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On the Area of a Polygon in the Hyperbolic Plane

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1. Introduction

The usual definition of the area of a polygon in the euclidean plane is in conformity with our intuition, since by the decomposition theorem of Farkas Bolyai [2] two polygons of the same area can be decomposed into pairwise congruent polygons.

The definition of the measure of a nonbounded polyeder in two- and in three-dimensional euclidean space [7] is simple and natural, because the notion of area has an inner characterization through decomposition-equality and completion-equality.

In this paper we define the area of any (bounded or unbounded) proper polygon in hyperbolic plane on the basis, only that the measure of the plane is $-2\pi k^2$, and we require decomposition-equality and completion-equality (this suggestion is from E. Molnár). Our definitions and many of our theorems coincide – often verbatim – with definitions and theorems already known for euclidean space or for finite polyeders. When this is the case, then we can conduct our proofs by a rethinking of the old ones.

2. Preliminary notions and theorems

Here we summarize freely of dimension some well-known definitions and properties (theorems) [3] which, however, are important for us.

A *convex polyeder* is an intersection of finitely many halfspaces.

A point-set P, which can be represented as a union of finitely many convex polyeders is called a *polyeder*.

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2.1. A polyeder can be decomposed in the sense of elementary geometry into finitely many convex polyeders P_i $(i = 1, 2, \ldots, n)$ in symbols

$$P = \cup_{i=1}^{n} P_i \quad (\mathrm{int}(P_i \cap P_j) = \emptyset,\ i \neq j). \tag{$*$}$$

Two polyeders A and B are called *decomposition-equal*, denoted by $A \sim B$, if there exist polyeders A_i, B_i $(i = 1, 2, \ldots, m)$ so that the following conditions are satisfied ($\cong$ stands for congruence):

$$A = \cup_{i=1}^{m} A_i, \quad B = \cup_{i=1}^{m} B_i, \quad A_i \cong B_i \ \ (i = 1, 2, \ldots, m)$$
$$\mathrm{int}(A_i \cap A_j) = \emptyset, \quad \mathrm{int}(B_i \cap B_j) = \emptyset \ \ (i \neq j).$$

2.2. The decomposition-equality of polyeders is an equivalence relation.

Two polyeders A and B are called *completion-equal*, denoted by $A \overset{+}{\sim} B$, if there exist polyeders C, D, such that $\mathrm{int}(A \cap C) = \emptyset$, $\mathrm{int}(B \cap D) = \emptyset$, $C \sim D$, and $A \cup C \sim B \cup D$.

2.3. The completion-equality of polyeders is an equivalence relation.

3. Polygons in the hyperbolic plane

A polyeder in the hyperbolic plane is called a *polygon*. We wish to represent a polygon as a union of simplexes. In the hyperbolic plane the *simplexes* are the proper angles and the proper triangles known from elementary geometry.

A convex angle is given by the intersection of two halfplanes bounded by two halflines (rays) having a common vertex. We suppose that the size α of the angle $W(\alpha)$ statisfies the inequalities $0 < \alpha < \pi$.

A triangle is finite (bounded) or asymptotic (unbounded) with parallel sides. Let the angles of the triangle D be α, β, γ. If the triangle $D(\alpha, \beta, \gamma)$ is finite, then these numbers are positive. If the triangle $D(\alpha, \beta, \gamma)$ is 1-, 2- or 3-asymptotic, then one, two or three of these numbers are equal to zero.

Theorem 1. *A polygon P can be decomposed into finitely many simplexes S_i in the sense of elementary geometry, in symbols*

$$P = \cup_{i=1}^{n} S_i \quad (\mathrm{int}(S_i \cap S_j) = \emptyset, i \neq j). \tag{$**$}$$

Proof. Since, by the polyeder property 2.1, a polygon is the union $(*)$ of separate (i.e. having no common inner points) convex polygons, we shall prove the theorem only for convex polygons. A convex polygon is bounded by the segments $A_h B_h$ $(h = 1, 2, \ldots, k)$, by the half lines $H_l H_l'$ $(l = 1, 2, \ldots, m)$ and by the lines $G_r G_r'$ $(r = 1, 2, \ldots, s)$, where H_1', G_r, G_r' are points at the infinity or ends (Fig. 1).

Let the point $A \in \mathrm{int}\, P$ and the set $\{Q_i \ (i = 1, 2, \ldots, n)\}$

$$= \{A_h,\ B_h \ (h = 1, 2, \ldots, k),\ H_l,\ H_l' \ (l = 1, 2, \ldots, m),\ G_r,\ G_r' \ (r = 1, 2, \ldots, s)\}.$$

We can suppose that $\angle Q_1 A Q_i < \angle Q_1 A Q_{i+1}$ $(i = 2, \ldots, n-1)$. Since the triangle $Q_i A Q_{i+1}$ $(i = 1, 2, \ldots, n-1)$ is a simplex S_i $(S_n = Q_n A Q_1)$ and the equation $\cup_{i=1}^{n} S_i = P$ $(\mathrm{int}(S_i \cap S_j) = \emptyset, i \neq j)$ holds, Theorem 1 is proved.

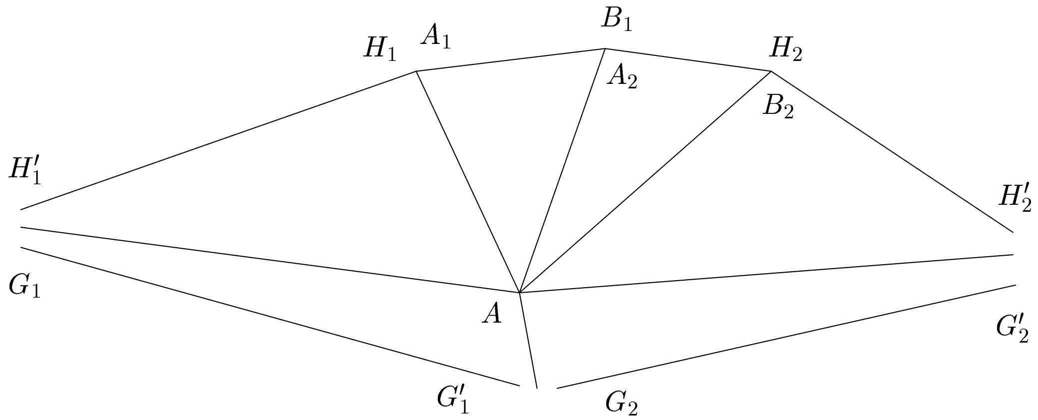

Figure 1

4. The area of the polygon in the hyperbolic plane

We define the area of a polygon as the sum of the areas of the simplexes which enter into the decomposition $(**)$ of that polygon.

If we wish to define the area $M(W(\alpha))$ of the angle $W(\alpha)$, then it is useful to choose a number which is proportional to the number α. If the area of the plane is, by our convention, $-2\pi k^2$ $(-k^2 = K$ is the sectional curvature, thus k is a natural distance unity) then let $M(W\alpha)) = -\alpha k^2$.

If we consider a finite triangle $D(\alpha,\beta,\gamma)$ and the outer angles $W(\pi-\alpha)$, $W(\pi-\beta)$, $W(\pi-\gamma)$ of this triangle (Fig. 2), then

$$D(\alpha,\beta,\gamma) \cup W(\pi-\alpha) \cup W(\pi-\beta) \cup W(\pi-\gamma) = W(2\pi).$$

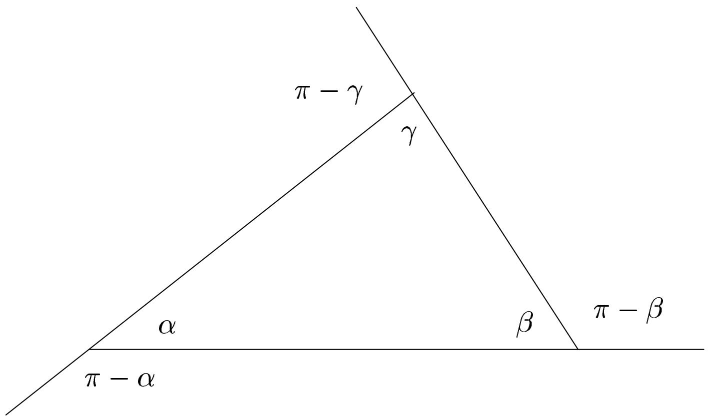

Figure 2

If we wish to make the area-function additive, then this equation implies

$$M(D(\alpha,\beta,\gamma)) = [\pi - (\alpha+\beta+\gamma)]k^2.$$

We arrive at the same result, if the triangle is 1-, 2- or 3-asymptotic. Thus the following definitions appear natural:

$$M(W(\alpha)) = -\alpha k^2, \quad M(D(\alpha,\beta,\gamma)) = [\pi - (\alpha+\beta+\gamma)]k^2,$$

$$M(P) = \textstyle\sum_{i=1}^{n} M(S_i) \ (P = \cup_{i=1}^{n} S_i, \mathrm{int}(S_i \cap S_j) = \emptyset (i \neq j),\ S_i \text{ are simplexes}).$$

These definitions can also be motivated by another method based on projective metrics [4].

5. Properties of the area-function

Theorem 2. *The area of the polygon is decomposition-invariant:*

$$P = \cup_{i=1}^{r} S_i' = \cup_{i=1}^{s} S_i'' \quad (\mathrm{int}(S_i' \cap S_j') = \emptyset, \quad \mathrm{int}(S_i'' \cap S_j'') = \emptyset, i \neq j)$$

implies $\sum_{i=1}^{r} M(S_i') = \sum_{i=1}^{s} M(S_i'')$.

Proof. Let $A_{ij} = S_i' \cap S_j'' = \cup_{k=1}^{m} A_{ijk} \quad (i,j,k : \mathrm{int}\, A_{ijk} \neq \emptyset, \quad \mathrm{int}(A_{ijk} \cap A_{ijl}) = \emptyset, k \neq l)$. $\cup A_{ijk}$ is a decomposition into simplexes of P.

Lemma. *The area of the simplex is decomposition-invariant:*

$$S = \cup_{i=1}^{n} S_i \ (\mathrm{int}(S_i \cap S_j) = \emptyset,\ i \neq j) \ \textit{implies}\ M(S) = \textstyle\sum_{i=1}^{n} M(S_i).$$

The proof of this lemma can be found for the triangle in [3].

Let the simplex S be an angle $W(\alpha)$. Since the set of the vertices of the simplexes S_i is bounded, there exists a line g, such that these vertices and the unbounded part of the angle $W(\alpha)$ are lying on different sides of the line (Fig. 3).

The line g dissects the simplexes $S_1, \dots, S_p, S_{p+1}, \dots, S_{p+q}$ $(p+q \leq n)$ which are angles for $1 \leq i \leq p$, and triangles for $p+1 \leq i \leq p+q$. If the line g dissects no angle and no triangle respectively, then we consider the simplexes $S_1, \dots, S_q$ and $S_1, \dots, S_p$. Let the parts of the simplex S_i $(i \leq p+q)$ be S_i', S_i'' where S_i' is bounded and S_i'' unbounded. In the case $1 \leq i \leq p$ we obtain, using algebraic transformations,

$$\begin{aligned} M(S_i) = -\alpha_i k^2 &= [\pi - (\alpha_i + \alpha_i' + \alpha_i'')]k^2 + [\pi - (\pi - \alpha_i') - (\pi - \alpha_i'')]k^2 \\ &= M(S_i') + [\pi - (\pi - \alpha_i') - (\pi - \alpha_i'')]k^2. \end{aligned}$$

In the case $p+1 \leq i \leq p+q$ we obtain, using algebraic transformations,

$$\begin{aligned} M(S_i) = (\pi - \beta_i - \gamma_i)k^2 &= [2\pi - (\beta_i + \gamma_i + \delta_i + \epsilon_i)]k^2 + [\pi - (\pi - \delta_i) - (\pi - \epsilon_i)]k^2 \\ &= M(S_i') + [\pi - (\pi - \delta_i) - (\pi - \epsilon_i)]k^2. \end{aligned}$$

Thus

$$\sum_{i=1}^{n} M(S_i) = \sum_{i=1}^{p} \{M(S_i') + [\pi - (\pi - \alpha_i') - (\pi - \alpha_i'')]k^2\}$$
$$+ \sum_{i=p+1}^{p+q} \{M(S_i') + [\pi - (\pi - \delta_i) - (\pi - \epsilon_i)]k^2\} + \sum_{i=p+q+1}^{n} M(S_i).$$

The line g intersects the halflines of the simplexes $S_1, \ldots, S_p, S_{p+1}, \ldots, S_{p+q}$ in $p+q+1$ points. Two of them are the points of intersection of the line g and the halflines of the angle $W(\alpha)$. Therefore

$$\sum_{i=1}^{p}[(\pi-\alpha_i')+(\pi-\alpha_i'')]+\sum_{i=p+1}^{p+q}[(\pi-\delta_i)+(\pi-\epsilon_i)]=\pi(p+q-1)+(\pi-\beta)+(\pi-\gamma),$$

and (see [3])

$$\sum_{i=1}^{p}\{M(S_i')+\sum_{i=p+1}^{p+q}M(S_i')+\sum_{i=p+q+1}^{n}M(S_i)=[\pi-(\alpha+\beta+\gamma)]k^2,$$

hence

$$\sum_{i=1}^{n}M(S_i)=[\pi-(\alpha+\beta+\gamma)]k^2+[\pi(p+q)-[\pi(p+q-1)+(\pi-\beta)+(\pi-\gamma)]]k^2$$

$$=-\alpha k^2=M(S).$$

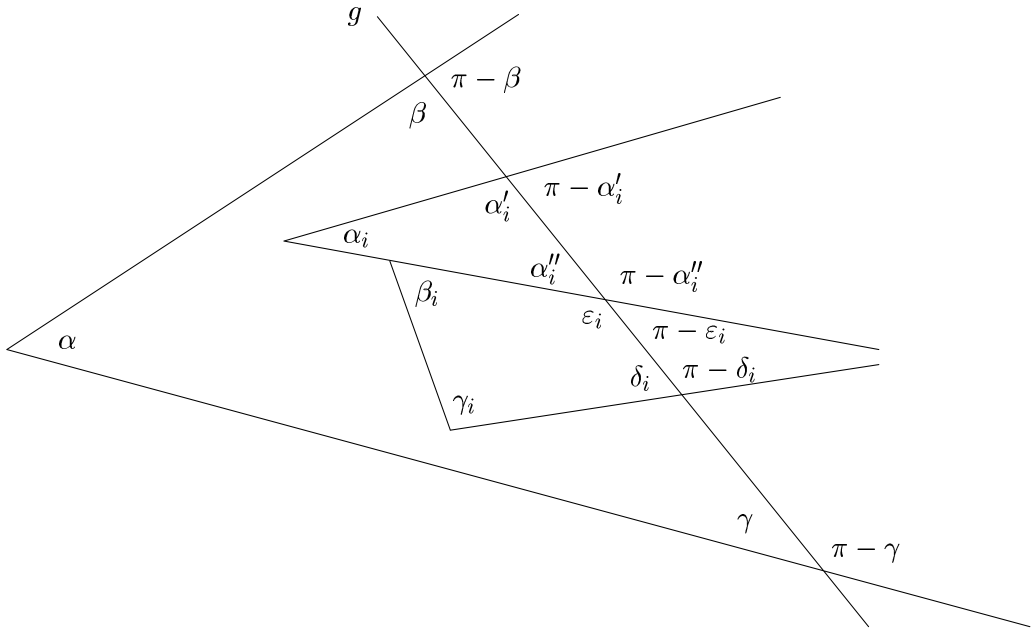

Figure 3

We made use of the fact that for a bounded polygon the theorem was valid. On the basis of the lemma we have the equations

$$\sum_{i=1}^{r}M(S_i')=\sum_{i=1}^{r}(\sum_{j,k}M(A_{ijk}))=\sum_{j=1}^{s}(\sum_{i,k}M(A_{ijk}))=\sum_{j=1}^{s}M(S_i''),$$

and so Theorem 2 is proved.

Since the area of the polygon is evidently isometry-invariant and additive, we have the following

Theorem 3. *If the polygons A, B satisfy the relation $A \sim B$ or $A \overset{+}{\sim} B$ then $M(A) = M(B)$.*

Now we are going to investigate the converse of this theorem. It is known [3] that two triangles with the same sum of angles are decomposition-equal, but either both of them must be asymptotic or both of them must be finite.

It is clear that a finite triangle and an asymptotic triangle can never be decomposition-equal, but the following theorem is important.

Theorem 4. *If a finite triangle D and an asymptotic triangle D' have the same sum of angles, then they are completion-equal: $D \overset{+}{\sim} D'$.*

Proof. We can suppose the asymptotic triangle D' to be 1-asymptotic, since for a 2-asymptotic triangle D'_2 there exists a 1-asymptotic triangle D'_1, such that the triangles D'_2 and D'_1 are decomposition-equal. If the relation $D \overset{+}{\sim} D'_1$ is already proved, then the relation $D \overset{+}{\sim} D'_2$ also holds by 2.3. A 3-asymptotic triangle and a finite triangle have never the same sum of angles.

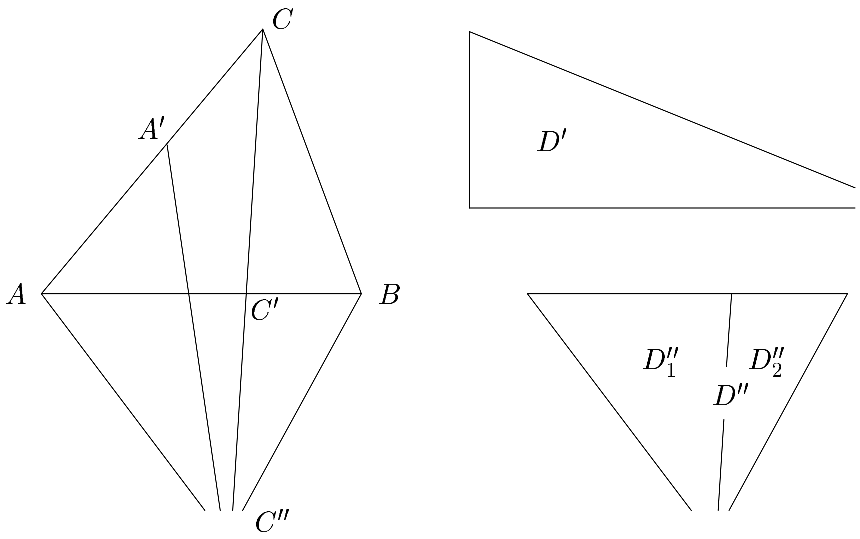

Figure 4

In our proof the completion-polygons are triangles, hence the theorems are valid in the ancient system of notions too.

Let the finite triangle D have the points A, B, C as its vertices (Fig. 4), let moreover be $C' \in \operatorname{int}(AB)$ and let C'' be the end of the halfline CC'. There exists a triangle D'' such that $\operatorname{int}(D' \cap D'') = \emptyset$, $D'' \cong \triangle ABC''$. We prove that $(\triangle ABC \cup \triangle ABC'') \sim D' \cup D''$. The completion is effected with the triangles ABC'' and D'', respectively.

We denote the sum of angles of a triangle K by $Ws(K)$.

Let $Ws(D') < Ws(\triangle ACC'')$. There exists a point A' on the segment AC, such that $Ws(\triangle AA'C'') = Ws(D')$, hence $\triangle AA'C'' \sim D'$.

In view of $Ws(D)+Ws(\triangle ABC'') = Ws(\triangle ACC'')+Ws(\triangle CBC'') = Ws(D')+Ws(D'')$ we have $Ws(\triangle CBC'') < Ws(D'')$. Now there exists a decomposition $D'' = D_1'' \cup D_2''$ of the triangle D'' into 1-asymptotic triangles D_1'' and D_2'' such that $Ws(\triangle CBC'') = Ws(D_1'')$. Hence we have the equality $Ws(\triangle A'CC'') = Ws(D_2'')$. These equalities imply the relations $\triangle CBC'' \sim D_1''$ and $\triangle A'CC'' \sim D_2''$.

The proof is similar, or simple, if we have $Ws(D') > Ws(\triangle ACC'')$ and $Ws(D') = Ws(\triangle ACC'')$, respectively.

Theorem 5. *If the polygons A, B satisfy $M(A) = M(B)$ then $A \sim B$, excepted the cases, when exactly one of the polygons A, B has either a) no angle or b) only finite triangles in its simplex representation. In these cases $A \overset{+}{\sim} B$ holds.*

Proof. We consider three simple decompositions. The existence of these decompositions can easily be seen.

Case 1. Let be given a finite triangle $D(\alpha, \beta, \gamma)$ and an angle $W(\alpha')$ such that $\alpha < \alpha'$.

Then there exists a decomposition $W = D_1 \cup D_2 \cup W_1 \cup W_2$ of the angle $W(\alpha')$ into separate simplexes, such that the part-simplex D_1 is a finite triangle with $D \cong D_1$, the part-simplex D_2 is a 1-asymptotic triangle, while the part-simplexes W_1, W_2 are angles statisfying $|M(W_2)| > \alpha k^2$ (Fig. 5).

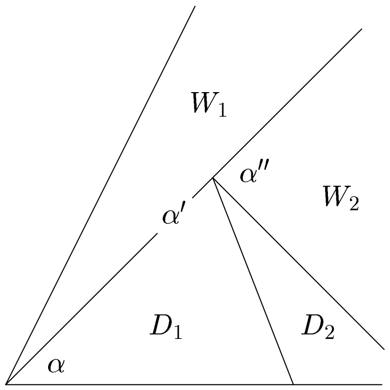

Figure 5

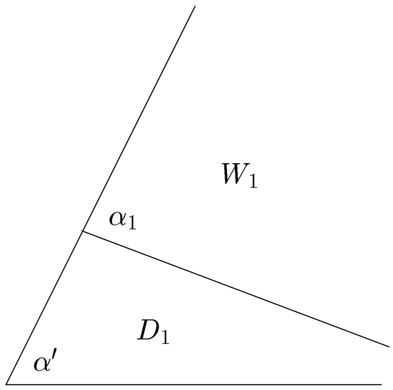

Figure 6

Case 2. Let be given a 1-asymptotic triangle $D(\alpha, \beta, \gamma = 0)$ and an angle $W(\alpha')$ such that $\alpha' < \alpha + \beta$.

Then there exists a decomposition $W = D_1 \cup W_1$ of the angle $W(\alpha')$ into separate simplexes, such that the part-simplex D_1 is a 1-asymptotic triangle with $D \sim D_1$, and the part-simplex W_1 is an angle $W_1(\alpha_1)$ with $\alpha_1 > \alpha'$ (Fig. 6).

Case 3. Let be given a (1-asymptotic or finite) triangle D and a finite triangle D' such that $M(D) > M(D')$.

Then there exists a decomposition $D = D_1 \cup D_2$ of the triangle D into separate simplexes, such than the part-simplex D_1 is a finite triangle with $D' \sim D_1$, and the part-simplex D_2 is a (1-asymptotic or finite) triangle.

Let $A = \cup_{i=1}^{h} S_i, B = \cup_{j=1}^{l} S'_j$ be the simplex-decompositions of the polygons A, B, where among the simplexes S_i, S'_j the asymptotic triangles are 1-asymptotic triangles. (2- or 3-asymptotic triangles can be decomposed into 1-asymptotic triangles.)

First we suppose that among the simplexes S_i, as well as among the simplexes S'_i, an angle can be found.

With the procedure of Case 1 we obtain that in the simplex decomposition of A resp. of B the finite triangles will be pairwise congruent. The statement $\alpha < \alpha'$ can be fulfilled by the decomposition of the finite triangle.

With the procedure of Case 2 we can obtain that in the simplex decomposition of A resp. of B, not only the finite triangles, but also the asymptotic triangles will be pairwise congruent. The statement $\alpha < \alpha + \beta$ can be fulfilled by the decomposition of the angle.

Because of $M(A) = M(B)$ we have $\sum_{i=1}^{r} M(S_i) = \sum_{j=1}^{s} M(S'_j)$, where S_i $(i = 1, 2, \ldots, r)$ and S'_j $(j = 1, 2, \ldots, s)$ are the angles in the last decomposition of A and B respectively. The relation $\cup_{i=1}^{r} S_i \sim \cup_{j=1}^{s} S'_j$ implies $A \sim B$.

Secondly, we suppose that among the simplexes S_i, S'_i there is no angle, but among the simplexes S_i, and also among the simplexes S'_j there is a 1-asymptotic one.

Let the simplexes $S_1, \ldots, S_r$ and $S'_1, \ldots, S'_s$ be the finite triangles, the simplexes $S_{r+1}, \ldots, S_h$ and $S'_{s+1}, \ldots, S'_l$, $(r < h, s < l)$ the 1-asymptotic triangles in the simplex-decomposition of A and of B, respectively.

If $\sum_{i=1}^{r} M(S_i) \le \sum_{j=1}^{s} M(S'_j)$ then we can obtain by the procedure of Case 3 that the polygon $\cup_{i=1}^{r} S_i$ and a subpolygon T' of the polygon $\cup_{j=1}^{s} S'_j$ are decomposition-equal.

Now we decompose the remainder-polygon $(\cup_{j=1}^{s} S'_j) \setminus T'$ and a subpolygon T of the polygon $\cup_{i=r+1}^{h} S_i$ by the procedure of Case 3 so that these polygons become decomposition-equal. This is possible because of

$$\begin{aligned} M((\cup_{j=1}^{s} S'_j) \setminus T') &< M((\cup_{j=1}^{l} S'_j) \setminus T') = M(B) - M(T') \\ &= M(A) - M(\cup_{i=1}^{r} S_i) = M(\cup_{i=r+1}^{h} S_i). \end{aligned}$$

Because of $M(A) = M(B)$ we have $\sum_{i=r+1}^{h} M(S''_i) = \sum_{j=s+1}^{l} M(S'_j)$, where the simplexes $S''_i (i = r+1, \ldots, h)$ are the 1-asymptotic triangles of the decomposition of the polygon $\cup_{i=r+1}^{h} S_i$. The relation $(\cup_{i=r+1}^{h} S''_i) \sim (\cup_{i=r+1}^{h} S''_i)$ implies $A \sim B$.

We can easily see that the relation $A \sim B$ is impossible in case exactly one of the polygons A, B has in its simplex-representation a) no angle, b) only finite triangles. In these cases the relation $A \overset{+}{\sim} B$ holds because of

$$M(A \cup W(\alpha)) = M(B \cup W(\alpha)) \quad (\mathrm{int}(A \cap W(\alpha) = \mathrm{int}(B \cap W(\alpha) = \emptyset),$$

where $W(\alpha)$ is an arbitrary angle.

Remark 1. The advantages of our definitions of measure are shown by the following examples, where the measurement of a domain is carried out with triangles or with angles (see e.g. [8] for the convenient formulas).

1.a. In computing the area of the n-sided regular polygon A_n, we use the triangles defined by the sides of the polygon and the center (Fig. 7).

$$M(A_n) = n(\pi - \alpha_n - \frac{2\pi}{n} k^2) = -2\pi k^2 + n(\pi - \alpha_n) k^2.$$

1.b. This area is the difference between the measure of the plane and that of exterior angle of the polygon:

$$M(A_n) = -2\pi k^2 - [-n(\pi - \alpha_n)k^2] = -2\pi k^2 + n(\pi - \alpha_n)k^2.$$

We compute the area of the circle domain as a limit:

$$\lim_{n\to\infty}[-2\pi k^2 + n(\pi - \alpha_n)k^2] = \lim_{n\to\infty}\{-2\pi k^2 + n[\pi - 2\operatorname{arcctg}(\operatorname{ch}\frac{r}{k}\operatorname{tg}\frac{\pi}{n})]k^2\} = 4\pi k^2 \operatorname{sh}^2\frac{r}{2k}.$$

(We obtained this result by using L'Hospital rule.)

2.a. The area of the domain bounded by a paracycle arc and two axes (Fig. 8), we compute as the limit of the area of the inscribed polygon P_n. $M(P_n)$ is to be computed with the help of the triangles: $n(\pi - 2\alpha_n)k^2$.

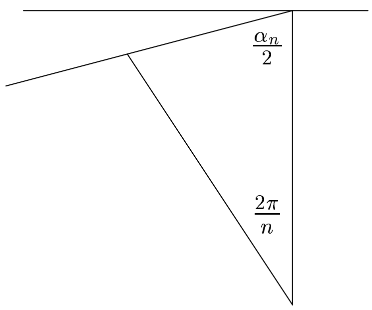

Figure 7

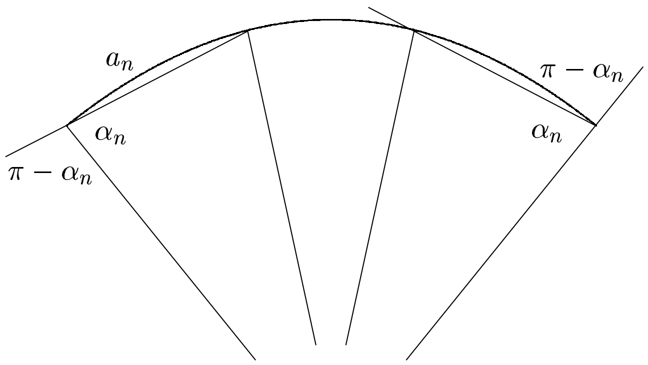

Figure 8

2.b. We compute $M(P_n)$ with the plane and with the exterior angles:

$$-\pi k^2 - [-(n-1)(\pi - 2\alpha_n - 2(\pi - \alpha_n)]k^2 = n(\pi - 2\alpha_n)k^2.$$

$$\lim_{n\to\infty} n(\pi - 2\alpha_n)k^2 = \lim_{n\to\infty} n(\pi - 4\operatorname{arctg} e^{-\frac{a_n}{k}})k^2$$

$$= \lim_{n\to\infty}[n(\pi - 4\operatorname{arctg} e^{-\frac{p}{2nk}}) + n(4\operatorname{arctg} e^{-\frac{p}{2nk}} - 4\operatorname{arctg} e^{-\frac{a_n}{2k}})]k^2 = \frac{p}{k}k^2 = pk,$$

where a_n is the finite side of a triangle and p the length of the paracycle arc. We obtained this result by relation between parallel angle and parallel distance and by L'Hospital rule again.

Remark 2. We are going to investigate the volume of the polyeder in hyperbolic 3-space in another paper. An interesting starting result is that for a tetraeder T and a halfspace H the relation $T \cup H \sim H$ holds.

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