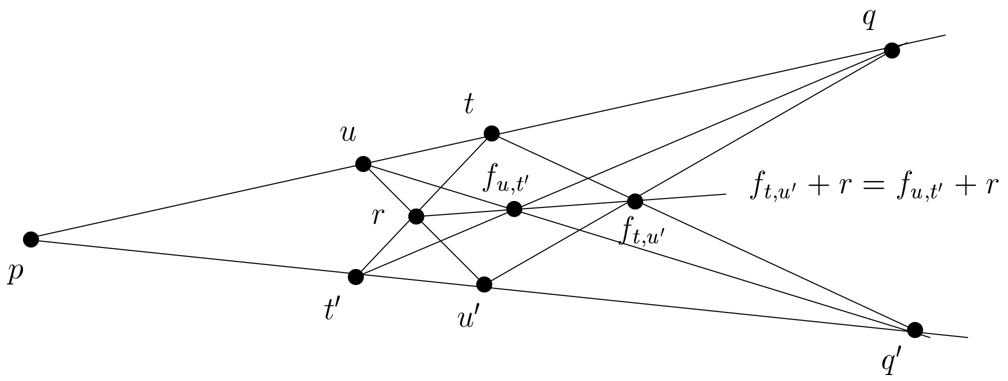

Fig. 4

Now from 2.2 we derive our main result:

Theorem 3.2. *Let $G = PG({}_RR^3)$ be a planar projective geometry. Then the following are equivalent:*

(a) *One regular triangle satisfies* (EP)*;*

(b) *Every regular triangle satisfies* (EP)*;*

(c) *The multiplication of R is commutative.*

References

[1] Artin, E.: *Geometric Algebra.* Interscience Publishers, New York 1957.

[2] Day, A.; Pickering, D.: *The coordinatization of arguesian lattices.* Trans. Amer. Math. Soc. **278** (1983) (2), 507–522.

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