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# Euclidean Superpythagorean Geometry

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## 1. Introduction

H. Lenz has studied in [8, p. 336–338] the conditions under which Hilbert's [5, p. 4–5] axiom of order II 3, which states that among three points of a line only one is between the other two, is (in)dependent of the remaining axioms for plane Euclidean geometry. He showed that II 3 is independent of the remaining plane axioms of the groups I–IV by showing that there are pythagorean fields which admit an 'almost-order' relation that is not an order relation.

He also showed that it is provable (hence redundant) from the remaining plane axioms I-IV (with a strengthened version of Pasch's axiom II 4, which excludes the possibility that a secant may pass through the interior of all three sides of a triangle) and the circle axiom, which states that a line that goes through a point that lies between the endpoints of the radius of a circle intersects that circle.

It is only natural to ask what the 'weakest' hypotheses are, under which II 3 can be proved. If we ask this question in field-theoretic terms, it becomes: What is the largest class of fields for which every almost-order is an order. The answer is: The class of Euclidean fields, so Lenz's result is best possible. However, if we replace Hilbert's order axiom II 2 (which states that, given two different points points $a$ and $c$, there is a point $b$ such that $c$ lies between $a$ and $b$) with the axiom that states that the midpoint of a segment lies between its endpoints, and add it to II 1 and the strengthened form of II 4, then the answer becomes: *superpythagorean* fields.

Superpythagorean fields were introduced and studied by R. Elman and T. Y. Lam [4] as well as by L. Bröcker [2] under the name "strictly pythagorean fields". They were defined in [4] as fields $F$ in which every set $S$, with $0 \notin S$ and $-1 \notin S$, for which $S$ is a subgroup of index 2 of $\dot{F}$, is an ordering on $F$.[1] In the terminology of half-orderings introduced in [11],

[1] $\dot{M}$ stands for $M \setminus \{0\}$.

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[12], superpythagorean fields were defined as fields in which every half-ordering that does not contain -1 is an ordering. An equivalent definition of superpythagorean fields is given by the following three conditions (cf. [7, Cor. 5.16]; also [3]):

(1) $-1 \notin F^2$, (2) $(\forall x)\, 1+x^2 \in F^2$, (3) $(\forall x)\, x \in F^2 \vee -x \in F^2 \vee 1+x \in F^2 \vee x(1+x) \in F^2$.

Conditions (1) and (2) are stating that $F$ is a formally real and Pythagorean field, whereas (3) is the additional condition that makes $F$ superpythagorean. We notice that condition (2) becomes redundant if we strengthen condition (1) to

(1′) $(\forall xy)\, x^2 + y^2 + 1 \neq 0$,

for, applying (3) to $x = -1 - z^2$ (with $z \neq 0$) and using (1′) we get (2).

The fact that (1′) and (3) alone suffice to axiomatize superpythagorean fields implies that we need, besides some order axioms, much less than the axioms of Hilbert's groups I, III, IV. We can start with an axiom system for Euclidean planes with bisectable right angles, add two axioms that imply that the coordinate field satisfies (1′) and (3) and then some order axioms. The aim of this note is to provide an axiom system for plane Euclidean geometry coordinatized by superpythagorean ordered fields.

## 2. The axiom system

As axioms for Euclidean planes we shall choose the axiom system presented in [9] (or [10]), since this axiom system uses very few axioms, all stated in the first order language $L_{\equiv}$ (with only one sort of variables, to be interpreted as 'points') using only one primitive notion, the quaternary predicate $\equiv$, standing for 'congruence'. The axioms[2] are:

**A1.** $ab \equiv cd \to cd \equiv ab$,

**A2.** $ab \equiv cd \wedge cd \equiv ef \to ab \equiv ef$,

**A3.** $ab \equiv ba$,

**A4.** $(\forall abca'b'c'xm)(\exists! x')\, [a \neq b \wedge c \neq m \wedge ac \equiv am \wedge bc \equiv bm \wedge a'b' \equiv ab$
$\wedge\, a'c' \equiv ac \wedge b'c' \equiv bc \to a'x' \equiv ax \wedge b'x' \equiv bx \wedge c'x' \equiv cx]$,

**A5.** $(\forall abcd)(\exists m)\, [a \neq b \wedge c \neq d \wedge ac \equiv ad \wedge bc \equiv bd \to am \equiv cm \wedge bm \equiv cm$
$\wedge\, m \neq a \wedge m \neq c]$,

**A6.** $(\exists abcd)\, a \neq c \wedge b \neq d \wedge ab \equiv bc \wedge bc \equiv bd \wedge cd \equiv da \wedge ac \equiv bd$.

Notice that the notion of collinearity ($L$) can be defined in terms of $\equiv$ by

$$L(abc) \leftrightarrow (\exists uv)\, u \neq v \wedge au \equiv av \wedge bu \equiv bv \wedge cu \equiv cv.$$

Every model of these axioms is isomorphic to a Euclidean plane with bisectable right angles.

We can express (1′) geometrically by the axiom

[2]The absence of quantifiers indicates that all the variables are to be considered as universally quantified.

**A7.** *The circle which has* $(1,0)$ *and* $(a^2+1,0)$ *(with* $a \neq 0$*) as endpoints of a diameter does not intersect the y-axis,*

whose formal expression in $L_{\equiv}$ can be obtained by using the definitions of addition and multiplication that allow the affine coordinatization of lines. Condition (3) becomes[3]

**A8.** *The circle which has* $(0,0)$ *and* $(a^2+a+1,0)$ *as endpoints of a diameter intersects at least one of the lines* $x=3a$, $x=-a$, $x=a^2$, $x=1$.

Thus A1–A8 form an axiom system for Euclidean planes with bisectable right angles and superpythagorean coordinate field.

As order axioms we need, besides the axiom A11, telling us that the midpoint of a segment lies between its endpoints[4], two axioms (A9 and A10) corresponding to the two statements that Hilbert's II 1 makes, and the strengthened form of the Pasch axiom (A12). To express them, we shall enlarge the language (which so far contained only $\equiv$) by a ternary relation $Z$ (with $Z(abc)$ to be read as '$b$ lies (strictly) between $a$ and $c$').

**A9.** $Z(abc) \to a \neq b \wedge b \neq c \wedge c \neq a \wedge L(abc)$

**A10.** $Z(abc) \to Z(cba)$,

**A11.** $a \neq b \wedge L(amb) \wedge ma \equiv mb \to Z(amb)$,

**A12.** $(\forall abcxx')(\exists y)(\forall y')\, \neg L(abc) \wedge \neg L(abx') \wedge \neg L(cxx') \wedge Z(axb) \to L(xx'y) \wedge \{[Z(byc) \wedge (L(xx'y') \to \neg Z(ay'c))] \vee [Z(ayc) \wedge (L(xx'y') \to \neg Z(by'c))]\}$.

The models of the axioms A1–A6, A9, A10 and A12 are Euclidean planes for which the underlying affine plane is geometrically half-ordered (in the sense of [1]). This follows from [6, Folgerung 1]. The addition of A7 ensures that every segment has a midpoint. If we now add to A1–A6, A9, A10, A12 and A7 the axiom A11 we achieve two objectives: We ensure that the half-order is not trivial (i. e. that $Z$ is not the empty relation), and we ensure that -1 is not contained in the multiplicative group of 'positive' elements of the coordinate field. Notice also that, in the presence of A9 and A11, requiring that $m \neq a \wedge m \neq c$ in A5 is no longer necessary. Let A5$'$ denote the axiom A5 with the requirement that $m \neq a \wedge m \neq c$ deleted from its consequent, and let $\Sigma = \{\mathrm{A1}, \ldots, \mathrm{A4}, \mathrm{A5}', \mathrm{A6}, \ldots, \mathrm{A12}\}$. We have thus shown the

**Theorem.** *$\mathfrak{M}$ is a model of $\Sigma$ iff $\mathfrak{M}$ is isomorphic to a Cartesian plane over a superpythagorean ordered field.*

By a *Cartesian plane* over an ordered field $(F, \leq)$ we understand the structure $\langle F \times F, \equiv_F, \mathbf{Z}_F \rangle$, where $\mathbf{ab} \equiv_F \mathbf{cd}$ iff $\|\mathbf{a}-\mathbf{b}\| = \|\mathbf{c}-\mathbf{d}\|$, where $\|\mathbf{x}\| = x_1^2 + x_2^2$ for $\mathbf{x} = (x_1, x_2)$, and $\mathbf{Z}_F(\mathbf{abc})$ iff $(\exists t \in F)\, 0 < t < 1, \mathbf{b}-\mathbf{a} = t(\mathbf{c}-\mathbf{a})$.

[3] A8 implies that the coordinate field satisfies (3$'$) $3a \in F^2 \vee -a \in F^2 \vee a+1 \in F^2 \vee a(a+1) \in F^2$. With $a=1$ in (3$'$) we conclude that either 2 or 3 is a square; if 2 is a square, then (3$'$) with $a=2$ implies that 3 is a square; whence (3).

[4] It would suffice, in fact, to postulate that there is a midpoint of a segment that lies between its endpoints (cf. [1]).

The class of superpythagorean ordered fields is both wider than that of Euclidean ordered fields (see [7, p. 46]) and smaller than that of pythagorean ordered fields. A pythagorean ordered field that is not superpythagorean is the pythagorean hull of $\mathbb{Q}$, since it admits proper almost-orders, as shown in [8, p. 337].

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