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Covering a Three-Dimensional Convex Body by Smaller Homothetic Copies

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Abstract. We show that every three-dimensional convex body C can be covered by 20 smaller positive homothetic copies. The homothety ratio here depends on C. Under the requirement that the homothety ratio is universal, such a covering is possible by 24 copies.

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Let C be a convex body in Euclidean d-space E^d. Following [4], we denote by $L(C)$ the smallest m such that C can be covered by m homothetic copies of a positive ratio smaller than 1. The well-known conjecture of Hadwiger says that $L(C) \leq 2^d$ for every convex body $C \subset E^d$ (see [5]). From the example of the parallelotope it follows that the number 2^d cannot be lessened. The problem remains open for $d \geq 3$. It is solved only for some special classes of convex bodies in E^d. The reader can find details in the surveys of results concerning Hadwiger's problem in [2] and [3].

From the d-dimensional estimate in Corollary 4 from [8] we see that $L(C) \leq 34$ for every three-dimensional convex body C. The aim of this paper is to improve this estimate up to $L(C) \leq 20$ for every convex body $C \subset E^3$. Recall that if C is centrally-symmetric, then $L(C) \leq 8$ (see [7]), and that if C is of constant width, then $L(C) \leq 6$ (see [9]).

Let l be a direction and let $\delta > 0$. Denote by $G_l(C)$ the union of all straight lines parallel to l which intersect a convex body C but have empty intersection with the interior of C. By the *δ-tube of C with respect to l* we mean the set of points of $\text{conv}(G_l(C))$ which are in the distance smaller than δ from $G_l(C)$. The symbol "conv" here means the operation of creating the convex hull. By *the part of C inside of the δ-tube of C with respect to l* we mean the set of points of C which do not belong to the δ-tube itself.

Lemma. *Let $C \subset E^d$ be a convex body. For every direction l and every $\delta > 0$ there are two smaller homothetic copies of C which cover the part of C inside of the δ-tube of C with respect to l.*

Proof. Denote by W the part of C inside of the δ-tube of C with respect to l. If $W = \emptyset$, our Lemma is obvious. Assume that $W \neq \emptyset$. Of course, W can be presented as the union of non-empty vertical segments lying in different vertical straight lines. Since W is a closed set, there is a shortest from those segments and its length η is greater than 0. We see that W is contained in the union of the interiors of the two translates C_1 and C_2 of C by $\frac{1}{3}\eta$ units in the directions l and $-l$. Since W is a closed set, it is also contained in the union of some smaller positive homothetic images C_1' and C_2' of C_1 and of C_2, respectively. Of course, C_1' and C_2' are smaller positive homothetic copies of C. □

Theorem. *Every three-dimensional convex body can be covered by* 20 *smaller homothetic copies.*

Proof. Without loss of generality, we may assume that the diameter of our body C is 1. Let a and b be points of C at the distance 1. Assume that the segment ab is vertical with a above b. Construct the horizontal planes H_a and H_b through a and b, respectively. Of course, the planes support C.

Denote by P_a and P_b the vertical projections of C onto H_a and H_b. Observe that C is contained in the cylinder $Q = \operatorname{conv}(P_a \cup P_b)$. Let K be the set of points of C whose projections on H_b belong to the relative boundary of P_b. We can present K as the union of vertical segments (including one-point segments) lying in different vertical straight lines. We take the highest point from every such vertical segment. The set of those highest points is denoted by M. We support M from the top and from the bottom by horizontal planes and we denote by ω the distance between them. We have $\omega < 1$ because otherwise the diameter of C would be greater than 1.

Put $\tau = \frac{1}{3+\omega} + \frac{1}{4}$. Since $0 \leq \omega < 1$, we have $\frac{1}{2} < \tau$. Let $\mu = \frac{5-\omega}{4}$. From $\omega < 1$, we obtain $\mu > 1$. We choose $\lambda < 1$ such that $\lambda\tau \geq \frac{1}{2}$ and $\lambda\mu \geq 1$. Take homothetic copies M_a and M_b of M with the homothety ratio τ and with homothety centers a and b, respectively. Put $D = \operatorname{conv}(M_a \cup M_b)$. Observe that $D \subset C$ and that D is the union of vertical segments of length at least $1 - \tau$.

Denote by $\mathbf{j}$ a unit vector parallel to the direction l of the segment ab, and by $\mathbf{v}$ an arbitrary vector. The union of the three translates of λD by vectors $\mathbf{v}$, $\mathbf{v} + \lambda(1-\tau)\mathbf{j}$ and $\mathbf{v} + 2\lambda(1-\tau)\mathbf{j}$ contains a right cylinder $R(\mathbf{v})$ of height $\lambda[3(1-\tau) - \omega\tau] = \lambda\frac{5-\omega}{4} = \lambda\mu$ whose base is a translate of $\lambda\tau P_b$.

Levi [10] proved that every plane convex body A can be covered by 7 homothetic copies of ratio $\frac{1}{2}$. This covering is obtained thanks to the well-known result that an affine-regular hexagon can be inscribed in A (see [1]). Homothety centers of 6 of them are in the vertices of the two triangles whose common part is the hexagon, and one in the center of the hexagon. The 6 copies cover the boundary of A together with a neighbourhood of it. One copy is in the interior of A.

We apply the above covering by 7 copies of ratio $\frac{1}{2}$ to P_b as A. Since $\lambda\tau \geq \frac{1}{2}$, we can also cover P_b by 7 homothetic copies of τP_b of ratio λ. Consider the 6 copies $A_1, \ldots, A_6$ which cover the boundary of P_b. Over every such copy A_i, we put 3 translates of λD in such a way that the base of the corresponding cylinder $R(\mathbf{v})$ is just A_i. Since the height $\lambda\mu$ of R($\mathbf{v}$) is at least 1, we see that there is a $\delta > 0$ such that the 18 translates of λD cover the part of Q which is a subset of the δ-tube of C of direction l. Since $C \subset Q$, they cover the part of C which is contained in the δ-tube of C of direction l. Since every translate of λD is a subset of a homothetic copy of C of ratio λ, we also obtain a covering by 18 smaller homothetic copies of C. Thanks to the Lemma, two additional smaller homothetic copies of C cover the part of C inside of the δ-tube of C. We see now that 20 smaller homothetic copies of C cover C. $\square$

A sketch of this proof was presented by the author in his talk during the conference "Convex Bodies" in Oberwolfach in 1984, comp. [2, p. 270], and [11, p. 389].

Since ω in the proof of the Theorem cannot be bounded from above by a constant smaller than 1, our τ and λ may be arbitrarily close to 1. Thus in the Theorem we do not establish a universal homothety ratio of our 20 homothetic copies. Of course, a fixed ratio smaller than 1 could be given for every particular convex body. The following Proposition guarantees a covering by homothetic copies with a fixed universal ratio smaller than 1. As a price, the number of the homothetic copies increases.

Proposition. *Every three-dimensional convex body can be covered by* 24 *homothetic copies of ratio* $\frac{7}{12}\sqrt{2} + \frac{1}{6} = 0.9916\ldots$

Proof. We apply the fact that every planar convex body can be covered by 4 homothetic copies of ratio $\frac{1}{2}\sqrt{2}$ (see [6]). The approach is similar to this in the proof of the Theorem. The difference is that now we put vertically 6 copies of λD in 4 columns and that we take $\tau = 6/(7+\sqrt{2})$ and $\lambda = \frac{7}{12}\sqrt{2} + \frac{1}{6} = 0.9916\ldots$ We have $\lambda\sqrt{2}\tau = \lambda[6(1-\tau) - \tau] = 1$. $\square$

Similarly we can show that *every three-dimensional convex body can be covered by* 28 *homothetic copies of ratio* $\frac{7}{8}$. We put vertically 4 copies of λD in 7 columns as in the proof of the Theorem and we take $\tau = \frac{4}{7}$ and $\lambda = \frac{7}{8}$. We have $\lambda 2\tau = \lambda[3(1-\tau) - \tau] = \lambda\frac{8}{7} = 1$.

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