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Unilateral Tilings of the Plane with Squares of Three Sizes

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Abstract. A tiling $\mathcal{H}$ of the Euclidean plane by squares of three distinct sizes is said to be a unilateral 3-tiling (or U3-tiling) if no two congruent tiles have a common side, and it is called equitransitive provided for any pair S, S' of congruent tiles of $\mathcal{H}$ there is a congruence transform of the plane mapping S onto S' and also mapping $\mathcal{H}$ onto itself. Contrary to the assertion by D. Schattschneider that there are only five equitransitive U3-tilings, we present three new such tilings (two of them are due to B. Grünbaum). We describe the local environments of each tile in a U3-tiling with their global extensions, as well as further global properties of U3-tilings.

Keywords: tiling of the plane, squares, unilaterality, equitransitivity
MSC 1991: 52C20

0. Introduction

Problems on tiling a square or a rectangle into squares of several distinct sizes have been studied by many mathematicians (see [2], Sections C2–C5, and [6], Section 2.4, for references).

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On the other hand, there are only a few results on tiling the whole plane into noncongruent squares (see, e.g., [3], [4], [5], [7], also [2], Section C10, and [6], Section 2.4).

One of the still unsettled (and attractive) problems here is to describe all possible tilings of the plane with squares of n given sizes. This problem has been investigated by Schattschneider [8] (see also Grünbaum and Shephard [6], pp. 76–81) for some special types of tilings.

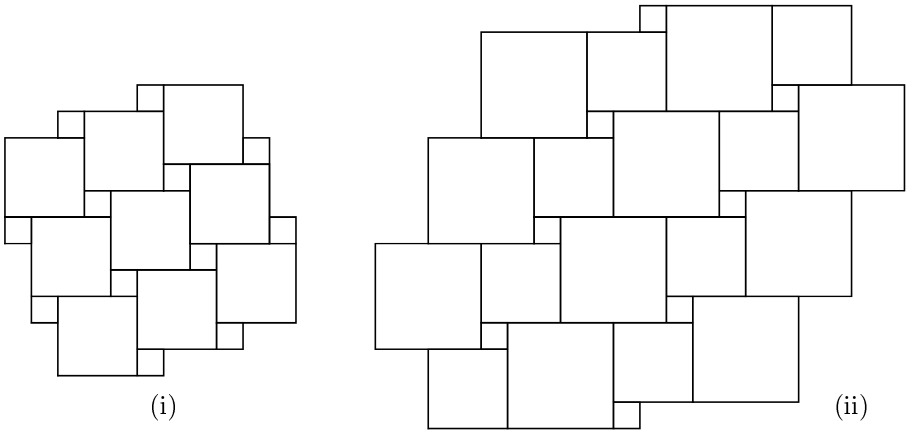

Figure 1

It is obvious that for any positive integer n there is a tiling of the Euclidean plane with squares of n distinct sizes. In Figure 1 one can see examples of such tilings by two and three distinct squares, respectively (cf. [6], Figures 2.4.2 and 2.4.5). These pictures can be easily redrawn for an arbitrary n (cf. [6], p. 81, Exercise 9.a).

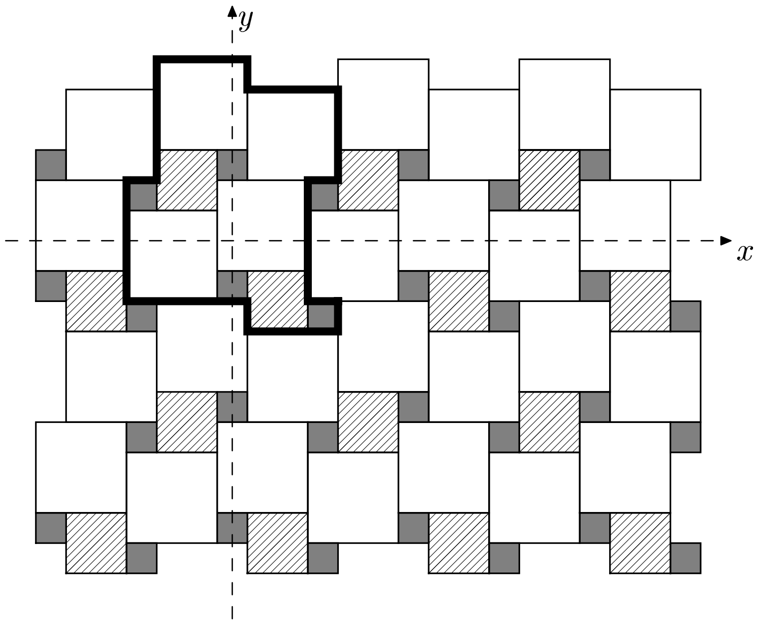

Figure 2

Since the whole variety of tilings of the plane into squares of n distinct sizes is too extensive, several authors have restricted their attention to special classes of tilings. According to [6], a tiling $\mathcal{H}$ by squares is called *unilateral* if no two congruent tiles (if they exist) have a common side, and it is called *equitransitive* provided for any pair S, S' of congruent tiles of $\mathcal{H}$ there is a congruence transform of the plane mapping S onto S' and also mapping $\mathcal{H}$ onto itself.

It is easy to verify (see [6], p. 81, Exercise 4) that there is only one (up to congruence) equitransitive, unilateral tiling of the plane with squares of two fixed sizes, as shown in Figure 1(i). Schattschneider [8] has described five equitransitive, unilateral tilings of the plane with squares of three sizes, claiming that there are exactly five such tilings (see [6], Figure 2.4.5). Contrary to her opinion (and therefore also to Exercise 2.4.6.b from [6]) we show in Figure 2 a new tiling of this type, which is a slight modification of Figure 2.4.5.d in [6]. Note that the symmetry group of this tiling contains all lattice translations with lattice basis $(6; 0)$, $(0; 8)$, and also reflections with respect to the coordinate axes, followed by translation along the half basis vector in the direction of the respective coordinate axis (i.e., glide reflections), as well as reflections with respect to the points $(\pm\frac{3}{2}; -2)$.)

In response to our Figure 2, B. Grünbaum sent us two further equitransitive U3-tilings (and kindly agreed to their inclusion in our paper, see Figure 3). These are also modifications of figures from [6], namely of Figures 2.4.5.d and 2.4.5.e, respectively.

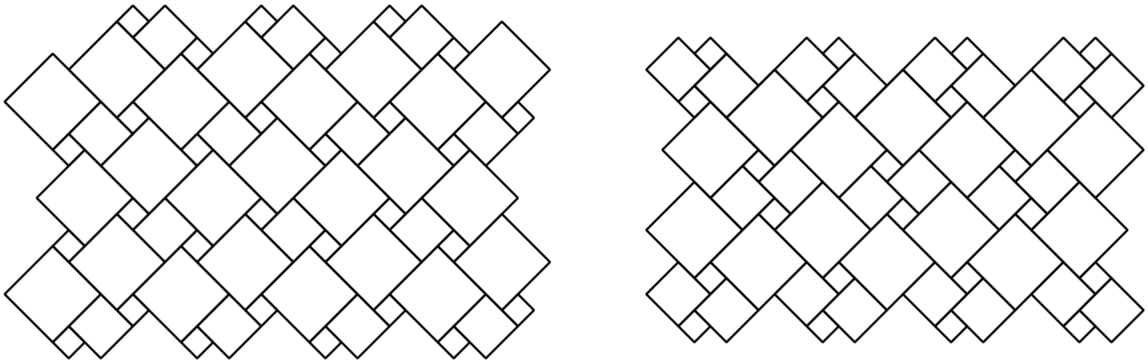

Figure 3

Thus the problem to describe all equitransitive, unilateral tilings of the plane with squares of three distinct sizes still remains open.

Our purpose here is to study some properties of unilateral – *not necessarily equitransitive* – tilings of the plane with squares of three sizes, which we will briefly call *U3-tilings*. To formulate our results, we need some definitions and notation. For a given U3-tiling $\mathcal{H}$, we denote by λ_1, λ_2, λ_3 the sizes of the tiling squares, assuming everywhere that $\lambda_1 < \lambda_2 < \lambda_3$. Thus by a λ_i-*tile* we mean a tile of size λ_i, $i = 1, 2, 3$. A side of $\mathcal{H}$ is any side of a tiling square, and $\operatorname{skel}\mathcal{H}$ (the skeleton of $\mathcal{H}$) is the union of all sides of $\mathcal{H}$. Every tile is supposed to be closed. Two tiles are *neighbourly* if their intersection is a non-degenerate line segment.

The paper is organized as follows. In Section 1 we introduce the notion of a *vortex* and discuss the possibility for a λ_i-tile to be a vortex. Section 2 is devoted to the study of neighbourly environments of a λ_i-tile, and in Section 3 some global properties of U3-tilings are exposed.

1. Vortices

In what follows, $\mathcal{H}$ stands for a tiling of the plane by squares.

Definition 1. *We say that a tile of $\mathcal{H}$ is a vortex if its sides are locally extendable in* skel $\mathcal{H}$ *according to an orientation of the tile, but none of them is extendable according to the other orientation.*

In other words, a tile T with the vertices a, b, c, d, taken according to some orientation of T, is a vortex provided the following two conditions are satisfied:

(i) *there are line segments $[a, b']$, $[b, c']$, $[c, d']$, and $[d, a']$, each of them properly containing the respective side of T and lying in* skel $\mathcal{H}$*,*

(ii) *every vertex of T belongs to the relative interior of a side of another tile* (see Figure 4).

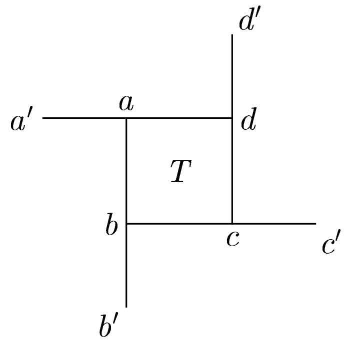

Figure 4

Theorem 1. *If $\mathcal{H}$ is a U3-tiling of the plane with squares of sizes λ_1, λ_2, λ_3, then every λ_1-tile is a vortex.*

Proof. Let T be a λ_1-tile. Denote by a, b, c, d the vertices of T, as shown in Figure 4. At least one of the vertices of T, say a, lies in the interior of a side of another tile, R, since otherwise every side of T would be a side of another tile. Then the right angle with apex a, lying outside $T \cup R$, is covered by a new tile, S. Since $\mathcal{H}$ is unilateral, S is larger than T, and b lies in the interior of a side of S. As above, the right angle with apex b, lying outside $T \cup S$, is covered by a new tile V, and the right angle with vertex c, lying outside $T \cup V$, is covered by a new tile W. Since W is larger than T, d lies in the interior of a side of W. If there were a new tile touching T and lying between R and W, this tile would be smaller than T, which is impossible. Hence $[a, d]$ lies in a side of R and d is a vertex of R. Summing up, we obtain that T is a vortex. $\square$

Theorem 2. *If $\mathcal{H}$ is a U3-tiling of the plane with squares of sizes λ_1, λ_2, λ_3, then every λ_2-tile is a vortex.*

Proof. Let T be a λ_2-tile. Denote by a, b, c, d the vertices of T, as shown in Figure 4. The vertex a is either a common vertex of four tiles in $\mathcal{H}$, or is a relative inner point of a side of a tile $S_1 \in \mathcal{H}$ and a vertex of a tile $S_2 \in \mathcal{H}$ different from T. Thus we may assume that a is a vertex of a tile $S \in \mathcal{H}$, and $S \cap]a, b[\neq \emptyset$.

From the unilaterality of $\mathcal{H}$ it follows that b cannot be a vertex of a tile different from T and having common points with $]a, b[$. Namely, in the contrary case $[a, b]$ would be a side of another λ_2-tile, or would be the union of sides of at least two λ_1-tiles. Hence $[a, b]$ is properly extendable to a segment $[a, b'] \subset \operatorname{skel} \mathcal{H}$, and $[b, c]$ is not properly extendable to a segment $[b', c] \subset \operatorname{skel} \mathcal{H}$. Similarly, there are proper extensions $[b, c']$, $[c, d']$, and $[d, a']$ in $\operatorname{skel} \mathcal{H}$ of the respective sides of T, and each of the points c, d, a is an interior point of a side of another tile. In other words, T is a vortex of $\mathcal{H}$. □

Remark 1. It is obvious that the assertions of Theorems 1 and 2 hold in the case of unilateral tilings with squares of an arbitrary number $n\,(\geq 2)$ of distinct sizes.

Lemma 1. *Let $\mathcal{H}$ be a U3-tiling of the plane with squares of sizes λ_1, λ_2, λ_3. Then at least one λ_1-tile has a common point with a pair of tiles of sizes λ_2 and λ_3, respectively.*

Proof. Assume, for contradiction, that no λ_1-tile in $\mathcal{H}$ has the required property, and choose a λ_1-tile $T \in \mathcal{H}$. Since T is a vortex (see Theorem 1), there are four tiles neighbourly to T. By the assumption, all these tiles are congruent to each other.

First consider the case when all these four tiles are of size λ_2. Since each of them is a vortex (see Theorem 2), there are four more λ_1-tiles, as shown in Figure 5(i). Again by the assumption, each of these new λ_1-tiles is surrounded by four λ_2-tiles. By continuing this consideration, we obtain the tiling of the whole plane with squares of sizes λ_1 and λ_2, as shown in Figure 1(i). The last statement contradicts the fact that $\mathcal{H}$ is a U3-tiling.

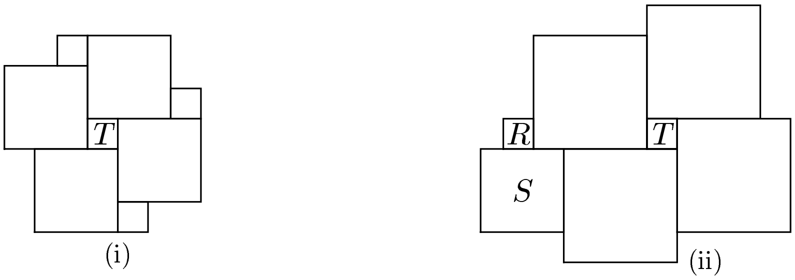

Figure 5

Now consider the case when all four tiles neighbourly to T are of size λ_3. We claim that also in this case there are four additional λ_1-tiles covering right exterior angles formed by the union of T and these four λ_3-tiles. Indeed, assume for a moment that one of these new tiles, say S, is of size λ_2, as shown in Figure 5(ii). Since S is a vortex, there is a λ_1-tile R neighbourly to S as shown in Figure 5(ii). In this case R satisfies the conclusion of Lemma 1, a contradiction. Then, by using the claim, we finish the proof as in the first case. □

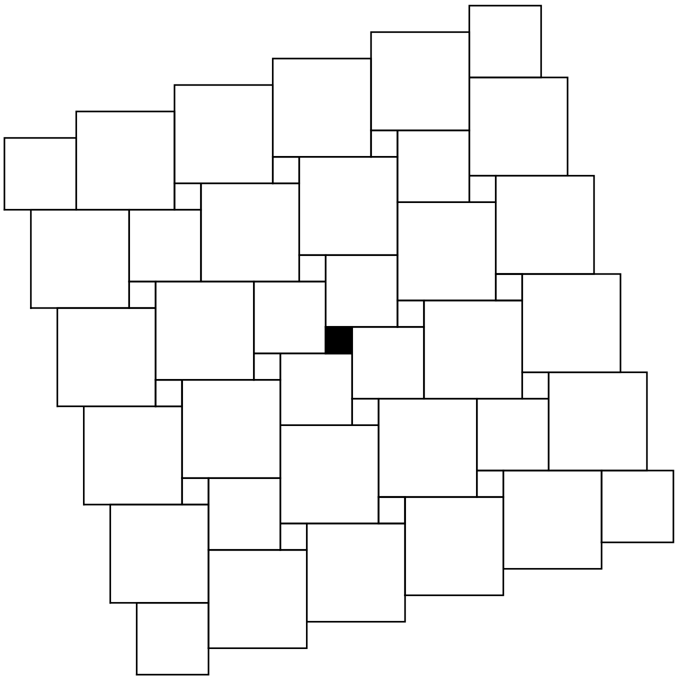

Figure 6

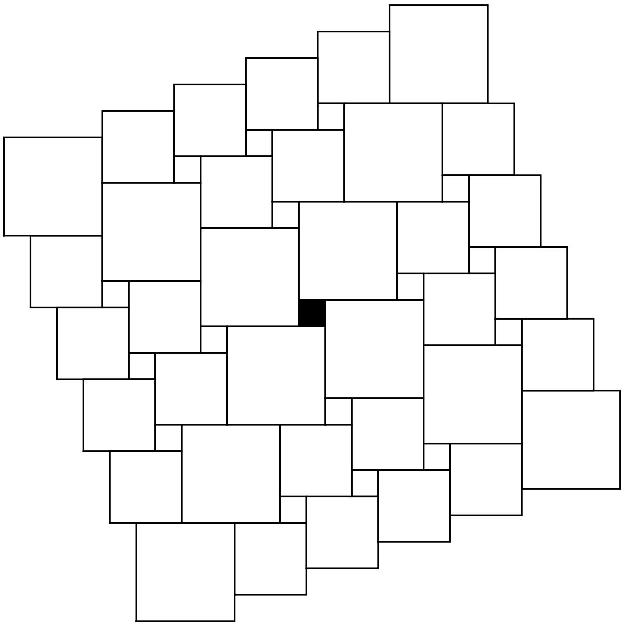

Figure 7

Remark 2. We cannot assert that under the hypothesis of Lemma 1 every λ_1-tile has a common point with two noncongruent neighbourly tiles. For example, all neighbourly tiles of the shaded λ_1-tile in Figures 6 and 7 are either of size λ_2 or of size λ_3, respectively. In Figure 6

(Figure 7, respectively), besides the central λ_1-tile we have four "rays" of λ_2-tiles (λ_3-tiles, respectively), and between any two neighbouring "rays" there is a packing of λ_1-tiles and λ_2-tiles (λ_3-tiles, respectively), as in Figure 1(i). (The "rays" in Figure 6 are discontinuous.)

Theorem 3. *Any U3-tiling of the plane with squares of sizes λ_1, λ_2, λ_3 has a λ_3-tile which is not a vortex.*

Proof. Due to Lemma 1, there is a λ_1-tile T_1 (that is a vortex by Theorem 1), having a common point with two noncongruent neighbourly tiles. Moreover, there is at least one such common point, with any given cyclic order of the sizes of squares adjacent to it. Thus there exists a side $[a, b]$ of T_1 which belongs to a side $[a, e]$ of a λ_3-tile T_3, and b is a vertex of a λ_2-tile T_2. Let $[b, f]$ be the side of T_2 having common relative interior points with $[a, e]$.

First we are going to prove that $[a, e]$ is the union of $[a, b]$ and $[b, f]$. Denote by $[f, g]$ the side of T_2 adjacent to $[b, f]$. Since T_2 is a vortex and since $[a, f]$ is a proper extension of $[b, f]$ in skel $\mathcal{H}$, there is a proper extension $[f', g]$ of $[f, g]$ in skel $\mathcal{H}$. Hence $[a, e]$ is covered by the union of $[a, b]$ and $[b, f]$. At the same time $[a, e]$ is not a proper part of $[a, f]$, since otherwise the segment $[e, f]$ would be of length smaller than λ_1, i.e., it could not be on a side of any tile in $\mathcal{H}$, different from T_2. Hence $[a, e] = [a, b] \cup [b, f]$.

From the fact that both T_1 and T_2 are vortices it follows that the segment $[a, e]$ is not properly extendable inside skel $\mathcal{H}$. Thus T_3 is not a vortex. $\square$

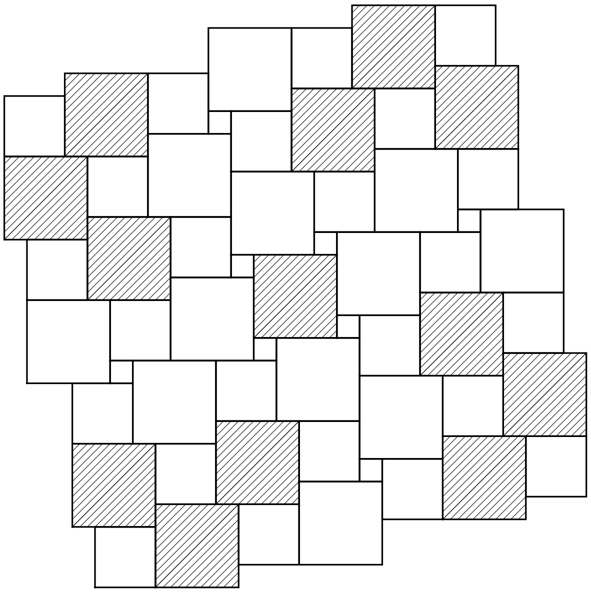

Figure 8

Remark 3. Observe that some U3-tilings can have vortices of the largest size. For example, the shaded squares in Figure 8 are vortices. In the figure, besides the central λ_3-tile we have four discontinuous "rays" of λ_1-tiles, and between any two neighbouring "rays" there is a packing of λ_2- and λ_3-tiles, as in Figure 1(i).

By considering the tiling shown in Figure 1(ii) and from the proof of Theorem 3 we conclude the following corollary, mentioned in [6], p. 81, without proof.

Corollary 1. *Let λ_1, λ_2, λ_3 be any three positive real numbers such that $\lambda_1 < \lambda_2 < \lambda_3$. There exists a U3-tiling of the plane with squares of sizes λ_1, λ_2, λ_3 if and only if $\lambda_1 + \lambda_2 = \lambda_3$.*

2. Neighbourly Environments

In this section we describe possible combinations of the placement of tiles in a U3-tiling, neighbourly to a given λ_i-tile, $i = 1, 2, 3$, that we will call *neighbourly environment* of the tile.

Lemma 2. *No λ_3-tile of a U3-tiling has a point in common with two distinct λ_2-tiles.*

Proof. Assume, for contradiction, that a λ_3-tile T has a point a in common with two distinct λ_2-tiles S_1, S_2. Since S_1 and S_2 have no common side, a is a vertex of T and of S_1, say. If a were an interior point of a side of S_2, then S_2 would not be a vortex. Hence a is also a vertex of S_2. Now, again using the fact that both S_1 and S_2 are vortices, we conclude that no tile covers the right angle with apex a exterior to $S_1 \cup T \cup S_2$. □

Theorem 4. *Let $\mathcal{H}$ be a U3-tiling of the plane with squares of sizes λ_1, λ_2, λ_3. Then Figures 9 and 10 describe all (up to congruence) neighbourly environments of a λ_1-tile and of a λ_2-tile, respectively.*

Proof. Since any of the neighbourly environments depicted in Figures 9 and 10 occurs in one of the tilings shown in this paper, it remains to prove that no other combination of neighbourly tiles exists.

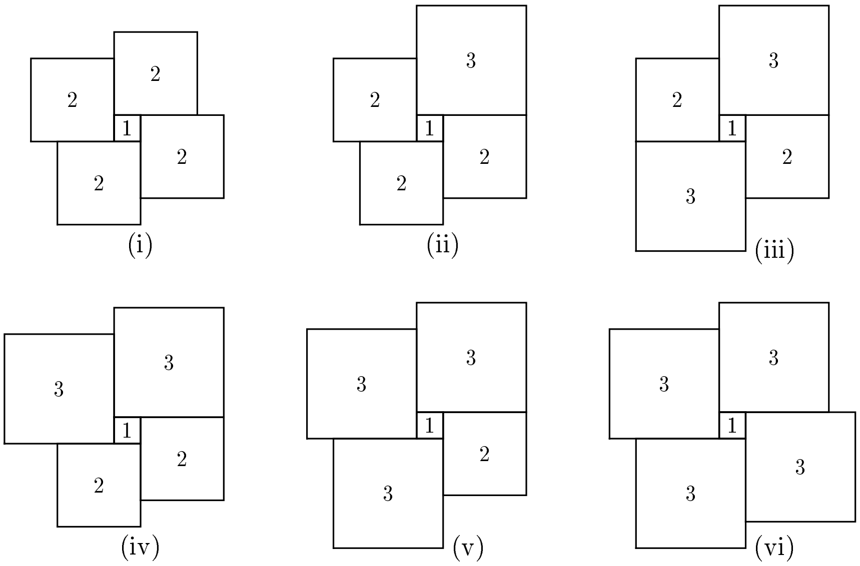

Figure 9

From the fact that any λ_1-tile is a vortex it easily follows that no two λ_1-tiles have a common point. Figures 9(i)–(vi) show that, for a given λ_1-tile, any combination of neighbourly λ_2-tiles and λ_3-tiles is possible.

Recall that any λ_2-tile is a vortex. With the notation of Figure 3, the neighbours of T along $[a, b]$, considered in the order from a to b, are either a λ_3-tile, or a λ_1-tile, or a λ_1-tile and a λ_3-tile, respectively. Together with Theorems 1 and 2, this condition implies that there are no more possible neighbourly environments of a λ_2-tile than those shown in Figures 10(i)–(vi). □

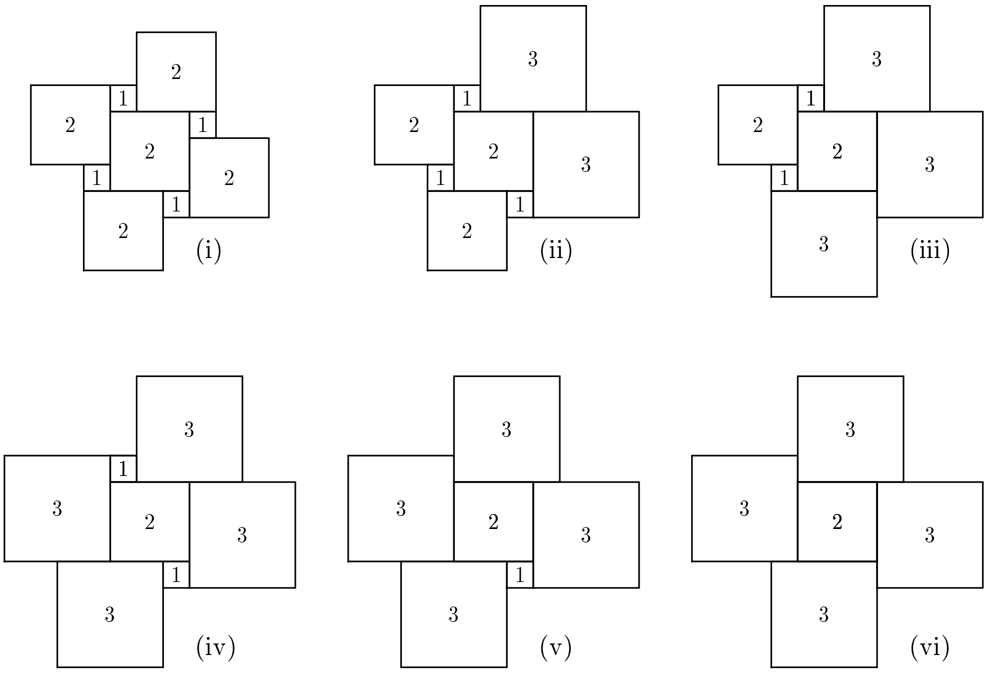

Figure 10

Lemma 3. *Let $\mathcal{H}$ be a U3-tiling of the plane with squares of sizes λ_1, λ_2, λ_3, and let $[a, b]$ be a side of a λ_3-tile T and S_1, S_2 be two λ_1-tiles such that their sides $[c, a]$ and $[b, d]$ together with $[a, b]$ form a line segment $[c, d]$. Then $\lambda_2 = 2\lambda_1$, and, up to congruence, there are exactly six neighbourly environments of T including T, S_1, S_2 as shown in Figure 11.*

Proof. First we prove that no other neighbourly environments exist than those in Figure 11. Since both S_1, S_2 are vortices, T, S_1, S_2 lie on the same side of the segment $[c, d]$, and $[c, d]$ is the union of sides of some tiles not overlapping $T \cup S_1 \cup S_2$. Clearly, none of these new tiles is of size λ_1. Hence their number is two, one of them being a λ_2-tile and the second being a λ_3-tile. This implies $\lambda_2 = 2\lambda_1$.

Denote by e, f the other two vertices of T such that a, b, e, f are ordered according to an orientation of T. Let V_1 and V_2 be the tiles neighbourly to both T, S_1 and T, S_2, respectively, like in Figure 11. If both V_1, V_2 are λ_3-tiles, then we have one of the configurations depicted in Figures 11(i) and 11(ii). If one of V_1, V_2 is a λ_2-tile and the second is a λ_3-tile, we have one of the configurations depicted in Figures 11(iii)–(vi).

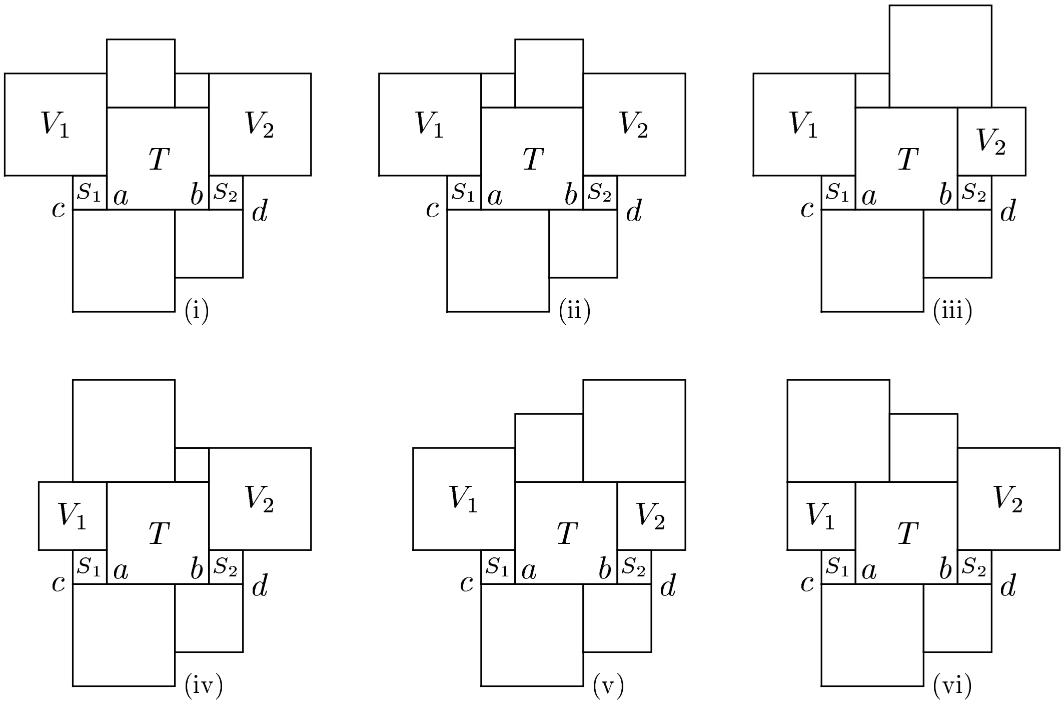

Figure 11

It remains to show that both V_1, V_2 cannot be of size λ_2. Indeed, assume that both V_1, V_2 are λ_2-tiles and $[g, f]$, $[e, h]$ are their sides such that $[g, h]$ is a line segment including $[f, e]$. Then, since both V_1, V_2 are vortices, $[g, h]$ is the union of sides of some tiles not overlapping $T \cup V_1 \cup V_2$. It is easy to see that these new tiles form an ordered sequence (starting from g) either of the form $\{\lambda_1\text{-tile}, \lambda_3\text{-tile}, \lambda_2\text{-tile}, \ldots\}$, contradicting Theorems 1 and 2, or of the form $\{\lambda_3\text{-tile}, \lambda_2\text{-tile}, \ldots\}$, contradicting the fact that V_2 is a vortex.

For finishing the proof of Lemma 3, we give examples of U3-tilings containing each of the neighbourly environments in Figure 11. Indeed, the environments (i), (ii), and (iii) occur in the U3-tilings from Figures 12(i)-(iii), respectively. The environment (iv) occurs in the U3-tiling from Figure 12(ii), and both environments (v) and (vi) occur in the U3-tiling from Figure 12(iv). □

Theorem 5. *Let $\mathcal{H}$ be a U3-tiling of the plane with squares of sizes λ_1, λ_2, λ_3.*

1) *If $\lambda_2 \neq 2\lambda_1$, then, up to congruence, Figure 13 describes all neighbourly environments of a λ_3-tile.*

2) *If $\lambda_2 = 2\lambda_1$, then, up to congruence, there are exactly twelve neighbourly environments of a λ_3-tile, namely the six ones shown in Figure 11, and the six further ones shown in Figure 13.*

Proof. Since any of the neighbourly environments in Figures 11 and 13 occurs in one of the tilings shown in this paper, it remains to prove that (except for those in Figures 11 and 13) no other neighbourly environments exist.

Let T be a λ_3-tile with the vertices a, b, c, d, taken according to an orientation of T. Clearly, every side of T can touch at most one λ_1-tile.

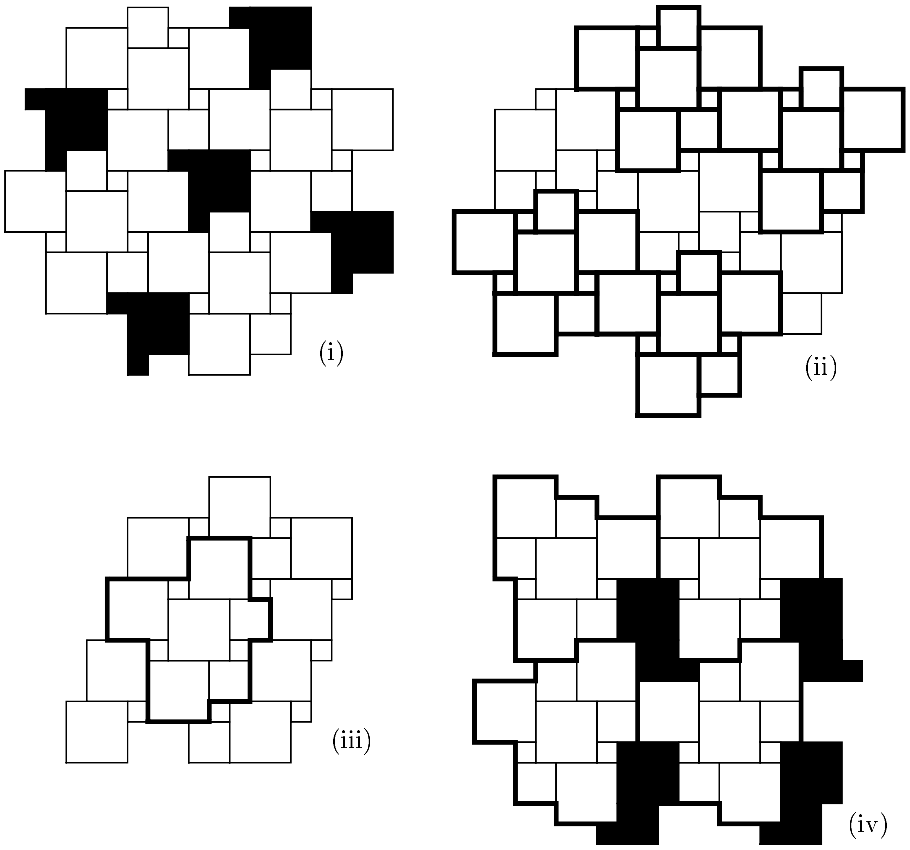

Figure 12

First we consider the case $\lambda_2 \neq 2\lambda_1$. From Lemma 3 it follows that no two λ_1-tiles can touch T as the squares S_1 and S_2 in Figure 11. (In fact, cases (a)–(e) below use only this consequence, but not the assumption $\lambda_2 \neq 2\lambda_1$ itself.)

(a) Assume first that T is neighbourly to four λ_1-tiles. By the above arguments, these λ_1-tiles are placed as shown in Figure 13(i). By Theorem 2 and Corollary 1, it is easily seen that in this case there are four additional λ_3-tiles neighbourly to T.

(b) Let T neighbour three λ_1-tiles and, without loss of generality, these three λ_1-tiles be placed as shown in Figure 13(ii). Then two λ_3-tiles touch T at $[a, d]$ and $[a, b]$, respectively.

We claim that T neighbours a λ_2-tile along the side $[b, c]$. Indeed, assume that T neighbours a λ_3-tile along $[b, c]$. Then, since there are only three λ_1-tiles neighbouring T, the vertex c belongs to a λ_2-tile touching $[c, d]$, and hence one additional λ_3-tile touches T at $[c, d]$. The last is impossible, since the λ_1-tile containing the vertex d and neighbouring T is a vortex.

Now we claim that along the side $[c, d]$ the tile T neighbours one λ_2-tile and one λ_3-tile, as shown in Figure 13(ii). Indeed, if this λ_2-tile and this λ_3-tile were neighbourly to T in the inverse order, this λ_2-tile would not be a vortex, which is impossible by Theorem 2.

(c) Let exactly two λ_2-tiles neighbour T. First we consider the case when these two λ_1-tiles are placed as shown in Figure 13(iii) or Figure 13(iv).

If they are placed as shown in Figure 13(iii), then, as above, T neigbours a λ_3-tile along

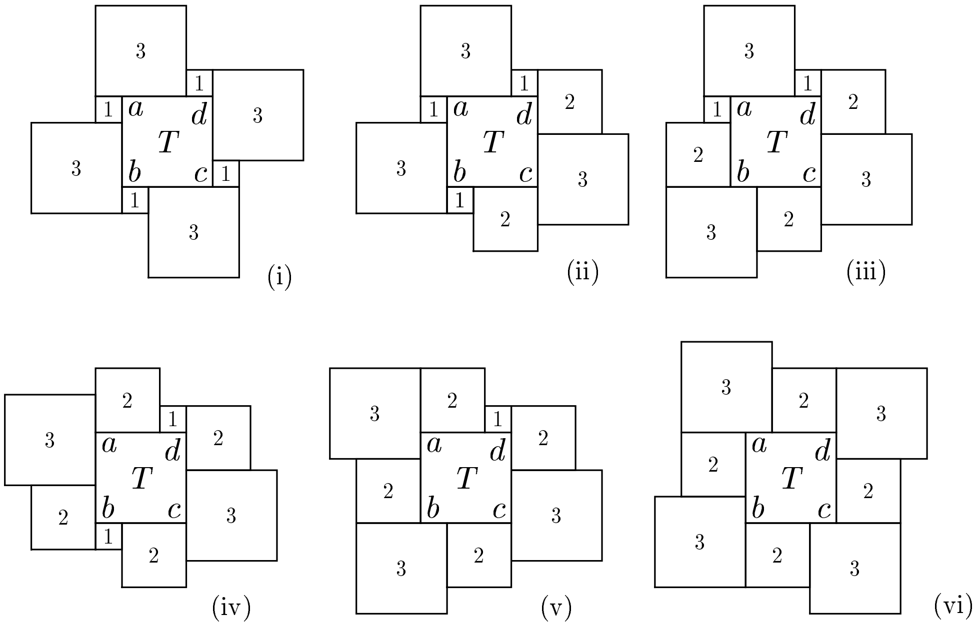

Figure 13

$[a, d]$, the side $[c, d]$ touches one λ_2-tile and one λ_3-tile (in the order from d to c), $[b, c]$ touches one λ_2-tile and one λ_3-tile and, finally, $[a, b]$ touches one λ_2-tile and one λ_1-tile.

If two λ_1-tiles are placed as shown in Figure 13(iv), then, by similar arguments, we obtain the neighbourly environment depicted in this figure.

It remains to show that Figure 13(iii) and 13(iv) describe the only possible placements of the two λ_1-tiles. Indeed, the only other possibility would be that, e.g., these λ_1-tiles (S_1 and S_2, say) have their lower right corners at b and d, respectively. Let $[e, b]$ be one of the horizontal sides of S_1. Since the λ_1-tiles are vortices, T neighbours at $[c, d]$ a λ_2-tile and a λ_3-tile (in some order), where one side of the lower tile contains c in its relative interior. Then $[e, c]$ is the union of sides of an alternating sequence of λ_2- and λ_3-tiles neighbouring T, which is impossible by $\lambda_2 < \lambda_3 < \lambda_1 + \lambda_3 < \lambda_2 + \lambda_3$.

(d) Let only one λ_1-tile touch T, as shown in Figure 13(v). As above, we have that $[c, d]$ touches one λ_2-tile and one λ_3-tile in this order. Then, necessarily, we have the configuration depicted in Figure 13(v).

(e) Finally, let no λ_1-tile neighbour T. In this case, analogously to Theorems 2 and 4 (see Figure 10(i)) we have the neighbourly configuration depicted in Figure 13(vi).

Now let $\lambda_2 = 2\lambda_1$. If no two λ_1-tiles touch T as the squares S_1 and S_2 in Figure 11, we can use the considerations from (a)–(e) above (as it has been mentioned at the beginning of the proof). On the other hand, the case when two such λ_1-tiles exist has been considered in Lemma 3; all the corresponding neighbourly environments are shown in Figure 11. □

3. Some Global Properties of U3-tilings

We will say that a tiling of the plane with squares is *simple* provided no proper subfamily of at least two tiles form a convex region, possibly unbounded. From the definitions it easily follows that any simple tiling is unilateral. At the same time, with the both-way unbounded strip formed by squares of four distinct sizes (cf. Figure 14) one can easily construct a unilateral tiling which is not simple. We show below that a unilateral tiling with squares of at most three distinct sizes is always simple.

From Theorems 1, 2, and Remark 1 we conclude the following assertion, implying that any unilateral tiling with squares of two sizes is simple.

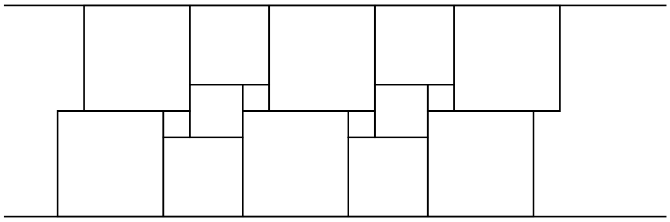

Figure 14

Theorem 6. *For any pair of real positive numbers* $\lambda_1 < \lambda_2$ *there is, up to congruence, a unique unilateral tiling of the plane with squares of sizes* λ_1 *and* λ_2*, respectively* (*this tiling is depicted in Figure* 1(i)).

Our next assertion shows that a similar fact holds for U3-tilings.

Theorem 7. *Any U3-tiling of the plane is simple.*

Proof. Assume the existence of a non-simple U3-tiling with squares of sizes λ_1, λ_2, λ_3. Then a proper subfamily of at least two tiles form a convex region R, possibly unbounded. The region R may be a rectangle, a strip (one- or both-way unbounded), a quarter-plane, or a half-plane. It is known (see [1]) that no rectangle can be simply partitioned into squares of at most four distinct sizes. Hence R is unbounded and contains a ray l in its boundary. We remark that no tile of size λ_1 or λ_2 has a side in the relative interior of l, since otherwise it would not be a vortex. Hence some subray of l is covered by infinitely many sides of λ_3-tiles, contradicting the unilaterality of $\mathcal{H}$. □

Theorem 8. *If* $\mathcal{H}$ *is a U3-tiling of the plane with squares of sizes* λ_1, λ_2, λ_3*, then tiles of each size occur infinitely many times in* $\mathcal{H}$*.*

Proof. Suppose that for some $i \in \{1, 2, 3\}$ the number of λ_i-tiles in $\mathcal{H}$ is finite. We will show that in this case $\mathcal{H}$ does not contain any tile of this size, getting a contradiction.

Thus, assume that the number of λ_i-tiles in $\mathcal{H}$ is finite and positive. Let j and k, $j < k$, be integers forming the set $\{1, 2, 3\} \setminus \{i\}$. According to Theorem 6, there is, up to congruence, a unique tiling of the plane with squares of sizes λ_j and λ_k, as shown in Figure 1(i). Moreover, even any local picture of such a tiling is unique. This implies the existence of an $m \times n$ frame $\mathcal{F}$ formed by λ_k-tiles such that all λ_i-tiles of $\mathcal{H}$ lie within $\mathcal{F}$ (see Figure 15). Clearly, $m, n \geq 3$. Without loss of generality, we may assume that $\mathcal{F}$ is a frame of minimal size containing all λ_i-tiles.

Consider the square S_1 depicted in Figure 15 ($S_1 \in \mathcal{H}$ by unilaterality and the assumption on $\mathcal{F}$). It is obvious that S_1 is a λ_j-tile. Let $[a, b]$ be the horizontal segment consisting of a side of S_1 and a side of V_2. In what follows, we distinguish two cases.

(a) $k = 3$, i.e., the frame $\mathcal{F}$ is formed by λ_3-tiles. Then the tile S_2, whose lower right vertex coincides with the upper right vertex of V_2 is of size either λ_1 or λ_2, and hence it is a vortex. The last is only possible if S_1 and S_2 are congruent.

Indeed, otherwise S_2 would be a λ_i-tile, with $\lambda_i > \lambda_j$, implying $i = 2$ and $j = 1$. Consider the part of the segment $[a, b]$ not covered by S_2, say L. This subinterval L is of length $\lambda_1 + \lambda_3 - \lambda_2$ ($= 2\lambda_1$ by Corollary 1). Then L would be covered from above either by one λ_2-tile or by two λ_1-tiles, both cases contradicting unilaterality.

Since S_1 and S_2 are congruent, the length of L equals λ_3. We claim that L is covered from above by a λ_3-tile. Indeed, otherwise it would be covered from above by a λ_i- and λ_j-tile (in some order), contradicting unilaterality.

Repeating these considerations by moving along the squares $V_1, V_2, \ldots, V_m$, we obtain that $\mathcal{F}$ can be substituted by a smaller $m \times (n-1)$ frame containing all the λ_i-tiles, a contradiction.

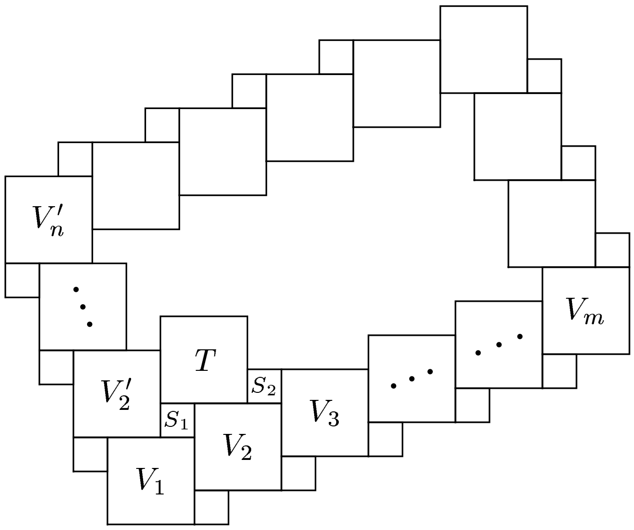

Figure 15

(b) $k = 2$, i.e., $j = 1$ and $i = 3$, and the frame $\mathcal{F}$ is formed by λ_2-tiles. Then V_3 is a vortex. That implies that S_2 as defined above is a λ_1-tile, and L as defined above is covered from above by a λ_2-tile. As in case (a), we can substitute $\mathcal{F}$ by a smaller $m \times (n-1)$ frame, getting a contradiction. □

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