Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 39 (1998), No. 2, 329-348

Higher Order Codazzi Tensors on Conformally Flat Spaces

H. L. Liu* **U. Simon**** **C. P. Wang****

Technische Universität Berlin, MA 8-3
Straße des 17. Juni 136, D-10623 Berlin
e-mail: liu@math.tu-berlin.de / simon@sfb288.math.tu-berlin.de

Department of Mathematics, Beijing University
Beijing 100871, P. R. China
e-mail: wang@sxx0.math.pku.edu.cn

Abstract. We generalize the concept of Codazzi tensors to higher order tensors and present examples. We derive second order PDE's of Weitzenboeck type and give local and global applications to hypersurfaces in space forms, minimal submanifolds and affine hypersurfaces.

MSC 1991: 53B21, 53A15, 53A30, 53B20, 53B25, 53C20, 53C21, 53C40, 53C42
Keywords: Codazzi tensors of higher order, conformally flat spaces, Weitzenboeck formulas, affine hypersurfaces, minimal submanifolds in spheres

1. Introduction

The well-known notion of a Codazzi-tensor Φ relative to a connection ∇ on a differentiable manifold $\mathbf{M}$ is contained as a special case, namely for $m = 2$, in the following definition.

Definition. *Let* $\mathbf{M}$ *be a connected, oriented* C^∞*-manifold of dimension* $n = \dim \mathbf{M} \geq 2$, ∇ *an affine connection and* Φ *a totally symmetric* $(0,m)$*-tensor field,* $m \geq 1$.

(i) *We call* Φ *a* $(0,m)$*-Codazzi tensor relative to* ∇*, or a* ∇*-Codazzi tensor of order* m*, if the covariant derivative* $\nabla\Phi$ *is totally symmetric. In this case we call* $\{\nabla, \Phi\}$ *also a Codazzi pair. If* ∇ *is the Levi-Civita connection* $\nabla = \nabla(g)$ *of a semi-Riemannian metric* g *we call* Φ *a Codazzi tensor relative to* g.

(ii) *We call* Φ *traceless relative to a semi-Riemannian metric* g *if* $\mathrm{trace}_g\Phi = 0$.

* Supported by a DFG-project at the TU Berlin
** Partially supported by a DFG-project at the TU Berlin

For order $m = 2$ there are many well-known examples and results, see 2.1.2 below.

For order $m = 3$, affine differential geometry gives examples of Codazzi tensors, namely the Codazzi property of certain cubic forms characterizes special classes of affine hypersurfaces like affine spheres; see 2.1.3.

In section 2 we list also new examples of higher order Codazzi tensors.

In this and a complementary paper [29] we extend results which are known for Codazzi order $m = 2$ and partly for $m = 3$. There are well-known PDE's and Weitzenboeck formulas for the square norm of a Codazzi-tensor; we refer to [50], [57], [45] for $m = 2$, and to Blaschke's original PDE for the Pick invariant (see [2] §76; Blaschke's PDE is much older than Simons') and its generalizations [6], [7], [42], [46] for $m = 3$; these led to our paper [25]. The latter paper was the starting point for our investigations on Codazzi tensors of arbitrary order m in [29] and the present note. Due to the complex methods the strong results in [29] are restricted to the dimension $n = 2$, here we study the higher dimensional case. In particular, we unify known results and generalize the Weitzenboeck-type formulas for Codazzi tensors to arbitrary order m. A typical result is the following surprisingly simple PDE.

1.1. Theorem. *Let $(\mathbf{M}, g)$ be a Riemannian manifold, where $\mathbf{M}$ is connected and oriented of dimension $n > 2$. Assume $(\mathbf{M}, g)$ to be conformally flat and Φ to be a* traceless $(0, m)$-*Codazzi tensor relative to g. Then*

$$\frac{1}{2}\Delta\|\Phi\|^2 = \|\nabla\Phi\|^2 + \frac{n+2m-4}{n-2}\alpha_{ij}(R^{ij} - \frac{(m-1)n}{n+2m-4}\kappa g^{ij}).$$

Here R_{ij} denote the components of the Ricci tensor in standard local notation, κ the normed scalar curvature, $g^{ij}R_{ij} = n(n-1)\kappa$, and α a symmetric, semi-positive definite $(0, 2)$-tensor defined in terms of Φ (see (4.1.2) below).

From section 5 on we give a series of exemplary local and global applications for Riemannian manifolds, their submanifolds as well as affine hypersurfaces. Our list of examples in section 2 admits additional applications of our main results in section 4. It is the aim of our examples and our applications to demonstrate that higher order Codazzi tensors naturally appear in many problems in differential geometry.

2. (0,m)-Codazzi tensor fields, examples

2.1. Examples. In this section we list examples and references.

2.1.1. (0,1)-Codazzi tensors. For $m = 1$, every exact 1-form is a (0,1)-Codazzi tensor.

2.1.2. (0,2)-Codazzi tensors. For $m = 2$ there are many results, examples and references.

2.1.2.1. Codazzi tensors on semi-Riemannian manifolds.

(a) The conference proceedings [16], pp. 243–299, contain a section of 10 contributions on Codazzi tensors. The papers [8] and [37] contain further examples, the papers [4], [20], [24], [40], [43], [44], [45], [56], [57], [58] give additional results and methods.

(b) [8] in particular investigates the situation of Codazzi tensors on Riemannian manifolds with exactly two distinct eigenvalue functions. These results were generalized by [18].

(c) [26] generalized the Cartan identities on isoparametric hypersurfaces to Codazzi tensors.

(d) Codazzi tensors appear in the theory of generalized curvature tensor fields; see e.g. [33].

(e) On a space $(\mathbf{M}, g)$ of constant curvature κ, every $f \in C^\infty(\mathbf{M})$ generates a Codazzi tensor

$$H(f) := \mathrm{Hess} f + \kappa f g$$

where Hess$f$ denotes the covariant Hessian of the function. Moreover, if $\mathbf{M}$ is simply connected, any Codazzi tensor Φ on $(\mathbf{M}, g)$ can be generated by an appropriate function f such that $\Phi = H(f)$. See [15] and section 2.1.2.2 below.

2.1.2.2. Codazzi tensors on projectively flat manifolds. Let ∇ be a torsion free connection on $\mathbf{M}$. We call ∇ projectively flat if $(\mathbf{M}, \nabla)$ locally is projectively equivalent to the projective space $\mathbb{R}P^n$ with its canonical projectively flat connection. One can generalize the properties of Codazzi tensors from constant curvature spaces to projectively flat spaces (see e.g. [41], p. 232, or [47]).

Norden's Lemma. *Let* $(\mathbf{M}, \nabla)$ *be torsion free and projectively flat with symmetric Ricci tensor* Ric.

(i) *If* $f \in C^\infty(\mathbf{M})$ *define*

$$H(f) := \mathrm{Hess} f + \frac{1}{n-1} \mathrm{Ric} f,$$

where Hess$f$ *is the* ∇*-covariant Hessian of* f *and* $\mathrm{Ric} := \mathrm{Ric}(\nabla)$ *its Ricci tensor (our sign convention for the curvature tensor and the Ricci tensor of* ∇ *follows* [21]*). Then* $\{\nabla, H(f)\}$ *is a Codazzi pair. In particular, for* $f = constant \neq 0$*, this includes the well-known fact that* $\{\nabla, \mathrm{Ric}\}$ *is a Codazzi pair for* $(\mathbf{M}, \nabla)$ *projectively flat.*

(ii) *If* $\mathbf{M}$ *is simply connected , any Codazzi tensor* Φ *on* $(\mathbf{M}, \nabla)$ *can be generated by an appropriate function* f *such that* $\Phi = H(f)$*.*

2.1.2.3. Conjugate connections. A triple $\{\nabla, g, \nabla^*\}$ of two connections and a semi-Riemannian metric g is called conjugate if

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$

for all vector fields X, Y, Z on $\mathbf{M}$. Let ∇ be torsion free. Then $\{\nabla, g\}$ is a Codazzi pair if and only if ∇^* is a torsion free connection. For details and references see section 3 below.

2.1.3. (0,3)-Codazzi tensors. For $m = 3$ we list the following examples.

2.1.3.1. Centroaffine hypersurfaces.([48], 6.3) Let C be the centroaffine cubic Pick form of a non-degenerate hypersurface and h be the centroaffine metric. C is a (0,3)-Codazzi tensor relative to h. This follows from (4.12.2.1.iii) together with (6.3.2) in [48].

2.1.3.2. Relative spheres.([48], 7.2) Let C be the relative cubic Pick form of a relative sphere. Then C is a (0,3)-Codazzi-tensor relative to any of the following 3 connections in relative geometry: the induced connection ∇, the Levi-Civita connection $\widehat{\nabla}$ of the relative metric, the conormal connection ∇^*. One easily verifies the fact that any of these Codazzi properties characterizes relative spheres. See [25], Corollary 1.4.6. For affine spheres this was proved in [3].

2.1.3.3. Tchebychev hypersurfaces.([25]) Consider a relative Tchebychev hypersurface $x : \mathbf{M} \to \mathbf{A}$ as in [25] together with its relative cubic Pick form C and the symmetric (0,3)-field $\widetilde{C}$ which is the traceless part of C

$$\widetilde{C}(u,v,w) := C(u,v,w) - \frac{n}{n+2}[h(u,v)\widehat{T}(w) + h(v,w)\widehat{T}(u) + h(w,u)\widehat{T}(v)];$$

here the relative Tchebychev form $\widehat{T}$ and the relative Tchebychev vector field T are defined by

$$n\hat{T}(v) := \mathrm{trace}\{u \to C(u,v)\}, \quad h(T,v) := \widehat{T}(v),$$

and h is the relative metric. x is called a relative Tchebychev hypersurface if the relative Tchebychev operator

$$L := \frac{1}{2}S - \frac{n}{n+2}\nabla(h)T$$

satisfies

$$L = \lambda\,\mathrm{id}.$$

It is proved in [25] that, for a non-degenerate hypersurface, $\widetilde{C}$ is a (0,3)-Codazzi tensor with respect to the relative metric h if and only if x is a relative Tchebychev hypersurface.

2.1.3.4. Totally real submanifolds. Let $<\ ,\ >$ be the inner product on $\mathbb{C}^n = \mathbb{R}^{2n}$ given by

$$< z, w >= \frac{1}{2}(z\bar{w} + \bar{z}w), \quad z, w \in \mathbb{C}^n.$$

An n-dimensional submanifold $\mathbf{M}$ of $\mathbb{C}^n$ is called totally real, if $< iX, Y > = 0$ for all $X, Y \in TM$. Let $x : \mathbf{M} \to \mathbb{C}^n$ be a totally real submanifold. We define a cubic form C on $\mathbf{M}^n$ by

$$C(X,Y,Z) :=< \bar{\nabla}_X Y, iZ >, \quad X, Y, Z \in T_p\mathbf{M}^n,$$

where $\bar{\nabla}$ is the standard connection on $\mathbb{C}^n$. It is easy to verify that C is a Codazzi tensor on $\mathbf{M}$. If $\mathbf{M}$ is totally real and minimal, then C is a traceless Codazzi tensor. We have $C \equiv 0$ if and only if the submanifold is totally geodesic.

2.1.3.5. (0,3)-Codazzi tensors on constant curvature spaces. Like for (0,2)-Codazzi tensors there is a procedure to generate (0,3)-Codazzi tensors by functions, and as before this procedure is suggested by results about spherical harmonics, now of second order ([52], [17]).

Proposition. *Let $(\mathbf{M}, g)$ be a space of constant curvature κ. For $f \in C^\infty(\mathbf{M})$, denote by $f_i := \partial_i f = \mathrm{d}f(\partial_i)$ the local components of $\mathrm{d}f$. Define the $(0,3)$-tensor $C(f)$ by*

$$C(f)_{ijk} := \nabla_k \nabla_j \nabla_i f + \kappa\{2f_k g_{ij} + f_i g_{jk} + f_j g_{ik}\}.$$

Then
(a) *$C(f)$ is a $(0,3)$-Codazzi tensor on $(\mathbf{M}, g)$;*
(b) *$C(f)^j_{ij} = \nabla_i(\Delta f + 2(n+1)\kappa f)$.*
(c) *In particular, if $(\mathbf{M}, g)$ is a Riemannian sphere $(\mathbf{M}, g) \cong (S^n(\kappa), g_{can})$, $f^\# := f - c$ is a second eigenfunction, where c is an appropriate constant.*

The proof uses the Ricci-identities (see [13], p. 30); it is straightforward.

2.1.3.6. Projectively flat spaces. Let $(\mathbf{M}, \nabla)$ be projectively flat as in 2.1.2.3. Then the Ricci tensor of ∇ is a (0,2)-Codazzi tensor, that means the covariant derivative is totally symmetric. One easily verifies that also the second covariant derivative is totally symmetric, that means ∇Ric is a (0,3)-Codazzi tensor ([23], p. 39).

2.1.4. (0,4)-Codazzi tensors.

2.1.4.1. (0,4)-Codazzi tensors on constant curvature spaces. We extend the procedure from (2.1.2.1.e) and (2.1.3) to generate (0,4)-tensors by functions. Similar constructions were used in [38] and [22] for results in spectral theory and on minimal submanifolds in spheres. Again, the constructions originate from results on spherical harmonics (see [17]).

Proposition. *Let $(\mathbf{M}, g)$ be a space of constant curvature κ. For $f \in C^\infty(\mathbf{M})$, define the $(0,4)$-tensor $U := U(f)$ in local terms by*

$$\begin{aligned} U_{ijkl} :=& \nabla_l \nabla_k [\nabla_j f_i + \kappa f g_{ij}] + 2\kappa [\nabla_l f_k g_{ij} + \nabla_l f_j g_{ik} + \nabla_l f_i g_{jk}] \\ &+ \kappa [\nabla_j f_i g_{kl} + \nabla_k f_j g_{il} + \nabla_k f_i g_{jl}] + 3\kappa^2 f [g_{ij} g_{kl} + g_{jk} g_{il} + g_{ik} g_{jl}]. \end{aligned}$$

Then

(i) *U is a $(0,4)$-Codazzi tensor.*

(ii) $g^{ij} U_{ijkl} = \nabla_l \nabla_k (\Delta f + 3(n+2)\kappa f) + \kappa g_{kl} (\Delta f + 3(n+2)\kappa f)$.

(iii) *If $(\mathbf{M}, g)$ is part of a sphere and f is a third order spherical harmonic, then* $g^{ij} U_{ijkl} = 0$.

(iv) *If $(\mathbf{M}, g)$ is part of a sphere and $F := \Delta f + 3(n+2)\kappa f$ is a first order spherical harmonic, then $g^{ij} U_{ijkl} = 0$.*

2.1.4.2. Remark on higher order Codazzi tensors. If f is a spherical harmonic of order k, we know from [17] that f satisfies PDE systems of order $(k+1)$. The foregoing results suggest that such systems admit a general procedure for the construction of (0,k)-Codazzi tensor on constant curvature spaces.

2.1.4.3. Improper affine spheres. Consider an affine sphere $x : \mathbf{M} \to \mathbf{A}$ which is not a quadric, equipped with the unimodular normalization; denote by C its equiaffine cubic form and by ∇ its induced connection. Then ∇C is totally symmetric if and only if x is an affine sphere. Moreover, it was proved in section 2 of [3] that x is improper if and only if $\nabla\nabla C$ is totally symmetric. Thus x from above is an improper affine sphere if both C and ∇C are Codazzi tensors. One similarly can consider ∇^* and C, i.e. the total symmetry of $\nabla^* C$ and $\nabla^* \nabla^* C$.

2.1.5. Higher order Codazzi tensors.

2.1.5.1. Parallel forms. In Euclidean geometry there are investigations on hypersurfaces with the property that a higher order covariant derivative of the second fundamental form II vanishes, let say

$$\nabla^k II = 0.$$

In particular, $\nabla^2 II = 0$ implies that $\nabla II - \frac{n}{n+2}\mathrm{sym}\{\mathrm{d}H \otimes g\}$ is a traceless Codazzi tensor of order 3 with constant squared norm; here H denotes the mean curvature and sym$\{\ldots\}$

the total symmetrization of the expression in brackets. See [9] and [10] for higher order parallel forms.

Similar investigations are known from affine differential geometry. A survey on such results is given by F. Dillen in A 2.2 in [27]; see also [12].

2.1.5.2. Reduction of the order.

2.1.5.2.1. Traces. Let $(\mathbf{M}, g)$ be semi-Riemannian and Φ be a Codazzi tensor of order $m \geq 3$ relative to g. The contraction of Φ gives a Codazzi tensor of order $m-2 \geq 1$.

2.1.5.2.2. Ejiri's method. Let ∇ be torsion free on $\mathbf{M}$ and Φ a ∇-Codazzi tensor of order $m \geq 2$. If X is a differentiable vector field, then $\Phi(X, ., \ldots, .)$ is a totally symmetric tensor of order $m-1 \geq 1$. In particular geometric situations one uses a vector field X with particular properties such that $\Phi(X, ., \ldots, .)$ is a Codazzi tensor. This procedure was used in [54] for the cubic form as (0,3)-Codazzi-tensor on affine spheres, but the method was introduced in an earlier paper of Ejiri [14].

3. Conjugate connections and Codazzi equations

In this section we summarize recent results on conjugate connections and give some applications for our purposes. We refer to [11], [34], [35] or [48]. Recall that a connection ∇ is called metric if it is the Levi-Civita connection of some semi-Riemannian metric g, $\nabla = \nabla(g)$.

3.1. Definition. *Consider on* $\mathbf{M}$ *a semi-Riemannian metric* h *and two torsion free connections* ∇, ∇^*. *We call* ∇, ∇^* *conjugate relative to* h *if*

$$X(h(Y,Z)) = h(\nabla_X Y, Z) + h(Y, \nabla^*_X Z)$$

for all tangent fields X, Y, Z *on* $\mathbf{M}$. *We also call* $\{\nabla, h, \nabla^*\}$ *a conjugate triple.*

3.2. Lemma. *Let* $\{\nabla, h, \nabla^*\}$ *be conjugate. Then*

(i) $\{\nabla, h\}$ *and* $\{\nabla^*, h\}$ *are Codazzi pairs.*

(ii) *The Levi-Civita connection* $\nabla(h)$ *satisfies*

$$\nabla(h) = \frac{1}{2}(\nabla + \nabla^*).$$

(iii) ∇ *is a metric connection,* $\nabla = \nabla(g)$, *if and only if* ∇^* *is a metric connection,* $\nabla^* = \nabla(g^*)$.

(iv) *If* ∇ *and* ∇^* *are metric in* (iii), *then there exists a regular operator* L *which is* h-*selfadjoint such that*

$$g(X,Y) = h(L^{-1}X, Y), \quad g^*(X,Y) = h(LX, Y).$$

(v) [19], [57]. *Let* $\lambda_1, \ldots, \lambda_n$ *be the eigenvalues of* L *in* (iv) *with* g-*orthonormal eigenvectors* $E_1, \ldots, E_n$, *then the sectional curvatures of* g *and* g^*, *resp., satisfy*

$$\lambda_i \lambda_j \kappa^*(E_i, E_j) = \kappa(E_i, E_j).$$

In particular, if L *from* (iv) *is definite on* $\mathbf{M}$, *i.e. all eigenvalues of* L *have the same sign, then the sectional curvatures of* g *and* g^* *have the same sign at any point.*

3.3. Proposition.([11]; [48], (4.4.5)). *Let $\{\nabla, h, \nabla^*\}$ be conjugate, φ an h-selfadjoint operator and Φ the associated symmetric bilinear form, $\Phi(X,Y) = h(\varphi X, Y)$. Then φ is a ∇-Codazzi operator if and only if Φ is a ∇^*-Codazzi tensor.*

3.4. Examples.

(i) Consider a hypersurface in Euclidean space $\mathbf{E}_{n+1}$ with nondegenerate second fundamental form II. Then $g := I$, $L = S$ (Weingarten operator) and $g^* = III$ give the conjugate triple $\{\nabla(I), II, \nabla(III)\}$.

(ii) Consider a space form $\bar{\mathbf{M}}(\bar{\kappa})$ with $\bar{\kappa} \neq 0$ and a (hyper-)surface immersion $x : \mathbf{M} \to \bar{\mathbf{M}}$ with non-zero Gauss-Kronecker curvature. Following the terminology in affine geometry we call x a non-degenerate immersion. According to (3.2) we can consider a conjugate triple $\{\nabla(I), II, \nabla^*\}$ where $g^*(X,Y) := II(SX, Y)$, $\nabla^* = \nabla(g^*)$. Then $\{\nabla^*, II\}$ is a Codazzi pair, and $\mathrm{trace}_{g^*} II = n\frac{H_{n-1}}{H_n} =: nP_1$.

(iii) The normed scalar curvature κ^* of $(\mathbf{M}, g^*)$ satisfies (see [5])

$$\kappa^* = \bar{\kappa} P_2 + 1, \qquad \text{where} \qquad P_2 := \frac{H_{n-2}}{H_n}.$$

Thus $\bar{\kappa} P_2$ is an intrinsic invariant of $(\mathbf{M}, g^*)$.

Below we will use the following curvature relations for $(\mathbf{M}, g^*)$:

3.5. Proposition. *Let $x : \mathbf{M} \to \bar{\mathbf{M}}(\bar{\kappa})$ a non-degenerate hypersurface. Assume $II_{ij} = k_i \delta_{ij}$ at $p \in \mathbf{M}$. Then the curvature tensor R^*, the Ricci tensor Ric^* and the normed scalar curvature $n(n-1)\kappa^* = \mathrm{trace}_{g^*}\mathrm{Ric}^*$ satisfy:*

$$R^{*l}{}_{rij} = (g^*_{rj}\delta^l_i - g^*_{ri}\delta^l_j) + \bar{\kappa}(II_{rj} I_{ki} - II_{ri} I_{kj}) II^{(kl)}, \tag{3.5.1}$$

where $II^{(kl)} II_{lj} = \delta^k_j$ defines the inverse matrix $(II^{(kl)})$;

$$R^*{}_{ij} = (n-1) g^*_{ij} + \bar{\kappa}(n\frac{H_{n-1}}{H_n} II_{ij} - I_{ij}), \tag{3.5.2}$$

$$n(n-1)\kappa^* = n(n-1) + \bar{\kappa}((n\frac{H_{n-1}}{H_n})^2 - \mathrm{trace}_{g^*} I); \tag{3.5.3}$$

the last equation implies (iii) *in* (3.4).

Proof. From [35] we have

$$II(R(u,v)w, z) + II(w, R^*(u,v)z) = 0.$$

Since

$$R^i{}_{jkl} = \bar{\kappa}(\delta^i_k I_{jl} - \delta^i_l I_{jk}) + (S^i_k II_{jl} - S^i_l II_{jk}),$$

by a direct computation, we get (3.5.1)–(3.5.3). □

4. Codazzi tensors on semi-Riemannian spaces, local results

We generalize PDE's known for (0,2)-Codazzi tensors and (0,3)-Codazzi tensors on Riemann spaces; see the references in the introduction.

In this section we use the following notation in Riemannian geometry: let $(\mathbf{M}, g)$ be a connected, oriented semi-Riemannian space of dimension $n \geq 3$, $\omega(g)$ its Riemannian volume form, ∇ its Levi-Civita connection, R its curvature tensor with the sign convention from [21], Ric its Ricci tensor, $\mathrm{trace}_g \mathrm{Ric} = n(n-1)\kappa$ its scalar curvature. Δ denotes the Laplacian, $\Delta f = \mathrm{trace}_g(\mathrm{Hess} f)$, and $(\mathrm{Hess}\, f)$ the covariant Hessian of f. W denotes the conformal Weyl curvature tensor. If convenient we use the standard local notation and adopt the Einstein convention.

4.1. Proposition. *Let $(\mathbf{M}, g)$ be as above and of dimension $n \geq 3$. Let Φ denote a $(0, m)$-Codazzi field, $m \geq 2$. Then*

$$
\begin{aligned}
\frac{1}{2}\Delta\|\Phi\|^2 = & \|\nabla\Phi\|^2 + (m-1)\Phi^{jkk_3\ldots k_m}\Phi^{il}_{k_3\ldots k_m}W_{ijkl} \\
& + \frac{n+2m-4}{n-2}\alpha_{ij}\left(R^{ij} - \frac{(m-1)n\kappa}{n+2m-4}g^{ij}\right) \\
& + \Phi^{k_1\ldots k_m}g^{ij}\nabla_{k_1}\nabla_i\Phi_{jk_2\ldots k_m} \\
& - \frac{2(m-1)}{n-2}\Phi^{i}_{ik_3\ldots k_m}\Phi^{jkk_3\ldots k_m}R_{jk} \\
& + \frac{n\kappa(m-1)}{n-2}\Phi_i^{ik_2\ldots k_m}\Phi^j_{jk_2\ldots k_m};
\end{aligned}
\tag{4.1.1}
$$

W_{ijkl} denote the components of the Weyl conformal curvature tensor, R_{ij} the components of the Ricci curvature tensor. We define the symmetric (0,2)-tensor α by

$$\alpha_{ij} := \Phi_{ik_2\ldots k_m}\Phi_j^{k_2\ldots k_m}. \tag{4.1.2}$$

Proof.

$$
\begin{aligned}
\frac{1}{2}\Delta\|\Phi\|^2 &= \|\nabla\Phi\|^2 + \Phi^{k_1\ldots k_m}g^{ij}\nabla_i\nabla_j\Phi_{k_1\ldots k_m} \\
&= \|\nabla\Phi\|^2 + \Phi^{k_1\ldots k_m}g^{ij}\nabla_i\nabla_{k_1}\Phi_{jk_2\ldots k_m},
\end{aligned}
$$

since Φ and $\nabla\Phi$ are totally symmetric. From the Ricci identity we get

$$\nabla_i\nabla_{k_1}\Phi_{jk_2\ldots k_m} = \nabla_{k_1}\nabla_i\Phi_{jk_2\ldots k_m} + \Phi_{\lambda k_2\ldots k_m}R^{\lambda}_{jk_1i} + \sum_{l=2}^{m}\Phi_{jk_2\ldots\lambda\ldots k_m}R^{\lambda}_{k_lk_1i}.$$

For the Riemannian curvature tensor of the metric g we have the well-known orthogonal decomposition

$$
\begin{aligned}
R_{ijkl} = W_{ijkl} &+ \frac{1}{n-2}(g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il}) \\
&- \frac{n\kappa}{n-2}(g_{ik}g_{jl} - g_{il}g_{jk}).
\end{aligned}
$$

Therefore

$$\frac{1}{2}\Delta\|\Phi\|^2 = \|\nabla\Phi\|^2 + \Phi^{k_1\ldots k_m}g^{ij}\nabla_i\nabla_{k_1}\Phi_{jk_2\ldots k_m}$$

$$
\begin{aligned}
=&\|\nabla\Phi\|^2 + \Phi^{k_1\dots k_m}\Phi_{\lambda k_2\dots k_m} g^{ij} R^{\lambda}_{\ jk_1 i} + \Phi^{k_1\dots k_m}\sum_{r=2}^{m} g^{ij}\Phi_{jk_2\dots\lambda\dots k_m} R^{\lambda}_{\ k_r k_1 i}\\
&+ \Phi^{k_1\dots k_m} g^{ij}\nabla_{k_1}\nabla_i\Phi_{jk_2\dots k_m}\\
=&\|\nabla\Phi\|^2 + \alpha_{ij}R^{ij} + (m-1)\Phi^{jkk_3\dots k_m}\Phi^{il}_{\ \ k_3\dots k_m} R_{ijkl}\\
&+ \Phi^{k_1\dots k_m} g^{ij}\nabla_{k_1}\nabla_i\Phi_{jk_2\dots k_m}\\
=&\|\nabla\Phi\|^2 + (m-1)\Phi^{jkk_3\dots k_m}\Phi^{il}_{\ \ k_3\dots k_m} W_{ijkl} + \alpha_{ij}R^{ij} + \frac{2(m-1)}{n-2}\alpha_{ij}R^{ij}\\
&- \frac{(m-1)n\kappa}{n-2}\alpha_{ij}g^{ij} + \Phi^{k_1\dots k_m} g^{ij}\nabla_{k_1}\nabla_i\Phi_{jk_2\dots k_m}\\
&- \frac{2(m-1)}{n-2}\Phi^{i}_{\ ik_3\dots k_m}\Phi^{jkk_3\dots k_m}R_{jk} + \frac{n\kappa(m-1)}{n-2}\Phi_{i}^{\ ik_2\dots k_m}\Phi^{j}_{\ jk_2\dots k_m}\\
=&\|\nabla\Phi\|^2 + (m-1)\Phi^{jkk_3\dots k_m}\Phi^{il}_{\ \ k_3\dots k_m} W_{ijkl} + \Phi^{k_1\dots k_m} g^{ij}\nabla_{k_1}\nabla_i\Phi_{jk_2\dots k_m}\\
&+ \frac{n+2m-4}{n-2}\alpha_{ij}\Big(R^{ij} - \frac{(m-1)n\kappa}{n+2m-4}g^{ij}\Big)\\
&- \frac{2(m-1)}{n-2}\Phi^{i}_{\ ik_3\dots k_m}\Phi^{jkk_3\dots k_m}R_{jk} + \frac{n\kappa(m-1)}{n-2}\Phi_{i}^{\ ik_2\dots k_m}\Phi^{j}_{\ jk_2\dots k_m}.
\end{aligned}
$$

This completes the proof of Proposition 4.1. □

4.2. Corollary. *If, additionally to the assumptions in Proposition 4.1, Φ is traceless, we have*

$$
\begin{aligned}
\frac{1}{2}\Delta\|\Phi\|^2 =&\|\nabla\Phi\|^2 + (m-1)\Phi^{jkk_3\dots k_m}\Phi^{il}_{\ \ k_3\dots k_m} W_{ijkl}\\
&+ \frac{n+2m-4}{n-2}\alpha_{ij}\Big(R^{ij} - \frac{(m-1)n\kappa}{n+2m-4}g^{ij}\Big).
\end{aligned}
\tag{4.2.1}
$$

4.3. Corollary. *If, additionally to the assumptions in Corollary 4.2, the metric g is conformally flat, we have*

$$
\frac{1}{2}\Delta\|\Phi\|^2 = \|\nabla\Phi\|^2 + \frac{n+2m-4}{n-2}\alpha_{ij}\Big(R^{ij} - \frac{(m-1)n\kappa}{n+2m-4}g^{ij}\Big).
\tag{4.3.1}
$$

4.4. Corollary. *Let $(\mathbf{M}, g)$ be Ricci-pinched:*

$$
\mathrm{Ric} > \frac{(m-1)n\kappa}{n+2m-4}g.
$$

Let Φ denote a traceless $(0,m)$-Codazzi tensor field, $m \geq 2$, and assume $\|\Phi\|^2 =$ constant.
(i) *If dim $\mathbf{M} \geq 4$ and $(\mathbf{M}, g)$ is conformally flat then $\Phi \equiv 0$.*
(ii) *If dim $\mathbf{M} = 3$ then $\Phi \equiv 0$.*

4.5. Corollary. *Let $(\mathbf{M}, g)$ be Ricci pinched*

$$
\mathrm{Ric} \geq \frac{(m-1)n\kappa}{n+2m-4}g.
$$

Let Φ be a traceless $(0,m)$-Codazzi tensor field, $m \geq 2$, and assume $\|\Phi\|^2 =$ constant.

(i) *If* $\dim\mathbf{M} \geq 4$ *and* $(\mathbf{M}, g)$ *is conformally flat, then* $\nabla\Phi = 0$.
(ii) *If* $\dim\mathbf{M} = 3$, *then* $\nabla\Phi \equiv 0$.
(iii) $\nabla\Phi \equiv 0$ *in* (i) *and* (ii), *resp., implies the following properties* (*see* [49], [56]):
   (1) *the eigenvalues of* α (*relative to* g) *are constant on* $\mathbf{M}$;
   (2) *the eigenvalues define eigendistributions*; *these are integrable and parallel*;
   (3) $\mathbf{M}$ *is locally the product of the integral manifolds* $(\mathbf{M}_k, g(k))$ *from* (2);
   (4) *let* $\lambda_1, \ldots, \lambda_r$ *be the different eigenvalues of* α *with multiplicities* $\mu_1, \ldots, \mu_r$, $\sum_{k=1}^{r} \mu_k = n$; *then there exist appropriate local coordinates associated to the product structure stated in* (3), *and*

$$\dim\mathbf{M}(k) = \mu_k, \quad g = \sum_{k=1}^{r} g(k), \quad \alpha = \sum_{k=1}^{r} \lambda_k g(k). \tag{4.5.1}$$

(iv) *If* $\mathbf{M}$ *is simply connected and* $(\mathbf{M}, g)$ *complete,* $(\mathbf{M}, g)$ *is globally a product of complete manifolds* $(\mathbf{M}_k, g(k))$.

Proof. The proofs of (i) and (ii) follow directly from (4.3.1). To prove (iii) and (iv) we note that $\nabla\Phi \equiv 0$ implies $\nabla\alpha \equiv 0$ where α was defined in (4.1.2). Now the product structure is a consequence of the de Rham decomposition theorem and the parallelity of α according to the properties (1)–(4). □

4.6. Remark. If Φ is of order 2, then the statements (1)–(4) hold for Φ. This fact is of particular importance for applications.

4.7. Corollary. *Let* $(\mathbf{M}, g)$ *be of constant nonnegative curvature* κ, $n \geq 2$. *Assume* Φ *is a traceless Codazzi tensor of order* $m \geq 2$, *satisfing* $\|\Phi\|^2 =$ *constant on* $\mathbf{M}$. *Then* $\Phi \equiv 0$ *or* $\kappa \equiv 0$.

Proof. $(\mathbf{M}, g)$ has constant curvature if and only if it is conformally flat and Einstein. If $\kappa > 0$, the latter implies, together with $n \geq 3$, $m \geq 2$,

$$\mathrm{Ric} = (n-1)\kappa g > (n-1)\frac{2(m-1)}{n+2m-4}\kappa g > \frac{n(m-1)}{n+2m-4}\kappa g.$$

We can now apply Corollary 4.4. For the proof in case of dimension $n = 2$ see [29]. □

4.8. Remark. Corollary 4.7 originates from affine geometry. See [54] and Proposition 8.3 below.

5. Global results for Codazzi tensors

There are many global results on Codazzi tensors for $m = 2$; see the references in (2.2.2.1). As a particular example we recall (see e.g. [57], [44], [45])

5.1. Theorem. *Let* $(\mathbf{M}, g)$ *be a closed* (*compact without boundary*), *connected, oriented Riemannian manifold of dimension* $n \geq 2$, *let* Φ *be a traceless Codazzi tensor of order* $m = 2$.
(i) *If the sectional curvature of* $(\mathbf{M}, g)$ *is positive then* $\Phi \equiv 0$.
(ii) *If the sectional curvature of* $(\mathbf{M}, g)$ *is non-negative, then* $\nabla\Phi \equiv 0$.

In this section, from now on, we consider $\mathbf{M}$ to be oriented, connected, closed with $\dim \mathbf{M} \geq 3$.

5.2. Proposition. *Let $\mathbf{M}$ be closed, connected, oriented and Φ be a traceless Codazzi tensor of order m relative to the Riemannian metric g, $m \geq 2$. Assume $(\mathbf{M}, g)$ to be Ricci-pinched*

$$\mathrm{Ric} > \frac{(m-1)n\kappa}{n+2m-4} g.$$

(i) *If* $\dim \mathbf{M} \geq 4$ *and* $(\mathbf{M}, g)$ *is conformally flat, then* $\Phi \equiv 0$.
(ii) *If* $\dim \mathbf{M} = 3$, *then* $\Phi \equiv 0$.

5.3. Proposition. *Let $(\mathbf{M}, g)$ be as above and Φ a traceless Codazzi tensor of order m, $m \geq 2$. Assume $(\mathbf{M}, g)$ to be Ricci pinched*

$$\mathrm{Ric} \geq \frac{(m-1)n\kappa}{n+2m-4} g.$$

(i) *If* $\dim \mathbf{M} \geq 4$ *and* $(\mathbf{M}, g)$ *is conformally flat, then* $\nabla\Phi \equiv 0$.
(ii) *If* $\dim \mathbf{M} = 3$, *then* $\nabla\Phi \equiv 0$.
Moreover, $\nabla\Phi \equiv 0$ in (i) *and* (ii), *resp., implies that $(\mathbf{M}, g)$ globally is a product of Riemannian manifolds $(\mathbf{M}_k, g(k))$ and*

$$g = \sum_{k=1}^{r} g(k), \quad r \geq 1.$$

It is well-known that Theorem 5.1 implies uniqueness theorems in global differential geometry, namely particular applications of the fact that, in the above situation, the trace of a Codazzi tensor determines the Codazzi tensor itself. It is obvious that (5.2) and (5.3) have similar consequences.

5.4. Corollary. *Let $(\mathbf{M}, g)$ be as in* (5.2). *Let Φ, $\Phi^{\#}$ be $(0, m)$-Codazzi tensors such that* $\mathrm{trace}_g \Phi = \mathrm{trace}_g \Phi^{\#}$. *Then* $\Phi = \Phi^{\#}$.

One can repeat this argument, using the following

5.5. Observation. Let Φ be a $(0, m)$-Codazzi tensor relative to a semi-Riemannian metric g on $\mathbf{M}$ for $m = 2k + l$, $l = 0, 1$. Let $1 \leq s \leq k$. Define by recursion

$$\mathrm{trace}_g^s \Phi := \mathrm{trace}_g(\mathrm{trace}_g^{s-1}\Phi), \qquad \text{where} \qquad \mathrm{trace}_g^0 \Phi := \Phi.$$

Then $\mathrm{trace}_g^s \Phi$ is again a $(0, 2(k-s)+l)$-Codazzi tensor.

5.6. Corollary. *Let $(\mathbf{M}, g)$ be as in* (5.2) *and let Φ, $\Phi^{\#}$ be $(0, m)$-Codazzi tensors where $m = 2k + l$ and $1 \leq k$, $l = 0, 1$. Let $1 \leq s \leq k$. Then*

$$\mathrm{trace}_g^s \Phi = \mathrm{trace}_g^s \Phi^{\#}$$

implies $\Phi = \Phi^{\#}$.

6. Hypersurfaces in space forms

We consider hypersurfaces in space forms and present applications of the results from section 5 to (0,2)-Codazzi tensors.

6.1. Curvature relations. Let $x : \mathbf{M} \to \bar{\mathbf{M}}(\bar{\kappa})$ be a hypersurface with Riemannian metric g in a real space form $\bar{\mathbf{M}}(\bar{\kappa})$. Let S be the Weingarten operator of x and H the mean curvature. Choose an orthonormal basis for TM such that, at p, $S_{ij} = k_i\delta_{ij}$, where $k_1, \ldots, k_n$ denote the principal curvatures at p. Recall the Gauss equations

$$R_{ijkl} = \bar{\kappa}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + S_{ik}S_{jl} - S_{il}S_{jk},$$

and the corollaries

$$R_{ik} = (n-1)\bar{\kappa}\delta_{ik} + nHS_{ik} - \sum_j S_{ij}S_{jk},$$

$$n(n-1)\kappa = n(n-1)\bar{\kappa} + n^2H^2 - \|S\|^2.$$

It is well-known that, within all (normed) elementary symmetric functions H_r of the principal curvatures, the mean curvature $H = H_1$ is a distinguished curvature function by the theorema egregium, as H_{2k} and $|H_{2k+1}|$, for $k \geq 1$, are intrinsic invariants. Define

$$f_m := \sum_i (k_i - H)^m, \quad m \in \mathbb{N}. \tag{6.1.a}$$

We call f_m the curvature momentum function of order m. While $f_1 = 0$, for $m \geq 2$ the functions f_m give appropriate measures for the deviation of the principal curvatures k_i (i=1,...,n) from the value of H. This is illustrated by the results in section 6.2 below which give integral relations between such extrinsic curvature functions and the Euler characteristic.

6.2. Theorem. *Let $x : \mathbf{M}^4 \to \bar{\mathbf{M}}^5(\bar{\kappa})$ be a compact hypersurface with Euler characteristic $\chi(\mathbf{M})$ and Riemannian metric g in a real space form $\bar{\mathbf{M}}^5(\bar{\kappa})$ such that the mean curvature H of x is constant. Then we have the following integral formula*

$$\int_{\mathbf{M}} \{\frac{5}{2}f_2^2 - (6\bar{\kappa}+10H^2)f_2 + 12(\bar{\kappa}+H^2)^2 - 3f_4\}\omega(g) - \int_{\mathbf{M}} \|\nabla\Phi\|^2\omega(g) = 16\pi^2\chi(\mathbf{M}), \tag{6.2.1}$$

where $\Phi_{ij} := S_{ij} - Hg_{ij}$.

Proof. From the Gauss equations we have

$$\begin{aligned}\|R_{ijkl}\|^2 &= 24(\bar{\kappa}+H^2)^2 + 2f_2^2 + 4(H^2-\bar{\kappa})f_2 - 2f_4 - 8Hf_3,\\ \|R_{ij}\|^2 &= f_4 - 4Hf_3 - (6\bar{\kappa}+2H^2)f_2 + 36(\bar{\kappa}+H^2)^2,\\ (12\kappa)^2 &= f_2^2 - 24(\bar{\kappa}+H^2)f_2 + 144(\bar{\kappa}+H^2)^2.\end{aligned}$$

The foregoing expressions give the identity

$$\|R_{ijkl}\|^2 - 4\|R_{ij}\|^2 + (12\kappa)^2 = -6f_4 + 8Hf_3 + 3f_2^2 - (4\bar{\kappa}+12H^2)f_2 + 24(\bar{\kappa}+H^2)^2.$$

The left hand side is the integrand of the Gauss Bonnet formula which in dimension $n = 4$ reads (c.f. [1], p. 82)

$$\int_{\mathbf{M}} \big(\|R_{ijkl}\|^2 - 4\|R_{ij}\|^2 + (12\kappa)^2\big)\omega(g) = 32\pi^2\chi(\mathbf{M}).$$

On the other hand, we have

$$\begin{aligned}\frac{1}{2}\Delta\|\Phi\|^2 &= \|\nabla\Phi\|^2 + \sum \Phi_{ij}\Phi_{\lambda k}R_{\lambda ijk} + \sum \Phi_{ij}\Phi_{i\lambda}R_{j\lambda} \\ &= \|\nabla\Phi\|^2 + \{(\bar{\kappa} + H^2)f_2 + f_4 + 2Hf_3 - f_2^2\} + \{-f_4 + 2Hf_3 + 3(\bar{\kappa} + H^2)f_2\} \\ &= \|\nabla\Phi\|^2 + 4(\bar{\kappa} + H^2)f_2 + 4Hf_3 - f_2^2.\end{aligned}$$

This gives a second integral formula

$$0 = \int_{\mathbf{M}} \{\|\nabla\Phi\|^2 + 4(\bar{\kappa} + H^2)f_2 + 4Hf_3 - f_2^2\}\omega(g).$$

We integrate the identity derived above and insert both integral formulas. This proves the assertion. □

6.2.2. Lemma. *The curvature functions f_2, f_4 satisfy the following inequalities*

$$f_4 \le f_2^2 \le 4f_4.$$

Equality on the left hand side or the right hand side at a point $p \in \mathbf{M}$ holds if and only if all principal curvatures coincide: $k_1 = k_2 = k_3 = k_4$ and p is an umbilic; in particular, equality on one side of the inequalities in (ii) *implies equality on the other, and in this case $f_k = 0$ for all k.*

Proof. (i) From the definition (6.1.a) of f_i, $i = 1, 2, 3, 4$, we have

$$\begin{aligned}f_2^2 &= [(k_1 - H)^2 + (k_2 - H)^2 + (k_3 - H)^2 + (k_4 - H)^2]^2 \\ &\ge (k_1 - H)^4 + (k_2 - H)^4 + (k_3 - H)^4 + (k_4 - H)^4 \\ &= f_4.\end{aligned}$$

Equality holds if and only if

$$(k_1 - H)^2 = (k_2 - H)^2 = (k_3 - H)^2 = (k_4 - H)^2 = 0;$$

that means $k_1 = k_2 = k_3 = k_4$.
(ii) To prove the second inequality consider the relation

$$\begin{aligned}f_2^2 &= [(k_1 - H)^2 + (k_2 - H)^2 + (k_3 - H)^2 + (k_4 - H)^2]^2 \\ &= [(k_1 - H)^2 \times 1 + (k_2 - H)^2 \times 1 + (k_3 - H)^2 \times 1 + (k_4 - H)^2 \times 1]^2 \\ &\le [(k_1 - H)^4 + (k_2 - H)^4 + (k_3 - H)^4 + (k_4 - H)^4] \times [1^2 + 1^2 + 1^2 + 1^2] \\ &= 4f_4;\end{aligned}$$

here we used the Cauchy-Schwarz inequality. Equality holds if and only if

$$(k_1 - H) = (k_2 - H) = (k_3 - H) = (k_4 - H).$$

This proves (ii). □

6.3. Corollary. *Let $x : \mathbf{M}^4 \to \bar{\mathbf{M}}^5(\bar{\kappa})$ be as above. Then*

(i) $$\int_{\mathbf{M}} \{\frac{5}{2} f_2^2 - (6\bar{\kappa} + 10H^2) f_2 + 12(\bar{\kappa} + H^2)^2 - 3f_4\}\omega(g) \geq 16\pi^2\chi(\mathbf{M}),$$

(ii) $$\int_{\mathbf{M}} \{\frac{7}{4} f_2^2 - (6\bar{\kappa} + 10H^2) f_2 + 12(\bar{\kappa} + H^2)^2\}\omega(g) - \int_{\mathbf{M}} \|\nabla\Phi\|^2\omega(g) \geq 16\pi^2\chi(\mathbf{M}) \geq$$
$$\geq \int_{\mathbf{M}} \{-\frac{1}{2} f_2^2 - (6\bar{\kappa} + 10H^2) f_2 + 12(\bar{\kappa} + H^2)^2\}\omega(g) - \int_{\mathbf{M}} \|\nabla\Phi\|^2\omega(g).$$

For a compact constant mean curvature immersion $x : \mathbf{M}^4 \to \bar{\mathbf{M}}^5(\bar{\kappa})$, the integral inequalities in (ii) relate the curvature momentum function f_2 with the topology of $\mathbf{M}$, giving lower and upper bounds for the Euler characteristic.

6.4. Corollary. *Let $x : \mathbf{M}^4 \to \bar{\mathbf{M}}^5(\bar{\kappa})$ be as in Theorem 6.1.*
(a) *Equality in the relation* (6.3.i) *holds if and only if the Weingarten operator S is parallel. Then, in particular, the principal curvatures must be constant, thus x is isoparametric.*
(b) *Equality on one side in the relation* (ii) *holds if and only if x is totally umbilical.*

6.5. Non-degenerate hypersurfaces. In this section we sketch how to get results similar to 6.1–6.4 in terms of the geometry of $(\mathbf{M}, g^*)$ on a non-degenerate hypersurface $x : \mathbf{M}^4 \to \bar{\mathbf{M}}^5(\bar{\kappa})$; see (3.4). In analogy to (6.1.a) we consider the curvature momentum functions of order m

(6.5.a) $$f_m^* := \sum_i (\frac{1}{k_i} - P_1)^m, \quad m \in \mathbb{N}.$$

Again we have $f_1^* = 0$, and for $m > 1$ the curvature functions f_m^* give a measure for the deviation of the principal radii of curvature from P_1.

6.5.1. Theorem. *Let $x : \mathbf{M}^4 \to \bar{\mathbf{M}}^5(\bar{\kappa})$ be a compact, non-degenerate hypersurface. Assume that the curvature function P_1 is constant. Then we have the following integral formula:*

$$32\pi^2\chi(\mathbf{M}) = -6\bar{\kappa} \int_{\mathbf{M}} \|\nabla^*\Phi\|^2\omega(g^*)$$
$$+ \bar{\kappa}^2 \int_{\mathbf{M}} \{5f_2^{*2} - 2f_4^* - 4f_2^*(11P_1^2 + \frac{5}{\bar{\kappa}}) - 24(P_1^2 + \frac{1}{\bar{\kappa}})^2\}\omega(g^*),$$

where $\Phi := II - P_1 g^*$.

Proof. Step 1. We use the curvature relations from (3.5) for $n = 4$. A lengthy, but straightforward calculation gives

$$\|R^*\|_*^2 = \bar{\kappa}^2\{2f_4^* + 8P_1 f_3^* - 2f^*2_2 + 4(\frac{1}{\bar{\kappa}} - P_1^2)f_2^* - 24(\frac{1}{\bar{\kappa}} + P_1^2)^2\},$$

$$\|\mathrm{Ric}^*\|_*^2 = \bar{\kappa}^2\{f_4^* - 4P_1 f_3^* - (\frac{6}{\bar{\kappa}} + 2P_1^2)f_2^* + 36(\frac{1}{\bar{\kappa}} + P_1^2)^2\},$$

$$(12\kappa^*)^2 = \bar{\kappa}^2\{f_2^{*2} - 24f_2^*(\frac{1}{\bar{\kappa}} + P_1^2) + 144(\frac{1}{\bar{\kappa}} + P_1^2)^2\},$$

where the $*$-tensor-norm $\| \quad \|_*$ is defined by the metric g^*. Thus

$$\text{(6.5.1.a)} \quad \|R^*\|_*^2 - 4\|\mathrm{Ric}^*\|_*^2 + (12\kappa^*)^2 = \\ = \bar{\kappa}^2[-24(\frac{1}{\bar{\kappa}} + P_1^2)^2 + (4\frac{1}{\bar{\kappa}} - 20P_1^2)f_2^* - f_2^{*2} + 24P_1f_3^* - 2f_4^*].$$

Step 2. $\Phi := II - P_1 g^*$ is a traceless g^*-Codazzi tensor. Then

$$\text{(6.5.1.b)} \qquad \frac{1}{2}\Delta^*\|\Phi\|_*^2 = \|\nabla^*\Phi\|_*^2 + \bar{\kappa}[4(\frac{1}{\bar{\kappa}} + P_1^2)f_2^* + 4P_1f_3^* - f_2^{*2}].$$

Step 3. Integrate (6.5.1.a), (6.5.1.b) and use the Gauss-Bonnet formula as in (6.2). □

6.5.2. Corollary. *Let* $x : \mathbf{M}^4 \to \bar{\mathbf{M}}^5(\bar{\kappa})$ *be as in* (6.5.1). *Then*

$$32\pi^2\chi(\mathbf{M}) \leq \bar{\kappa}^2 \int_{\mathbf{M}} \{5f_2^{*2} - 2f_4^* - 4(11P_1^2 + \frac{5}{\bar{\kappa}})f_2^* - 24(P_1^2 + \frac{1}{\bar{\kappa}})^2\}\omega(g^*);$$

equality holds if and only if x is isoparametric.

6.5.3. Corollary. *Let* $x : \mathbf{M}^4 \to \bar{\mathbf{M}}^5(\bar{\kappa})$ *be as in* (6.5.1). *Then we have the following integral estimates in terms of P_1 and f_2 for the Euler characteristic:*

$$\bar{\kappa}^2 \int_{\mathbf{M}} \{\frac{9}{2}f_2^{*2} - 4f_2^*(11P_1^2 + \frac{5}{\bar{\kappa}}) - 24(P_1^2 + \frac{1}{\bar{\kappa}})^2\}\omega(g^*) \geq 32\pi^2\chi(\mathbf{M}) \geq \\ \geq \bar{\kappa}^2 \int_{\mathbf{M}} \{3f_2^{*2} - 4f_2^*(11P_1^2 + \frac{5}{\bar{\kappa}}) - 24(P_1^2 + \frac{1}{\bar{\kappa}})^2\}\omega(g^*) - 6\bar{\kappa} \int_{\mathbf{M}} \|\nabla^*\Phi\|_*^2\omega(g^*).$$

Equality holds on the left or on the right hand side if and only if x is totally umbilical. In this case we have

$$4\pi^2\chi(\mathbf{M}) = -3\bar{\kappa}^2(P_1^2 + \frac{1}{\bar{\kappa}})^2 \mathrm{vol}(\mathbf{M}, g^*).$$

7. Minimal submanifolds of spheres

Consider an isometric immersion $\tilde{x} : \mathbf{M} \to \mathbb{S}^{n+p}(1)$ of a Riemannian manifold $(\mathbf{M}, g)$ and the canonical inclusion $j : \mathbb{S}^{n+p}(1) \to \mathbb{E}^{n+p+1}$ of the unit sphere into Euclidean space, $n \geq 2$, $p \geq 1$. We choose the center of the sphere as origin of the canonical coordinate system and denote by x also the position vector of the isometric immersion $x := j \circ \tilde{x}$.

The mean curvature normals ξ, $\tilde{\xi}$ of x, $\tilde{x}$ satisfy

$$\text{(7.1)} \qquad \Delta x = \xi = \tilde{\xi} + n(-x),$$

where Δ denotes the Laplacian of the metric and here is applied to the vector valued mapping x. It is a well-known result of Takahashi [51] that $\tilde{x}$ is a minimal immersion if and only if $\tilde{\xi} = 0$.

If ξ is parallel in the normal bundle, say $D\xi = 0$ (D denotes the induced connection in the normal bundle of x), then its induced second fundamental form $II(\xi)$ is a $\nabla(I)$-Codazzi tensor with constant trace, $\mathrm{trace}_g II = \langle \xi, \xi \rangle$, and

$$\Phi := II(\xi) - \frac{1}{n}\big(\mathrm{trace}_g II(\xi)\big)g$$

is traceless. If $\Phi \equiv 0$ then integration of the Weingarten equation $D\xi = 0$ and comparison with (7.1) gives $\tilde{\xi} = 0$; that means $\tilde{x}$ is a minimal immersion; see e. g. [43], section 9.

We combine the foregoing remarks with the results from section 5.

7.2. Theorem. *Let* $(\mathbf{M}, g)$ *be closed. As above, denote by* $\tilde{x}$ *an isometric immersion into a sphere such that the mean curvature vector* ξ *of* $x = j \circ \tilde{x}$ *is parallel in the normal bundle,* $D\xi = 0$.

- (i) *If* $n \geq 2$ *and the sectional curvature of* $(\mathbf{M}, g)$ *is positive then* $\tilde{x}$ *is a minimal isometric immersion.*
- (ii) *If* $n = 3$ *and the Ricci tensor is pinched as in Proposition* (5.2) *then* $\tilde{x}$ *is a minimal isometric immersion.*
- (iii) *If* $n \geq 4$, $(\mathbf{M}, g)$ *is conformally flat and the Ricci tensor is pinched as in* (5.2) *then* $\tilde{x}$ *is a minimal isometric immersion.*

Analogously we can apply Proposition 5.3 and prove that $(\mathbf{M}, g)$ has a product structure, $(\mathbf{M}, g) \cong \bigoplus_k (\mathbf{M}_k, g(k))$. Then $\Phi = \bigoplus_k \lambda_k g(k)$, where λ_k is a constant eigenvalue of Φ with multiplicity $\mu_k = \dim \mathbf{M}_k$. Moreover, $\sum \lambda_k \mu_k = 0$. This implies a decomposition of $II(\xi)$

$$II(\xi) = \bigoplus_k \Big(\lambda_k + \frac{1}{n} < \xi, \xi > \Big) g(k).$$

8. Affine hypersurfaces

There are several papers on cubic forms in affine differential geometry. Particular topics are PDE's for cubic forms and classifications of hypersurfaces. We refer e.g. to F. Dillen's survey in [27] and to [53].

In [25] we studied Codazzi properties of cubic forms; this paper initiated our study of Codazzi tensors of arbitrary order. It contains a detailed study of PDE's for cubic forms and applications of our general results to affine differential geometry.

In the following we give some further applications in centroaffine hypersurfaces theory. We recall basic notations from centroaffine hypersurfaces theory (see [48], section 6.3, or [25], [30–32], [28,29]). Let g denote the centroaffine metric of a hypersurface, C its cubic form, $\hat{T}$ its Tchebychev form and T its Tchebychev vector field. Define (compare (2.1.3.3))

$$\widetilde{C}^k_{ij} := C^k_{ij} - \frac{n}{n+2}(T_i \delta^k_j + T_j \delta^k_i + T^k g_{ij}).$$

A centroaffine Tchebychev hypersurface is defined by

$$\nabla T = \lambda \mathrm{id}, \quad \lambda \in C^\infty(\mathbf{M}).$$

One can find results on Tchebychev hypersurfaces in the papers we referred to above.

8.1. Theorem. *Let* $x : \mathbf{M}^n \to \mathbf{R}^{n+1}$ *be a compact centroaffine Tchebychev hypersurface,* $\dim \mathbf{M} \geq 3$. *If the centroaffine metric* g*, its Ricci curvature and its normed scalar curvature* κ *satisfy*

$$\mathrm{Ric} > \kappa g,$$

then x *is an ellipsoid.*

Proof. From [28], Corollary 5.1, we know that a compact centroaffine Tchebychev hypersurface is conformally flat for $n \geq 3$. Define $\Phi_{ij} := \widetilde{C}_{ijl}T^l$, then Φ_{ij} is symmetric and traceless. Since

$$\nabla_k \Phi_{ij} = \nabla_k(\widetilde{C}_{ijl}T^l) = (\nabla_k \widetilde{C}_{ijl})T^l + \widetilde{C}_{ijk}(\nabla_k T^l)$$

and x is Tchebychev, from [28], (4.2.1) and (4.1) we get that $\nabla\Phi$ is totally symmetric. Hence Φ is a traceless Codazzi tensor. From Corollary 4.2 above, if $\mathrm{Ric} > \kappa g$, we get $\Phi \equiv 0$. On the other hand, for a centroaffine Tchebychev hypersurface, we have ([28], (4.3))

$$\rho(e)\big(\hat{S}(e) - H(e)g\big) = -\frac{2n}{n+2}\Phi,$$

where the mark (e) denotes equiaffine quantities of x and $\hat{S}$ is the Weingarten form, H the mean curvature and ρ the support function. Therefore, from $\Phi = 0$, x is an equiaffine sphere. Now the Theorem of Blaschke and Deicke completes the proof of Theorem 8.1. □

8.2. Remark. Theorem 8.1 partially improves Theorem 6.2 from [25]. Realize that the pinching condition for the Ricci tensor in (8.1) is weak.

Corollary 4.7 extends a result from affine geometry to arbitrary Codazzi tensors, but the extent is restricted to non-negative sectional curvature while the original affine result on the cubic Pick form yields for any sign of the sectional curvature (compare [54]).

8.3. Proposition. *Let $x : \mathbf{M} \to \mathbb{R}^{n+1}$ be a locally strongly convex affine hypersphere with non-negative constant sectional curvature. Then x is a quadric or $\kappa \equiv 0$ and x is equiaffinely equivalent to the hypersurface $x_1 x_2 \ldots x_{n+1} = 1$.*

Proof. The equiaffine cubic form C is traceless (apolarity) and satisfies Codazzi equations (see 2.1.3.2). Apply 4.7. If $C \equiv 0$ then x is a quadric. If C is not a quadric then the Pick invariant J is a positive constant from the assumptions and $\kappa = J + H$ (see [48]), thus $\kappa \equiv 0$ from 4.7. Then it is a well-known result of A. M. Li that x is (modulo equivalences) the hypersurface $x_1 \ldots x_{n+1} = 1$ (compare [54]). □

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Received May 21, 1997