Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 39 (1998), No. 2, 433-446.

On the Cohen-Macaulayness of the Associated Graded Ring of Certain Monomial Curves

To the memory of Professor Wolfgang Vogel

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Abstract. Let K be a field and let $m_0, \dots, m_{e-1}$ be a sequence of positive integers. Let $\mathcal{C}$ be an algebroid monomial curve in the affine e-space $\mathbb{A}_K^e$ defined parametrically by $X_0 = T^{m_0}, \dots, X_{e-1} = T^{m_{e-1}}$ and let A be the coordinate ring of $\mathcal{C}$. In this paper we discuss when exactly the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay. If some $e-1$ terms of $m_0, \dots, m_{e-1}$ form an arithmetic sequence then we give necessary and sufficient conditions for the Cohen-Macaulayness of the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(A)$.

1. Introduction

It is important and often difficult to compute the Hilbert function of a local ring $(A, \mathfrak{m})$, which is a good measure of singularity of $(A, \mathfrak{m})$ at $\mathfrak{m}$. However, if the associated graded

*During this work the second author was supported by Consiglio Nazionale delle Ricerche, Italy. This work was done during June 1994 and July 1996, while the second author was visiting Genova University, Genova, Italy, whose hospitality is gratefully acknowledged.

ring $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay then the computation of the Hilbert function of A can be reduced to the computation of the Hilbert function of an Artinian local ring, which is simple. Therefore, the question whether the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay is important and considered by many authors. For example, in [8], [9] and [10], Sally proved that if either $\mathrm{emdim}(A) = \mathrm{mult}(A)$ or $\mathrm{mult}(A) - 2 \le \mathrm{emdim}(A)$ and A is Gorenstein then $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay, where $\mathrm{emdim}(A)$ and $\mathrm{mult}(A)$ denote the embedding dimension and multiplicity of A respectively. There are examples (see (5.1)) such that $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay with A non-Gorenstein and $\mathrm{mult}(A) >> \mathrm{emdim}(A)$.

In this article we assume that A is the coordinate ring of an algebroid monomial curve in the affine e-space $\mathbb{A}^e_K$ defined parametrically by $X_0 = T^{m_0}, \ldots, X_{e-1} = T^{m_{e-1}}$ with $0 < m_0 < m_1 < \cdots < m_{e-1}$ and $\gcd(m_0, \ldots, m_{e-1}) = 1$. In special case when the sequence $m_0, \ldots, m_{e-1}$ is an arithmetic sequence then $\mathrm{gr}_{\mathfrak{m}}(A)$ is always Cohen-Macaulay (see [3 ; Proposition (1.1)]). In the case of monomial space curves, that is, $e = 3$, Robbiano and Valla (see [7; Section 3]) gave necessary and sufficient conditions for $\mathrm{gr}_{\mathfrak{m}}(A)$ to be Cohen-Macaulay. In [2] Herzog also considered the case of monomial space curves. However, in concrete examples (particularly if $\mathrm{mult}(A)$ is large) to check these conditions is tedious and requires long computation.

Our aim of this paper is to give simple necessary and sufficient condition for $\mathrm{gr}_{\mathfrak{m}}(A)$ to be Cohen-Macaulay and which is easy to check. We consider the case when the sequence $m_0, \ldots, m_{e-1}$ is an almost arithmetic sequence, that is, $m_0 < \cdots < m_p$, $p := e-2$ is an arithmetic sequence and $n := m_{e-1}$ is arbitrary. We prove that (in most cases) $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay if and only if exactly one inequality between two positive integers holds. This inequality involves the non-negative integers $\lambda, \mu, \nu, u, v, z, w$ which were defined in [6], using the explicit description of the standard basis S_{m_0} of the semigroup $\Gamma := \sum_{i=0}^{e-2} \mathbb{Z}^+ m_i + \mathbb{Z}^+ n$. The explicit description of S_{m_0} and the properties of the non-negative integers $\lambda, \mu, \nu, u, v, z, w$ were used to find explicit minimal sets of generators for the derivation module and the relation ideal of A (see [5] and [6]). Further in [4] the explicit set of generators for the relation ideal of A and properties of the non-negative integers $\lambda, \mu, \nu, u, v, z, w$ were used to prove that these curves are set-theoretic complete intersections. Given integers $m_0, \ldots, m_p, n$, it is easy to find the non-negative integers $\lambda, \mu, \nu, u, v, z, w$ and then we only have to check only one inequality for $\mathrm{gr}_{\mathfrak{m}}(A)$ to be Cohen-Macaulay.

For an integer $t \in \mathbb{Z}^+$, let $q_t \in \mathbb{Z}$, $r_t \in \{1, 2, \ldots, p\}$ be the unique integers defined by the equation $t = q_t p + r_t$. Then, in 3.4 and 4.4 we prove the following results:

Case I: $m_p < n$: $\mathrm{gr}_{\mathfrak{m}}(A)$ *is Cohen-Macaulay if and only if* $\lambda + w \ge q_u + 1$ *and* $v \le q_z + 1 + \mu$.

Case II: $m_0 < n, m_p$: *if* $\mu \ne 0$ *then* $\mathrm{gr}_{\mathfrak{m}}(A)$ *is Cohen-Macaulay if and only if* $\lambda + w \ge q_u + 1$ *and* $v \le q_z + 1 + \mu$.

Case III: $n < m_0$:

(1) *if either* $z = 0$ *or* $\nu \le q_{u-z} + 1 + v - w$ *then* $\mathrm{gr}_{\mathfrak{m}}(A)$ *is Cohen-Macaulay.*

(2) *if* $\mu \ne 0$ *and* $z \ne 0$ *then* $\mathrm{gr}_{\mathfrak{m}}(A)$ *is Cohen-Macaulay if and only if* $\nu \le q_{u-z} + 1 + (v - w)$.

Therefore for the case of monomial space curves, we can determine when exactly (see the Case I above) the associated graded ring is Cohen-Macaulay, since any three integers are in almost arithmetic sequence with $m_0 < m_1 < n$. This is much more easier to check than, for instance, the necessary and sufficient conditions given by Robbiano and Valla. We also give many examples to illustrate.

For many values of $m_0, \ldots, m_p, n$, we used the Computer algebra program CoCoA to test whether or not the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay. This helped us to find the right inequalities.

2. Key-Lemma

In this section we recall the explicit description of the standard basis from [6; §3]. Our proof depends mainly on this description and so we call this description as a Key-Lemma.

We fix the following notations throughout this paper.

2.1. Notations. Let $\mathbb{Z}$ (respectively $\mathbb{Z}^+$) denote the set of all (respectively non-negative) integers. For $a, b \in \mathbb{Z}$, let $[a, b] := \{i \in \mathbb{Z} \mid a \le i \le b\}$. Unless mentioned otherwise, the symbols $a, b, c, d, e, i, j, k, l, m, n, p, q, r, s, t, u, v, w, z$ denote integers.

Let $m_0, \ldots, m_{e-1}$ be a sequence of positive integers with $m_0 < \cdots < m_{e-1}$ and $\gcd(m_0, \ldots, m_{e-1}) = 1$. We assume that $m_0, \ldots, m_{e-1}$ is a minimal set of generators for the semigroup $\Gamma := \sum_{i=0}^{e-1} \mathbb{Z}^+ m_i$.

- Let $\mathcal{E} := \left(\mathbb{Z}^+\right)^e$.
- For $i \in [0, e-1]$, we put $\mathbf{e}_i := (\delta_{ij})_{0 \le j \le e-1}$, where δ_{ij} denote the Kronecker delta.
- For $\alpha = \sum_{i=0}^{e-1} a_i \mathbf{e}_i$, let $\partial(\alpha) := \sum_{i=0}^{e-1} a_i m_i$.
- For $h \in \Gamma$, let $\mathcal{E}(h) := \{\alpha \in \mathcal{E} \mid \partial(\alpha) = h\}$.
- For $\alpha = \sum_{i=0}^{e-1} a_i \mathbf{e}_i \in \mathcal{E}(h)$, we put $\deg(\alpha) := \sum_{i=0}^{e-1} a_i$.
- For $\alpha, \beta \in \mathcal{E}(h)$, we write $\alpha \le_{\deg} \beta$ if $\deg(\alpha) \le \deg(\beta)$.

Then $\le_{\deg}$ is an order on $\mathcal{E}(h)$ and since $\mathcal{E}(h)$ is a finite set, $\mathcal{E}(h)$ has maximal elements with respect to the order $\le_{\deg}$. Let $\max(\mathcal{E}(h))$ denote the set of all maximal elements in $\mathcal{E}(h)$. Note that all elements of $\max(\mathcal{E}(h))$ have the same degree, therefore this degree we shall denote by $\operatorname{maxdeg}(h)$.

2.2. Proposition. *Let $A := K[[T^{m_0}, \ldots, T^{m_{e-1}}]]$ and let $\mathfrak{m}$ be the maximal ideal of A. Let $\tau_0, \ldots \tau_{e-1}$ denote the images of $T^{m_0}, \ldots, T^{m_{e-1}}$ in $\operatorname{gr}_{\mathfrak{m}}(A)$ respectively. Then*

(1) $\operatorname{gr}_{\mathfrak{m}}(A) = K[\tau_0, \ldots, \tau_{e-1}]$ *and a monomial* $\tau_0^{a_0} \cdots \tau_{e-1}^{a_{e-1}}$ *is non-zero in* $\operatorname{gr}_{\mathfrak{m}}(A)$ *if and only if* $\sum_{i=0}^{e-1} a_i = \operatorname{maxdeg}\left(\sum_{i=0}^{e-1} a_i m_i\right)$.

(2) *For a positive integer t and a non-zero monomial $\tau_0^{a_0} \cdots \tau_{e-1}^{a_{e-1}}$ in $\operatorname{gr}_{\mathfrak{m}}(A)$, the monomial $\tau_0^t\,(\tau_0^{a_0} \cdots \tau_{e-1}^{a_{e-1}})$ is non-zero in $\operatorname{gr}_{\mathfrak{m}}(A)$ if and only if* $\operatorname{maxdeg}(\sum_{i=0}^{e-1} a_i m_i + t m_0) = \operatorname{maxdeg}(\sum_{i=0}^{e-1} a_i m_i) + t$.

(3) *The following statements are equivalent:*

(i) $\operatorname{gr}_{\mathfrak{m}}(A)$ *is Cohen-Macaulay.*

(ii) $\operatorname{maxdeg}(h + t m_0) = \operatorname{maxdeg}(h) + t$ *for every* $h \in \Gamma$ *and* $t \in \mathbb{Z}^+$.

Proof. (1) is immediate from definition and (2) follows from (1).

(3) Since $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay if and only if τ_0^t is a non-zero divisor in $\mathrm{gr}_{\mathfrak{m}}(A)$ for every $t \geq 1$ by [1; Theorem 7], the equivalence follows from (2). □

To give necessary and sufficient conditions so that the condition (ii) of (3) holds, we shall use the explicit description of the standard basis from [6; §3], which we recall below.

2.3. Standard basis. The set $S_{m_0} := \{z \in \Gamma \mid z - m_0 \notin \Gamma\}$ is called the *standard basis* or *Apèry set of* Γ *with respect to* m_0. It is clear that S_{m_0} depends on Γ and m_0, but for simplicity we put $S := S_{m_0}$. It is easy to see that $S = \{s_0 := 0, s_1, \ldots, s_{m_0-1}\}$, where $s_1, \ldots, s_{m_0-1} \in \Gamma$ are positive integers with the following properties:

(i) $s_i \equiv i \pmod{m_0}$ for every $i \in [0, m_0 - 1]$.

(ii) If $z \in \Gamma$ then $z \equiv i \pmod{m_0}$ for a unique $i \in [0, m_0 - 1]$ and $z \geq s_i$.

2.4. Explicit description of the standard basis of a semigroup generated by almost arithmetic sequence. Put $p := e - 2$ and assume that $m_0, \ldots, m_p$ is an arithmetic sequence, that is, there is a positive integer d with $m_i = m_0 + id$ for all $0 \leq i \leq p$. Let n be an arbitrary positive integer with $\gcd(m_0, d, n) = 1$. Let $A := K[[T^{m_0}, \ldots, T^{m_p}, T^n]]$, $\Gamma' := \sum_{i=0}^{p} \mathbb{Z}^+ m_i$ and $\Gamma = \Gamma' + \mathbb{Z}^+ n$. Let $S := S_{m_0}$ be the standard basis of Γ with respect to m_0. For $t \in \mathbb{Z}^+$, let $q_t \in \mathbb{Z}$, $r_t \in [1, p]$ and $g_t \in \Gamma'$ be defined by $t = q_t p + r_t$ and $g_t = q_t m_p + m_{r_t}$.

The Key-Lemma from [6; §3] (see also [5; §1]) is the following:

2.5. Key-Lemma. *With the notations as above*

(1) $g_s + g_t = \varepsilon m_0 + g_{s+t}$ *with* $\varepsilon = 1$ *or* 0 *according as* $r_s + r_t \leq p$ *or* $r_s + r_t > p$.

(2) *Let* $u := \min\{t \in \mathbb{Z}^+ \mid g_t \notin S\}$ *and* $v := \min\{b \geq 1 \mid bn \in \Gamma'\}$. *Then there exist unique integers* $w \in [0, v - 1]$, $z \in [0, u - 1]$, $\lambda \geq 1$, $\mu \geq 0$ *such that*

  (i) $g_u = \lambda m_0 + wn$;

  (ii) $vn = \mu m_0 + g_z$;

  (iii) $g_{u-z} + (v - w)n = \nu m_0$. *Moreover,* $\nu = \lambda + \mu + \varepsilon$ *where* $\varepsilon = 1$ *or* 0 *according as* $r_{u-z} < r_u$ *or* $r_{u-z} \geq r_u$.

(3) *Let* $V := [0, u - 1] \times [0, v - 1]$, $W := [u - z, u - 1] \times [v - w, v - 1]$ *and* $U := V \setminus W$. *Then* $S = \{g_s + bn \mid (s, b) \in U\}$. *In particular, if* $(s, b), (t, c) \in U$ *with* $g_s + bn \equiv g_t + cn \pmod{m_0}$ *then* $(s, b) = (t, c)$.

(4) *Every element of* Γ *can be expressed uniquely in the form* $am_0 + g_s + bn$ *with* $a \in \mathbb{Z}^+$ *and* $(s, b) \in U$.

(5) *The map* $\mathcal{E} \to \left(\mathbb{Z}^+\right)^2$ *defined by* $\sum_{i=0}^{p+1} a_i \mathbf{e}_i \mapsto (\sum_{i=0}^{p} a_i m_i, a_{p+1} n)$ *is a bijection between* S *and* U.

Proof. See [6; §3]. □

In the sequel the symbols $u, v, w, z, \lambda, \mu, \nu, U$ and W have the meaning assigned to them by the above Key-Lemma.

In addition, below we prove few technical lemmas which are consequences of the above Key-Lemma and are needed for the proof of the main result.

2.6. Lemma.

(1) $\deg(\gamma_j) = q_j + 1$ *for every* $j \in \mathbb{Z}^+$.

(2) *For* $i, j \in \mathbb{Z}^+$, *there is* $\varepsilon \in \{0, 1\}$ *such that* $\gamma_i + \gamma_j = \varepsilon \mathbf{e}_0 + \gamma_{i+j}$ *and* $q_i + q_j + 1 = \varepsilon + q_{i+j}$.

(3) *For* $h \in \Gamma$ *and* $\alpha \in \mathcal{E}(h)$, *there exist* $a, b, j \in \mathbb{Z}^+$ *such that* $\deg(\alpha) = a + q_j + 1 + b$.

Proof. Clear from the definitions and the Key-Lemma 2.5. □

2.7. Lemma.

(1) *If* $n > m_p$ *then* $v < \mu + q_z + 1$.

(2) *If* $n > m_0$ *then* $\nu > q_{u-z} + 1 + v - w$.

(3) *If* $n < m_p$ *then* $\lambda + w \geq q_u + 1$.

Proof. (1) and (2) are clear from (2)(ii) and (2)(iii) of the Key-Lemma 2.5 respectively.

(3) If $\lambda + w < q_u + 1$ then $g_u = q_u m_p + m_{r_u} \geq (\lambda + w)m_p + m_{r_u} > \lambda m_0 + wn = g_u$, a contradiction. □

2.8. Lemma. *Suppose that* $\lambda + w \geq q_u + 1$. *Then*

(1) *If* $j \in \mathbb{Z}^+, j \geq u$ *then* $q_j \leq \lambda - \varepsilon + q_{j-u} + w$ *with* $\varepsilon \in \{0, 1\}$.

(2) *For every* $h \in \Gamma$ *and* $\alpha \in \mathcal{E}(h)$, *there exist* $a, b \in \mathbb{Z}^+$ *and* $i \in [0, u-1]$ *such that* $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ *and* $\deg(\alpha) \leq a + q_i + 1 + b$.

(3) *Let* $h \in \Gamma$ *and let* α *be a maximal element in* $\mathcal{E}(h)$. *Then there exist* $a, b \in \mathbb{Z}^+$ *and* $i \in [0, u-1]$ *such that* $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ *and* $\deg(\alpha) = \deg(a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1})$.

Proof. (1) Since $\gamma_j + \varepsilon \mathbf{e}_0 = \gamma_u + \gamma_{j-u}$ with $\varepsilon \in \{0, 1\}$ and by assumption $\deg(\gamma_u) = q_u + 1 \leq \lambda + w$, we have $q_j = q_u + q_{j-u} + 1 - \varepsilon \leq \lambda + w + q_{j-u} - \varepsilon$.

(2) By 2.6 (3) there exist $a', b', j \in \mathbb{Z}^+$ such that $\deg(\alpha) = a' + q_j + 1 + b' = \deg(a'\mathbf{e}_0 + \gamma_j + b'\mathbf{e}_{p+1})$. If $j < u$ then there is nothing to prove. If $j \geq u$ then by the above part (1), we have $\beta := (a' + \lambda - \varepsilon)\mathbf{e}_0 + \gamma_{j-u} + (w + b')\mathbf{e}_{p+1} \in \mathcal{E}(h)$ where $\gamma_j + \varepsilon \mathbf{e}_0 = \gamma_u + \gamma_{j-u}$ with $\varepsilon \in \{0, 1\}$ and $\deg(\alpha) \leq \deg(\beta)$. If $j - u \geq u$ then repeating we get $i \in [0, u-1]$ such that $\deg(\alpha) \leq \deg(a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}) = a + q_i + 1 + b$ for some $a, b \in \mathbb{Z}^+$.

(3) By the part (2) above there exist $a, b \in \mathbb{Z}^+$ and $i \in [0, u-1]$ such that $\beta := a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(h)$ and $\deg(\alpha) \leq \deg(\beta)$. Now by the maximality of α in $\mathcal{E}(h)$, it follows that $\deg(\alpha) = \deg(\beta)$. □

2.9. Lemma. *If* $a\mathbf{e}_0 + \gamma_i \in \mathcal{E}(bn)$ *for some* $a, b \in \mathbb{Z}^+$ *and* $i \in [1, u-1]$ *then* $z \neq 0$.

Proof. Suppose that $z = 0$. Then $vn = \mu m_0 \equiv 0 \pmod{m_0}$. Let $b_1 \in \mathbb{Z}^+$ and $b_0 \in [0, v-1]$ be the unique integers defined by the equation $b = b_1 v + b_0$. Then $g_i \equiv bn = b_1 vn + b_0 n \equiv b_0 n \pmod{m_0}$. This is a contradiction, since $i \neq 0$ and both $(i, 0)$, $(0, b_0)$ belong to U. □

2.10. Lemma. *Suppose that $z = 0$. If $\gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(am_0)$ for some $i \in [0, u-1]$ and $a, b \in \mathbb{Z}^+$ then $i = 0$.*

Proof. Let $b_1 \in \mathbb{Z}^+$ and $b_0 \in [0, v-1]$ be the unique integers defined by the equation $b = b_1 v + b_0$. Then from $\mu m_0 = vn$, it follows that $g_i + b_0 n \equiv 0 \pmod{m_0}$. Therefore, since $(i, b_0) \in U$, we must have $(i, b_0) = (0, 0)$. □

2.11. Lemma. *If $t \in [0, v-1]$ then $\mathcal{E}(tn) = \{t\mathbf{e}_{p+1}\}$. In particular,* $\operatorname{maxdeg}(tn) = t$ *for every $t \in [0, v-1]$.*

Proof. Let $t \in [0, v-1]$ and let $\alpha \in \mathcal{E}(tn)$. Since $\partial(\alpha) = tn \in S$, it follows that $\alpha = \gamma_i + b\mathbf{e}_{p+1}$ for some $i \in \mathbb{Z}^+$ and $b \in [0, t]$. Now, from $(t-b)n = g_i$ and the definition of v, it follows that $i = 0$ and $t = b$. In particular, $\alpha = t\mathbf{e}_{p+1}$. □

2.12. Lemma. *Let $j \in [0, u-1]$ and let*

$$I_j := \{(i,k) \in \left(\mathbb{Z}^+\right)^2 \mid j = i + kz \text{ and } g_i + kg_z = g_j\}.$$

Then

$$\mathcal{E}(g_j) = \begin{cases} \{\gamma_j\} & \text{if } \mu \neq 0, \\ \{\gamma_i + kv\mathbf{e}_{p+1} \mid (i,k) \in I_j\} & \text{if } \mu = 0. \end{cases}$$

In particular,

(1) *if $j < z$ then $\mathcal{E}(g_j) = \{\gamma_j\}$ and* $\operatorname{maxdeg}(g_j) = q_j + 1$.

(2) $\mathcal{E}(g_z) = \begin{cases} \{\gamma_z\}, & \text{if } \mu \neq 0, \\ \{\gamma_z, v\mathbf{e}_{p+1}\}, & \text{if } \mu = 0 \end{cases}$ *and*

$$\operatorname{maxdeg}(g_z) = \begin{cases} \{q_z + 1\}, & \text{if } \mu \neq 0, \\ \max\{q_z + 1, v\}, & \text{if } \mu = 0 \end{cases}.$$

(3) *if $\mu \neq 0$ and $z \neq 0$ then $\mathcal{E}(g_{u-z}) = \{\gamma_{u-z}\}$ and* $\operatorname{maxdeg}(g_{u-z}) = q_{u-z} + 1$.

Proof. Let $\alpha \in \mathcal{E}(g_j)$. Since $\partial(\alpha) = g_j \in S$, it follows that $\alpha = \gamma_i + b\mathbf{e}_{p+1}$ with $i \leq j$ and $b \in \mathbb{Z}^+$. If $b = 0$ then $i = j$ and there is nothing to prove, since $(j, 0) \in I_j$. Therefore assume that $b \geq 1$ and $i < j$. Then $g_{j-i} \equiv bn \pmod{m_0}$. Therefore, since $(j-i, 0) \in U$, we have $b \geq v$. Let $b_0 \in [0, v-1]$ and $b_1 \in \mathbb{Z}^+$ be the unique integers defined by the equation $b = b_1 v + b_0$. Then $g_j = (a + b_1\mu)m_0 + g_{i+b_1 z} + b_0 n \in S$, where $g_i + b_1 g_z = am_0 + g_{i+b_1 z}$, $a \in \mathbb{Z}^+$. Therefore $g_{j-(i+b_1 z)} = (a + b_1\mu + \varepsilon)m_0 + b_0 n$, where $g_j + \varepsilon m_0 = g_{j-(i+b_1 z)} + g_{i+b_1 z}$. Now, since both $(j - (i + b_1 z), 0) \in U$ and $(0, b_0) \in U$, it follows that $a = 0 = \varepsilon$, $\mu = 0$, $b_0 = 0$ and $j - (i + b_1 z) = 0$ and so $g_j = g_i + g_{b_1 z} = g_i + b_1 g_z$. Therefore $(i, b_1) \in I_j$ and $\alpha = \gamma_i + b_1 v\mathbf{e}_{p+1}$. □

3. The case $n > m_0$

Let $m_0, \ldots, m_p, n$ be an almost arithmetic sequence of positive integers with $\gcd(m_0, \ldots, m_p, n) = 1$. Let $A := K[[T^{m_0}, \ldots, T^{m_p}, T^n]]$ and let $\mathfrak{m}$ be the maximal ideal of A. Suppose that $m_0, \ldots, m_p, n$ is a minimal set of generators for the semigroup $\Gamma := \sum_{i=0}^{p} \mathbb{Z}^+ m_i + \mathbb{Z}^+ n$.

We continue using the same notations as in Section 2. In particular, remember the Key-Lemma. Sometimes giving reference of the Key-Lemma 2.5 is omitted.

In this section assume that $n > m_0$.

3.1. Proposition. *Suppose that* $v \leq \mu + q_z + 1$. *Then*

(1) *Let* $a, b, i \in \mathbb{Z}^+$ *be such that* $\gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(am_0)$. *Then* $a \geq q_i + 1 + b$.

(2) *Let* $a, b \in \mathbb{Z}^+$, $i \in [0, u-1]$ *be such that* $a\mathbf{e}_0 + \gamma_i \in \mathcal{E}(bn)$. *Then* $a + q_i + 1 \geq b$.

(3) *Let* $a, a', b, b' \in \mathbb{Z}^+$ *and* $i, i' \in [0, u-1]$ *be such that* $\alpha' = a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1} \in \mathcal{E}(am_0 + g_i + bn)$. *If* $b \leq b'$ *then* $\deg(\alpha) \geq \deg(\alpha')$, *where* $\alpha := a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$

Proof. (1) This is clear from $n > m_0$ and $am_0 = g_i + bn$.

(2) Since $am_0 + g_i = bn$, it follows from the definition of v (see the Key-Lemma) that $b \geq v$. Therefore

$$am_0 + g_i = \mu m_0 + g_z + (b - v)n. \tag{3.1.a}$$

and so $a \geq \mu$, since $(i, 0) \in U$. We now consider two cases $i \leq z$ and $i > z$ separately.

Case $i \leq z$: In this case from the equation (3.1.a) we get

$$(a - \mu + \varepsilon)m_0 = g_{z-i} + (b - v)n$$

with $\gamma_z + \varepsilon\mathbf{e}_0 = \gamma_i + \gamma_{z-i}$, $\varepsilon \in \{0, 1\}$ and so $a - \mu + \varepsilon > q_{z-i} + 1 + b - v$, since $n > m_0$. Therefore $a + q_i + 1 \geq q_i + 1 + q_{z-i} + 1 - \varepsilon + \mu - v + b = q_z + 1 + \mu - v + b \geq b$.

Case $i > z$: In this case, since $i - z < u$, we have $(i - z, 0) \in U$ and so from the equation (3.1.a) we get

$$(a - \mu - \varepsilon)m_0 + g_{i-z} = (b - v)n$$

with $\gamma_i + \varepsilon\mathbf{e}_0 = \gamma_{i-z} + \gamma_z$, $\varepsilon \in \{0, 1\}$ and $i - z \geq 1$. Now, since $z \neq 0$ by 2.9, it follows the case $i \leq z$ and by induction on i that $a - \mu - \varepsilon + q_{i-z} + 1 \geq b - v$. Therefore $a - \mu - \varepsilon + q_{i-z} + 1 + q_z + 1 \geq b - v + q_z + 1$. This proves that $a + q_i + 1 \geq b$, since $v \leq \mu + q_z + 1$ by assumption.

(3) Since $(i, 0) \in U$, $b \leq b'$ and $am_0 + g_i + bn = a'm_0 + g_{i'} + b'n$, it follows that $a' \leq a$ and so $(a - a')m_0 + g_i = g_{i'} + (b' - b)n$. Therefore

$$\begin{array}{lll} \varepsilon m_0 + (a - a')m_0 = g_{i'-i} + (b' - b)n & \text{if} & i \leq i' \\ (a - a')m_0 + g_{i-i'} = \varepsilon m_0 + (b' - b)n & \text{if} & i > i' \end{array}$$

with

$$\begin{array}{lll} \gamma_{i'} + \varepsilon\mathbf{e}_0 = \gamma_i + \gamma_{i'-i} & \text{if} & i \leq i', \\ \gamma_i + \varepsilon\mathbf{e}_0 = \gamma_{i'} + \gamma_{i-i'} & \text{if} & i > i',\ \varepsilon \in \{0, 1\} \end{array}$$

and so

$$\varepsilon + a - a' \geq q_{i'-i} + 1 + (b' - b) \text{ by the part (1) above,} \tag{3.1.b}$$

$$(a - a') + q_{i-i'} + 1 \geq \varepsilon + (b' - b) \text{ by the part (2) above.} \tag{3.1.c}$$

Now by using 2.6 (2), we get

$$\begin{array}{rcll} \varepsilon + q_i + 1 + a + b & \geq & a' + q_i + 1 + q_{i'-i} + 1 + b' & \\ & = & a' + \varepsilon + q_{i'} + 1 + b' & \text{if } i \leq i', \\ q_i + 1 + \varepsilon + a + b & = & q_{i'} + 1 + q_{i-i'} + 1 + a + b & \\ & \geq & a' + q_{i'} + 1 + \varepsilon + b' & \text{if } i > i'. \end{array}$$

In both the cases, this proves that $\deg(\alpha) \geq \deg(\alpha')$. □

3.2. Proposition. *Suppose that* $\lambda + w \geq q_u + 1$ *and* $v \leq \mu + q_z + 1$. *Let* $h \in \Gamma$ *and let* $a \in \mathbb{Z}^+, (i, b) \in U$ *be such that* $h = am_0 + g_i + bn$. *Then* $\operatorname{maxdeg}(h) = a + q_i + 1 + b$.

Proof. Let $\alpha' \in \mathcal{E}(h)$ be an element of maximal degree. Then $\deg(\alpha') = \operatorname{maxdeg}(h)$ and in view of 2.8 (3), we may assume that $\alpha' = a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1}$ with $a', b' \in \mathbb{Z}^+$ and $i' \in [0, u-1]$. Put $\alpha := a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$. Then $\alpha \in \mathcal{E}(h)$ and $\deg(\alpha) \leq \deg(\alpha')$ by the maximality of α'. Since $(i, b) \in U$ and $am_0 + g_i + bn = h = a'm_0 + g_{i'} + b'n$, we have $a' \leq a$. Further, $b' \geq b$. For, if $b' < b$ then $(i, b - b') \in U$ and $g_{i'} \equiv g_i + (b - b')n$, which contradicts the fact that $(i', 0) \in U$. Then $\deg(\alpha) \geq \deg(\alpha')$ by 3.1 (3). Therefore $\deg(\alpha) = \deg(\alpha')$ and so α must be a maximal element in $\mathcal{E}(h)$. In particular, $\operatorname{maxdeg}(h) = \deg(\alpha) = a + q_i + 1 + b$.

3.3. Corollary. *Suppose that* $\lambda + w \geq q_u + 1$ *and* $v \leq \mu + q_z + 1$. *Let* $h \in \Gamma$ *and let* $t \in \mathbb{Z}^+$. *Then* $\operatorname{maxdeg}(h + tm_0) = \operatorname{maxdeg}(h) + t$.

Proof. Let $a \in \mathbb{Z}^+$ and $(i, b) \in U$ be such that $h = am_0 + g_i + bn$. Then $h + tm_0 = (a+t)m_0 + g_i + bn$. Therefore by 3.2 we have $\operatorname{maxdeg}(h + tm_0) = a + t + q_i + 1 + b = \operatorname{maxdeg}(h) + t$. □

3.4. Theorem. *If either* $m_p < n$, *or* $m_0 < n < m_p$ *and* $\mu \neq 0$ *then the following statements are equivalent* :

(i) $\operatorname{gr}_{\mathfrak{m}}(A)$ *is Cohen-Macaulay.*

(ii) $\operatorname{maxdeg}(h + tm_0) = \operatorname{maxdeg}(h) + t$ *for all* $h \in \Gamma$ *and for all* $t \in \mathbb{Z}^+$.

(iii) $\lambda + w \geq q_u + 1$ *and* $v \leq \mu + q_z + 1$.

Proof. (i) $\iff$ (ii) : is proved in the proposition 2.2.

(ii) $\Rightarrow$ (iii) : From the equation $g_u = \lambda m_0 + wn$, (ii) and 2.11, it follows that $q_u + 1 = \deg(\gamma_u) \leq \lambda + \operatorname{maxdeg}(wn) = \lambda + w$. Similarly from the equation $vn = \mu m_0 + g_z$, (ii), 2.12 and either $m_p < n$, or $m_0 < n < m_p$ and $\mu \neq 0$, it follows that $v = \deg(vn) \leq \mu + \operatorname{maxdeg}(g_z) = \mu + q_z + 1$.

(iii) $\Rightarrow$ (ii) : is proved in the corollary 3.3. □

3.5. Proposition. *Suppose that* $m_0 < n < m_p$ *and* $\mu = 0$. *Let* $z_1 \in \mathbb{Z}^+$ *and* $z_0 \in [0, z-1]$ *be the unique integers defined by the equation* $u - z = z_1 z + z_0$. *Then there exists an integer* $\nu_1 \geq -1$ *such that* $(\nu + \nu_1)m_0 + g_{z-z_0} = (z_1 + 1)vn + (v - w)n$.

Proof. Since $g_{u-z} + bm_0 = g_{z_0} + z_1 g_z = g_{z_0} + z_1 vn$ for some $b \in \mathbb{Z}^+$, from the equation $\nu m_0 + g_z = g_{u-z} + vn + (v - w)n$, it follows that $(\nu + b - \varepsilon)m_0 + g_{z-z_0} = (z_1 + 1)vn + (v - w)n$ with $\varepsilon \in \{0, 1\}$. □

3.6. Corollary. *Suppose that* $m_0 < n < m_p$ *and* $\mu = 0$. *Let* $z_1 \in \mathbb{Z}^+$, $z_0 \in [0, z-1]$ *and* $\nu_1 \geq -1$ *be as in* 3.5. *If* $z_0 \neq 0$ *and* $\operatorname{gr}_{\mathfrak{m}}(A)$ *is Cohen-Macaulay then* $\nu + \nu_1 + q_{z-z_0} + 1 \geq (z_1 + 1)v + (v - w)$.

Proof. Since $m_0 < n$ and $0 \neq z_0 < z$, by 2.12 and 2.2 we have

$$\begin{aligned} \nu + \nu_1 + q_{z-z_0} + 1 &= \nu + \nu_1 + \operatorname{maxdeg}(g_{z-z_0}) \\ &= \operatorname{maxdeg}\left((\nu + \nu_1)m_0 + g_{z-z_0}\right) \geq (z_1 + 1)v + (v - w)\,. \end{aligned}$$ □

3.7. Questions. *Suppose that $m_0 < n < m_p$ and $\mu = 0$. Let $z_1 \in \mathbb{Z}^+$, $z_0 \in [0, z-1]$ and $\nu_1 \geq -1$ be as in* 3.5. *Then*

(1) *In the case $z_0 = 0$, is $\mathrm{gr}_{\mathfrak{m}}(A)$ always Cohen-Macaulay?*

(2) *Is the converse of* 3.6 *true? That is, in the case $m_0 < n < m_p$, $\mu = 0$, $z_0 \neq 0$ and $\nu + \nu_1 + q_{z-z_0} + 1 \geq (z_1+1)v + (v-w)$ where z_0, z_1, ν_1 are as in* 3.5, *is $\mathrm{gr}_{\mathfrak{m}}(A)$ Cohen-Macaulay?*

3.8. Remark. If $p = 1$, $m_0 < n < m_1$ and $z_0 = 0$ then in [7; Remark 3 of § 3], it is proved that the question 3.7 (1) has an affirmative answer. In all the examples 5.3 the converse of 3.6 holds.

4. The case $n < m_0$

In this section, we continue using the same notations as in Section 3 and assume that $n < m_0$. Therefore $\lambda + w \geq q_u + 1$ by 2.7 (3).

4.1. Lemma. *Suppose that $z = 0$. Then $\mu \neq 0$ and $\beta := \lambda_0 \mathbf{e}_0 + \lambda_1 v \mathbf{e}_{p+1}$ is a maximal element in $\mathcal{E}(\lambda m_0)$, where $\lambda_1 \in \mathbb{Z}^+$ and $\lambda_0 \in [0, \mu-1]$ be the unique integers defined by the equation $\lambda = \lambda_1 \mu + \lambda_0$. In particular,* $\mathrm{maxdeg}(\lambda m_0) = \lambda_1 v + \lambda_0$.

Proof. Since $z = 0$, clearly $\mu \neq 0$ by 2.5 (2)(ii) and $\partial(\beta) = \lambda_0 m_0 + \lambda_1 v n = \lambda_0 m_0 + \lambda_1 \mu m_0 = \lambda m_0$ and so $\beta \in \mathcal{E}(\lambda m_0)$. Let $\alpha \in \mathcal{E}(\lambda m_0)$ be an element of maximal degree. Then $\mathrm{maxdeg}(\lambda m_0) = \deg(\alpha)$ and in view of 2.8 (3), we may assume that $\alpha = a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$ with $a, b \in \mathbb{Z}^+$ and $i \in [0, u-1]$. Let $b_0 \in [0, v-1]$ and $b_1 \in \mathbb{Z}^+$ be the unique integers defined by the equation $b = b_1 v + b_0$. Then, since $b_1 v n = b_1 \mu m_0$, we have $0 \equiv \partial(\alpha) \equiv g_i + b_0 n \pmod{m_0}$. Now, $(i, b_0) \in U$ (see the Key-Lemma 2.5), since $z = 0$ and so $(i, b_0) = (0, 0)$. This proves that $\alpha = a\mathbf{e}_0 + b_1 v \mathbf{e}_{p+1}$. If $b_1 > \lambda_1$ then $vn \leq (b_1 - \lambda_1)vn = (\lambda_0 - a)m_0 < \mu m_0 = vn$, a contradiction. Therefore $b_1 \leq \lambda_1$ and $(\lambda_1 - b_1)vn = (a - \lambda_0)m_0$ and so $(\lambda_1 - b_1)v \geq a - \lambda_0$, since $n < m_0$. This shows, by the maximality of α in $\mathcal{E}(\lambda m_0)$ that we must have $\lambda_1 = b_1$ and $a = \lambda_0$. In particular, $\beta = \alpha$ is a maximal element in $\mathcal{E}(\lambda m_0)$. $\square$

4.2. Proposition. *Suppose that either $z = 0$ or $\nu \leq q_{u-z} + 1 + (v-w)$.*

(1) *Let $a, b, i \in \mathbb{Z}^+$ be such that $a\mathbf{e}_0 + \gamma_i \in \mathcal{E}(bn)$. Then $a + q_i + 1 \leq b$.*

(2) *Let $a, b \in \mathbb{Z}^+$, $i \in [0, u-1]$ be such that $\gamma_i + b\mathbf{e}_{p+1} \in \mathcal{E}(am_0)$. Then $a \leq q_i + 1 + b$.*

(3) *Let $a, a', b, b' \in \mathbb{Z}^+$ and $i, i' \in [0, u-1]$ be such that $\alpha' = a'\mathbf{e}_0 + \gamma_{i'} + b'\mathbf{e}_{p+1} \in \mathcal{E}(am_0 + g_i + bn)$. If $b \leq b'$ then $\deg(\alpha) \leq \deg(\alpha')$, where $\alpha := a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$*

Proof. (1) Clear from $n < m_0$ and $am_0 + g_i = bn$.

(2) If $z = 0$ then $i = 0$ by 2.10 and so $a \leq b$, since $n < m_0$. Now, assume that $\nu \leq q_{u-z} + 1 + v - w$. Since $g_i + bn = am_0 \equiv 0 \pmod{m_0}$, we have $(i, b) \notin U$ and so we have two cases $i < u - z$ and $i \geq u - z$ separately:

Case $i < u - z$: In this case $b \geq v$. Let $j \geq 1$ be an integer with $i + j = u - z$. Then $g_j + g_i = g_{u-z} + \varepsilon m_0$ with $\varepsilon \in \{0, 1\}$. Since $\nu m_0 = g_{u-z} + (v-w)n$, we have $am_0 + g_j =$

$(\nu+\varepsilon)m_0+(b-v+w)n$ and so $a \geq \nu+\varepsilon$, since $(j,0)\in U$. Now, from $(a-\nu-\varepsilon)m_0+g_j = (b-v+w)n$ we get $a-\nu-\varepsilon+q_j+1 < b-v+w$. Therefore $a < b-q_j-1-v+w+\nu+\varepsilon = b+q_i+1-(q_{u-z}+1+v-w-\nu) \leq b+q_i+1$, since $q_{u-z}+\varepsilon = q_i+q_j+1$ by 2.6 (2).

Case $i \geq u-z$: In this case we have $b \geq v-w$ and $am_0 = g_i+bn \geq g_{u-z}+(v-w)n = \nu m_0$ and so $a \geq \nu$. Now, since $g_i+\varepsilon m_0 = g_{u-z}+g_{i-u+z}$ with $\varepsilon\in\{0,1\}$ and $\nu m_0 = g_{u-z}+(v-w)n$, we get $(a-\nu+\varepsilon)m_0 = g_{i-u+z}+(b-v+w)n$. Now, since $u-z\neq 0$, it follows by the case $i < u-z$ and induction on i that $a-\nu+\varepsilon \leq q_{i-u+z}+1+(b-v+w)$ and so $a \leq b+q_i+1+(\nu-(q_{u-z}+1+v-w)) \leq b+q_i+1$, since $q_{u-z}+q_{i-u+z}+1 = \varepsilon+q_i$ by 2.6 (2).

(3) Since $(i,0)\in U$, $b\leq b'$ and $am_0+g_i+bn = a'm_0+g_{i'}+b'n$, it follows that $a'\leq a$ and so $(a-a')m_0+g_i = g_{i'}+(b'-b)n$. Therefore

$$\begin{aligned} \varepsilon m_0+(a-a')m_0 &= g_{i'-i}+(b'-b)n \quad \text{if} \quad i\leq i' \\ (a-a')m_0+g_{i-i'} &= \varepsilon m_0+(b'-b)n \quad \text{if} \quad i>i' \end{aligned}$$

with

$$\begin{aligned} \gamma_{i'}+\varepsilon\mathbf{e}_0 &= \gamma_i+\gamma_{i'-i} && \text{if} \quad i\leq i', \\ \gamma_i+\varepsilon\mathbf{e}_0 &= \gamma_{i'}+\gamma_{i-i'} && \text{if} \quad i>i', \quad \varepsilon\in\{0,1\} \end{aligned}$$

and so

(4.2.a) $\qquad \varepsilon+a-a' \leq q_{i'-i}+1+(b'-b)$ by the part (2) above,

(4.2.b) $\qquad (a-a')+q_{i-i'}+1 < \varepsilon+(b'-b)$ by the part (1) above.

Now by using 2.6 (2), we get

$$\begin{aligned} \varepsilon+q_i+1+a+b &\leq a'+q_i+1+q_{i'-i}+1+b' \\ &= a'+\varepsilon+q_{i'}+1+b' && \text{if } i\leq i', \\ q_i+1+\varepsilon+a+b &= q_{i'}+1+q_{i-i'}+1+a+b \\ &\geq a'+q_{i'}+1+\varepsilon+b' && \text{if } i>i'. \end{aligned}$$

In both the cases, this proves that $\deg(\alpha)\leq\deg(\alpha')$. □

4.3. Corollary. *Suppose that either $z=0$ or $\nu\leq q_{u-z}+1+v-w$.*

(1) *Let $a,b\in\mathbb{Z}^+$, $i\in[0,u-1]$ be such that $\alpha = a\mathbf{e}_0+\gamma_i+b\mathbf{e}_{p+1}$ is a maximal element in $\mathcal{E}(am_0+g_i+bn)$. Then $\alpha+t\mathbf{e}_{p+1}$ is also maximal element in $\mathcal{E}(am_0+g_i+bn+tn)$ for every $t\in\mathbb{Z}^+$.*

(2) $\operatorname{maxdeg}(h+tn) = \operatorname{maxdeg}(h)+t$ *for every $h\in\Gamma$ and $t\in\mathbb{Z}^+$.*

Proof. (1) Put $h := am_0+g_i+bn$. It is enough to prove the assertion for $t=1$. Let $\beta\in\mathcal{E}(h+n)$ be a maximal element. In view of 2.8 (3) we may assume that $\beta = a'\mathbf{e}_0+\gamma_{i'}+b'\mathbf{e}_{p+1}$ with $a',b'\in\mathbb{Z}^+$ and $i'\in[0,u-1]$. If $b'\leq b+1$ then $\deg(\beta)\leq\deg(\alpha+\mathbf{e}_{p+1})$ by 4.2 (3). If $b'>b+1$ then α and $\beta' = \beta-\mathbf{e}_{p+1}\in\mathcal{E}(h)$ and $\deg(\beta')\leq\deg(\alpha)$ by the maximality of α in $\mathcal{E}(h)$ and so $\deg(\beta) = \deg(\beta')+1\leq\deg(\alpha)+1 = \deg(\alpha+\mathbf{e}_{p+1})$. In any case we have

$\deg(\alpha + \mathbf{e}_{p+1}) = \deg(\beta)$ by the maximality of β in $\mathcal{E}(h+n)$. In particular $\alpha + \mathbf{e}_{p+1}$ is also maximal in $\mathcal{E}(h+n)$.

(2) Let $h \in \Gamma$. Then by 2.8 (3) there is a maximal element in $\mathcal{E}(h)$ of the form $a\mathbf{e}_0 + \gamma_i + b\mathbf{e}_{p+1}$ with $a, b \in \mathbb{Z}^+$ and $i \in [0, u-1]$ and the assertion follows from (1). □

4.4. Theorem. *Suppose that $n < m_0$.*

(1) *If either $z = 0$ or $\nu \leq q_{u-z} + 1 + v - w$ then $\operatorname{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay.*

(2) *If $\mu \neq 0$ and $z \neq 0$ then the following statements are equivalent :*

(i) $\operatorname{gr}_{\mathfrak{m}}(A)$ *is Cohen-Macaulay.*

(ii) $\operatorname{maxdeg}(h + tn) = \operatorname{maxdeg}(h) + t$ *for all $h \in \Gamma$ and for all $t \in \mathbb{Z}^+$.*

(iii) $\nu \leq q_{u-z} + 1 + v - w$.

Proof. (1) is immediate from 4.3 (2) and 2.2.

(2) (i) $\iff$ (ii) : is proved in the proposition 2.2.

(ii) $\Rightarrow$ (iii) : From $\nu m_0 = g_{u-z} + (v-w)n$, (ii) and 2.12, it follows that $\nu \leq \operatorname{maxdeg}(\nu m_0) \leq \operatorname{maxdeg}(g_{u-z}) + (v-w) = q_{u-z} + 1 + v - w$.

(iii) $\Rightarrow$ (ii) : is proved in the corollary 4.3 (2). □

4.5. Proposition. *Suppose that $n < m_0$ and $\mu = 0$. Let I_{u-z} be as in 2.12, $z_0 := \min\{i \in \mathbb{Z}^+ \mid (i,k) \in I_{u-z}$ for some $k \in \mathbb{Z}^+\}$ and let $z_1 \in \mathbb{Z}^+$ be such that $(z_0, z_1) \in I_{u-z}$. Then*

(1) $\mathcal{E}(g_{z_0}) = \{\gamma_{z_0}\}$. *In particular,* $\operatorname{maxdeg}(g_{z_0}) = q_{z_0} + 1$.

(2) *If $\operatorname{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay then $\nu \leq q_{z_0} + 1 + z_1 v + (v - w)$.*

Proof. (1) is immediate from 2.12.

(2) Since $n < m_0$, by 2.12, 2.2 and $\nu m_0 = g_{z_0} + z_1 vn + (v-w)n$, we have $\nu \leq \operatorname{maxdeg}(g_{z_0}) + z_1 v + (v-w) = q_{z_0} + 1 + z_1 v + (v-w)$. □

4.6. Question. *Is the converse of 4.5 (2) true? That is, in the case $n < m_0$, $\mu = 0$ and $\nu \leq q_{z_0} + 1 + z_1 v + (v-w)$ where $z_0, z_1 \in \mathbb{Z}^+$ are as in (4.5), is $\operatorname{gr}_{\mathfrak{m}}(A)$ Cohen-Macaulay?*

4.7. Remark. In all the examples 5.4 the converse of 4.5 (2) holds.

5. Examples

5.1. Example. Let $m_0, \ldots, m_p, m_{p+1}$ be an arithmetic sequence and let $n := m_{p+1}$. Then $u = p+1$, $\lambda = 1, w = 1$ and so $\lambda + w = 2 = q_u + 1$. Therefore $\operatorname{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by 3.4.

5.2. Examples. Let m_0, m_1, n be positive integers with $\gcd(m_0, m_1, n) = 1$. Suppose that m_0, m_1, n is a minimal set of generators for the semigroup $\Gamma := \mathbb{Z}^+ m_0 + \mathbb{Z}^+ m_1 + \mathbb{Z}^+ n$. Then

(1) If $m_0 = 3$ and $3 < m_1 < n$ then it is easy to see that $u = 2 = v$ and either $(\lambda, w) = (1,1)$ or $\lambda \geq 2$. Therefore $\lambda + w \geq 2 \geq u = q_u + 1$ and so by $\operatorname{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by 3.4.

(2) If $m_0 = 4$ and $4 < m_1 < n$ and the semigroup Γ is symmetric then it is easy to see that $1 < u < 4$, $1 < v \leq 3$ and one of the following cases occurs

(a) $u = 2$, $v = 2$ and $w = 0$ and so $\lambda \geq 2$.

(b) $u = 2$, $v = 2$ and $w = 1$.

In any case we have $\lambda + w \geq 2 = u = q_u + 1$ and so $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by 3.4.

(3) If $m_0 = 4$, $m_1 = 5$ and $n = 11$ then the semigroup Γ is not symmetric and $u = 3$, $\lambda = 1$, $w = 1$, $v = 2$ and so $q_u + 1 = u = 3 > 2 = \lambda + w$. Therefore $\mathrm{gr}_{\mathfrak{m}}(A)$ is not Cohen-Macaulay by 3.4.

(4) If $m_0 = 5$ and $5 < m_1 < n$ and the semigroup Γ is symmetric then it is easy to see that $1 < u < 5$, $1 < v \leq 3$, $w \neq 0$ and one of the following cases occurs

(a) $u = 2$, $v = 3$ and $w = 1$.

(b) $u = 3$, $v = 3$ and $w = 2$.

(c) $u = 3$, $v = 2$, $w = 1$, $z = 1$, $\mu = 0$ and so $\lambda = \nu \geq 2$.

(d) $u = 4$, $v = 2$, $w = 1$, $z = 3$, $\mu = 0$ and $\lambda = \nu \geq 3$.

In any case we have $\lambda + w \geq u = q_u + 1$ and so $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by 3.4.

(5) If $m_0 = 5$, $m_1 = 8$ and $n = 27$ then the semigroup Γ is not symmetric and $u = 4$, $\lambda = 1$, $w = 1$, $v = 2$ and so $q_u + 1 = u = 4 > 2 = \lambda + w$. Therefore $\mathrm{gr}_{\mathfrak{m}}(A)$ is not Cohen-Macaulay by 3.4.

(6) If $m_0 = 6$, $m_1 = 7$ and $n = 15$ then the semigroup Γ is symmetric and $u = 3$, $\lambda = 1$, $w = 1$, $v = 2$ and so $q_u + 1 = u = 3 > 2 = \lambda + w$. Therefore $\mathrm{gr}_{\mathfrak{m}}(A)$ is not Cohen-Macaulay by 3.4.

(7) (Sally). Let m_0 be any integer ≥ 4, $m_1 = m_0 + 1$ and let $n := (m_0 - 1)m_0 - 1$. Then from the equations $(m_0 - 1)m_1 = m_0 + n$, $2n = m_0(m_0 - 1) + (m_0 - 2)m_1$ and $m_0^2 = m_1 + n$, it follows that $u = m_0 - 1$, $\lambda = 1$, $w = 1$, $v = 2$, $\mu = m_0 - 1$, $z = m_0 - 2$ and $\nu = m_0$. Therefore, since $\lambda + w = 2 < m_0 - 1 = u = q_u + 1$, $\mathrm{gr}_{\mathfrak{m}}(A)$ is not Cohen-Macaulay by 3.4.

(8) Let $m_0 = 7$, $m_1 = 8$ and $n = 20$. Then $u = 6$, $\lambda = 4$, $w = 1$, $v = 2$, $\mu = 0$, $z = 5$, $\nu = 4$ and so $\lambda + w = 5 < 6 = u = q_u + 1$. Therefore $\mathrm{gr}_{\mathfrak{m}}(A)$ is not Cohen-Macaulay by 3.4.

(9) Let $m_0 = 30$, $m_1 = 39$ and $n = 50$. Then $u = 10$, $w = 0$, $\lambda = 13$, $v = 3$, $z = 0$, $\mu = 5$, $\nu = 18$ and so $\lambda + w = 13 > 10 = u = q_u + 1$. Therefore $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by 3.4.

5.3. Examples with $m_0 < n < m_p$ and $\mu = 0$.

(1) Let $m_0 = 5$, $m_1 = 9$ and $n = 6$. In this case, since the semigroup Γ generated by $5, 6, 9$ is symmetric, $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by the example 5.2 (4). Further, $u = 3$, $\lambda = 3$, $w = 2$, $v = 3$, $\mu = 0$, $z = 2$, $\nu = 3$, $z_1 = 0$, $z_0 = 1$, $\nu_1 = 0$ (see 3.5 for notations) and $\nu + \nu_1 + q_{z-z_0} + 1 = \nu + 1 = 4 = (z_1 + 1)v + v - w$.

(2) Let $m_0 = 7$, $m_1 = 20$ and $n = 8$. Then $u = 3$, $\lambda = 4$, $w = 4$, $v = 5$, $\mu = 0$, $z = 2$, $\nu = 4$, $z_1 = 0$, $z_0 = 1$, $\nu_1 = 0$ (see 3.5 for notations), $\nu + \nu_1 + q_{z-z_0} + 1 = 5 < 6 = (z_1 + 1)v + (v - w)$ and $\mathrm{gr}_{\mathfrak{m}}(A)$ is not Cohen-Macaulay by the example 5.2 (8).

5.4. Examples with $n < m_0$ and $\mu = 0$.

(1) Let $m_0 = 39$, $m_1 = 50$ and $n = 30$. Then $u = 9$, $w = 2$, $\lambda = 10$, $v = 5$, $z = 3$, $\mu = 0$, $\nu = 10$, $z_0 = 0$, $z_1 = 2$ (see 4.5 for notations) and so $q_{u-z} + 1 + (v - w) = 9 < \nu = 10 < 13 = q_{z_0} + 1 + z_1 v + v - w$ and $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by the example 5.1 (9).

(2) Let $m_0 = 15$, $m_1 = 17$, $m_2 = 19$ and $n = 12$. Then $u = 6$, $w = 1$, $\lambda = 3$, $v = 3$, $z = 3$, $\mu = 0$, $\nu = 4$, $z_0 = 0$, $z_1 = 1$ (see 4.5 for notations) and so $q_{u-z} + 1 + (v - w) = \nu = 4 < 5 = q_{z_0} + 1 + z_1 v + v - w$ and $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by 4.4 (1).

(3) Let $m_0 = 27$, $m_1 = 35$, $m_2 = 43$ and $n = 26$. Then $u = 10$, $w = 1$, $\lambda = 7$, $v = 3$, $z = 3$, $\mu = 0$, $\nu = 8$, $z_0 = 4$, $z_1 = 1$ (see (4.5) for notations), $\nu = 8 > 7 = q_{z_0} + 1 + z_1 v + v - w$ and $\mathrm{gr}_{\mathfrak{m}}(A)$ is not Cohen-Macaulay by 4.5 (2).

5.5. Examples with $n < m_0$ and $z = 0$.

Let p be any positive integer, $n := 2(p + 1)$ and let $m_0 := 3(p + 1)$.

(1) Let $m_i := 2m_0 + i$ for $1 \le i \le p$. Let Γ be the semigroup generated by $n, m_0, m_1, \ldots, m_p$. Then from the equations: $g_{p+1} = 3m_0 + 2n$, $3n = 2m_0$, $5m_0 = g_{p+1} + n$, it follows that $u = p + 1$, $\lambda = 3$, $w = 2$, $v = 3$, $\mu = 2$, $z = 0$ and $\nu = 5$. Therefore $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by 4.4 (1), but in this case $\nu = 5 > 3 = q_{u-z} + 1 + v - w$.

(2) Let $m_i := 2m_0 + 3i$ for $1 \le i \le p$. Let Γ be the semigroup generated by $n, m_0, m_1, \ldots, m_p$. Then from the equations: $g_{p+1} = 5m_0$, $3n = 2m_0$, $7m_0 = g_{p+1} + 3n$, it follows that $u = p + 1$, $\lambda = 5$, $w = 0$, $v = 3$, $\mu = 2$, $z = 0$ and $\nu = 7$. Therefore $\mathrm{gr}_{\mathfrak{m}}(A)$ is Cohen-Macaulay by 4.4 (1), but in this case $\nu = 7 > 5 = q_{u-z} + 1 + v - w$.

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Received September 5, 1997