Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 39 (1998), No. 2, 447-459.

Solution Branching of a General Perimeter Partition Problem

Anita Kripfganz
Institut für Mathematik, Universität Leipzig,
Augustusplatz 10, 04109 Leipzig, Germany
e-mail: kripfganz@mathematik.uni-leipzig.de

Abstract. In generalization of an isoperimetric partition problem m-tuples of plane figures are considered. Each of them is inscribed in a unit circular sector with the same central angle. Their total perimeter length is fixed while minimizing the total area. The appearing solution branching between symmetric and nonsymmetric perimeter partitions is investigated. It is based on a certain convexity defect of an optimal value function. The perimeter partition problem is treated as a special case of partition problems for a certain class of convex-concave functions.

MSC 1991: 52A40 (primary); 52A38, 90C90, 90C31 (secondary)
Keywords: isoperimetric problem, global optimization, parametric optimization, convexity defect

1. Introduction

Our aim is to solve an isoperimetric partition problem for m-tuples of sector indomains having least area. We investigate its solution structure in dependence on the given parameter value. This problem was studied for m-tuples of unit circle indomains by März in [5]. Because of its hierarchic structure it is based on the generalized Favard problem (see [3]) and the partition problem for pairs of sector indomains (see [4]).

Before formulating the optimization problem we summarize some basic definitions and notations. We denote by S_ω the unit circular sector with central angle $\omega \in [0, 2\pi]$, and by Δ_ω the corresponding isosceles triangle. Any closed plane set $\mathcal{B}$ with $S_\omega \cap \Delta_\omega \subseteq \mathcal{B} \subseteq S_\omega$ and convex set $\Delta_\omega \cup \mathcal{B}$ is called sector indomain of S_ω. The perimeter of the sector indomain $\mathcal{B}$ is composed by the legs of Δ_ω and a convex curve joining the free endpoints of these legs. This curve is the variable boundary part of domain $\mathcal{B}$. We denote its length by $\mathcal{L}_\omega(\mathcal{B})$, the area of $\mathcal{B}$ by $\mathcal{F}_\omega(\mathcal{B})$. Now, the isoperimetric partition problem for m ω-sector indomains $\mathcal{B}_1, \dots, \mathcal{B}_m$ can be formulated as follows:

$$\text{(P)}: \qquad \sum_{i=1}^{m} \mathcal{F}_\omega(\mathcal{B}_i) \to \text{Min!}$$

$$\sum_{i=1}^{m} \mathcal{L}(\mathcal{B}_i) = m\Lambda$$

with parameter $\Lambda \in [\underline{L}_\omega, \omega]$, $\underline{L}_\omega = 2(k + \sin\frac{\hat{\omega}}{2})$, $\hat{\omega} \in [0, \pi)$ and $\omega = k\pi + \hat{\omega}$ for k=1,2.
For $m = 1$ (generalized Favard problem) it was shown [3] that a uniquely determined almost regular inpolyeder of S_ω is area-minimal. The number n of according central angles is given by $L \in \left(l_{n-1}^\omega, l_n^\omega\right]$ with $l_1^\omega = \underline{L}_\omega$ and

$$l_n^\omega := 2n \cdot \sin\frac{\omega}{2n} \quad \text{for} \quad n \geq 2. \tag{1.1}$$

The minimal area depends parametrically on L. It is described by the piecewisely and implicitely given optimal value function (*fonction penetrante*)

$$f_\omega(L) := \frac{n-1}{2}\sin\alpha_\omega(L) + \frac{1}{2}\sin(\omega - (n-1)\alpha_\omega(L)). \tag{1.2}$$

In this, the central angle $\alpha_\omega(L)$ is given by

$$L = 2(n-1)\sin\frac{\alpha}{2} + 2\sin\frac{1}{2}(\omega - (n-1)\alpha) \tag{1.3}$$

for $L \in \left(l_{n-1}^\omega, l_n^\omega\right]$. This function f_ω is a continuous and monotonously increasing function. It is piecewise continuously differentiable on $(l_{n-1}^\omega, l_n^\omega)$ for $n \geq 2$. For $n \geq 3$ there exists a turning point ${}^1\lambda_n^\omega \in (l_{n-1}^\omega, l_n^\omega)$ such that f_ω is strictly convex on $[l_{n-1}^\omega, \lambda_n^\omega]$ and strictly concave on $[\lambda_n^\omega, l_n^\omega]$. On $[l_1^\omega, l_2^\omega]$ the function f_ω is strictly convex. The derivatives $f_\omega^{(3)}$ and $f_\omega^{(4)}$ are negative on $(l_{n-1}^\omega, l_n^\omega)$ for all $n \geq 3$.
The hierarchic structure of (P) and the solution of the generalized Favard problem lead to the equivalent perimeter partition problem

$$({}^{\mathrm{m}}\mathrm{P}^{\omega,\Lambda}): \qquad \sum_{i=1}^{m} f_\omega(L_i) \to \text{Min!}$$

$$\sum_{i=1}^{m} L_i = m \cdot \Lambda$$

$$\underline{L}_\omega \leq L_1 \leq \dots \leq L_m \leq \omega.$$

An optimal solution exists and is denoted by $(L_1^*, \dots, L_m^*)$.

For $m = 2$ (pair partition problem) a branching between symmetric ($L_1^* = L_2^*$) and certain nonsymmetric ($L_1^* = l_{n-1}^\omega$ or $L_2^* = l_n^\omega$) solutions occurs (see [4]). The number n is fixed by the condition $\Lambda \in (l_{n-1}^\omega, l_n^\omega]$. Hence,

$$L_i^* \in [l_{n-1}^\omega, l_n^\omega], \quad i = 1, \dots, m \tag{1.4}$$

is a necessary condition for an optimal perimeter partition of problem ($^{\mathrm{m}}\mathrm{P}^{\omega,\Lambda}$).

2. A general partition problem

Let a function $f : [0,1] \to \mathbf{R}^1$ be given with the following properties:

(A1) $f \in C^2(0,1)$,

(A2) there exists a unique turning point $\xi_1 \in (0,1)$ such that $f''(x) > 0$ for $x \in (0, \xi_1)$, $f''(x) < 0$ for $x \in (\xi_1, 1)$.

Under these assumptions on f we study the following partition problem:

$$({}^{\mathrm{m}}\mathrm{Q}^{\mathrm{c}}) : \qquad {}^mF({}^m\mathbf{x}) = \sum_{i=1}^m f(x_i) \to \mathrm{Min!}$$

$$\sum_{i=1}^m x_i = m \cdot c$$

$$0 \le x_1 \le \cdots \le x_m \le 1$$

for $c \in (0,1)$, ${}^m\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{R}^m$. The feasible domain $B_m(c)$ of this optimization problem is nonempty and compact. Optimal solutions exist.

2.1. Convexity defects. The solution of problem ($^{\mathrm{m}}\mathrm{Q}^{\mathrm{c}}$) is closely related to the curvature behavior of f.

Definition 1. *Let $\mu \in (0,1)$. The function*

$$^\mu\varphi_f(x) := \begin{cases} (1-\mu)f(0) + \mu f(\frac{1}{\mu}x) - f(x) & \text{for } x < \mu \\ \mu f(1) + (1-\mu) f\left(\frac{x-\mu}{1-\mu}\right) - f(x) & \text{for } x \ge \mu \end{cases}$$

is called μ-point convexity defect of f.

For a convex function all these defect functions are nonnegative.

Lemma 1. *Let $\mu \in (0,1)$ and f be a function with properties* (A1), (A2). *Then there exists a unique zero $\xi(\mu) \in (0,1)$ of $^\mu\varphi_f$ with $^\mu\varphi_f(x) > 0$ for $x \in (0, \xi(\mu))$ and*

a) $\xi(\mu) \in (\mu\xi_1, \mu)$ for $^\mu\varphi_f(\mu) \le 0$,

b) $\xi(\mu) \in (\mu, \xi_1 + \mu(1-\xi_1))$ for $^\mu\varphi_f(\mu) > 0$.

Proof. a) Let $x \in (0,\mu]$. For $x \in (0,\mu\xi_1)$ we get ${}^{\mu}\varphi'_f(x) > 0$ because $f''(x) > 0$ on $(0,\xi_1)$ implies $f'(\frac{1}{\mu}x) > f'(x)$. Hence, ${}^{\mu}\varphi_f(0) = 0$ yields ${}^{\mu}\varphi_f(x) > 0$ for $x \in (0,\mu\xi_1)$.

For $x \in (\mu\xi_1,\xi_1)$ it holds ${}^{\mu}\varphi''_f(x) < 0$ because of $f''(\frac{1}{\mu}x) < 0$ and $f''(x) > 0$. This means that ${}^{\mu}\varphi_f$ is strictly concave on $(\mu\xi_1,\xi_1)$.

Let ${}^{\mu}\varphi_f(\mu) \leq 0$. For $\xi_1 > \mu$ it follows the existence of a unique zero $\xi(\mu) \in (\mu\xi_1,\mu)$ of ${}^{\mu}\varphi_f$, immediately. For $\xi_1 \leq \mu$ we use that ${}^{\mu}\varphi'_f(x) < 0$ on (ξ_1,μ) because of $f''(x) < 0$ on $(\xi_1, 1)$.

b) Let $x \in [\mu, 1)$. We consider the reflection $g(y) = -f(1-y)$. This function has the properties (A1), (A2), too. Let $\eta_1 = 1 - \xi_1$ denote the unique turning point of g. Considering the $(1-\mu)$-proportional convexity defect of g we get ${}^{1-\mu}\varphi_g(y) = -{}^{\mu}\varphi_f(x)$ for $1 - x = y \in (0, 1-\mu]$.
Let ${}^{\mu}\varphi_f(\mu) > 0$. From ${}^{1-\mu}\varphi_g(1-\mu) = -{}^{\mu}\varphi_f(\mu) < 0$ and case a) it follows the existence of a unique zero of ${}^{1-\mu}\varphi_g$ on $((1-\mu)\eta_1, 1-\eta)$. Hence, the defect function ${}^{\mu}\varphi_f$ has a unique zero on $(\mu, \xi_1 + \mu(1-\xi_1))$. □

In the special case $\mu = \frac{1}{m}$ we denote this zero by $\xi_m = \xi(\frac{1}{m})$. For $\mu = \frac{1}{2}$, the convexity defect of this kind was introduced by J. Focke [1]. We call it *midpoint convexity defect.*

2.2. Partition problem for pairs. We consider the partition problem $({}^2\mathrm{Q}^c)$. An equivalent optimization problem is given by

$$(\mathrm{Q}): \qquad F(x) := f(x) + f(2c-x) \rightarrow \text{Min!} \qquad x \in [0,1] \cap [2c-1, 2c].$$

First, we study stationary points of the symmetric objective function F on the open interval $(0,1) \cap (2c-1, 2c)$.

Lemma 2. *Let ξ_1 be the unique turning point of $f \in C^2(0,1)$ with property* (A2). *Then*

1) *$x = c$ is a local minimum of F on $(0,1) \cap (2c-1, 2c)$ if $c < \xi_1$, and a local maximum if $c > \xi_1$,*
2) *$x = c$ is the only local minimum of F on $(0,1) \cap (2c-1, 2c)$ if $f \in C^4(0,1)$ and $f^{(3)}(x) < 0$, $f^{(4)}(x) < 0$ for all $x \in (0,1)$.*

Proof. 1) The statement follows immediately from (A2).

2) We show that $f'(x) - f'(2c-c) = 0$ implies $f''(x) + f''(2c-x) < 0$ for $x \in (0,c)$ if $f^{(3)}(x) < 0$, $f^{(4)}(x) < 0$ for $x \in (0,1)$.

Because the local maximum ξ_1 is the only stationary point of f' on (0,1) a stationary point $x \in (0,c)$ of F satisfies $x < \xi_1 < 2c - x$. It can be shown (see [4]) that the function $Q(x) := -2f'''(\xi_1)(f'(x) - f'(\xi_1)) + f''^2(x)$ is strictly monotonously increasing on $(0,1)$. This property yields the implication stated above. Hence, each stationary point $x \in (0,c)$ is a maximal point. □

Lemma 2 gives a sufficient criterion for the following additional condition on f:

(A3) For $c, x, 2c-x \in (0,1)$, $x \neq c$, relation $f'(x) = f'(2c-x)$ implies $f''(x) + f''(2c-x) < 0$.
Now, we investigate the solution behavior of problem $({}^2\mathrm{Q}^c)$ in dependence on the parameter c where f fulfills the conditions (A1), (A2) and (A3).

Lemma 3. *The zero ξ_2 of the midpoint convexity defect ${}^{\frac{1}{2}}\varphi_f$ is a branch point for the solution structure of $({}^2Q^c)$ according to*

1) ${}^2\mathbf{x} = (c, c)$ *is optimal for* $0 < c \leq \xi_2$,
2) ${}^2\mathbf{x} = (0, 2c)$ *is optimal for* $\xi_2 \leq c \leq \frac{1}{2}$,
3) ${}^2\mathbf{x} = (2c-1, 1)$ *is optimal for* $Max\{\xi_2, \frac{1}{2}\} \leq c < 1$.

It holds the sharpened inclusion $\xi_2 \in (\frac{1}{2}\xi_1, \xi_1)$.

Proof. We consider the equivalent problem (Q). For $\mu = \frac{1}{2}$, Lemma 1 implies the unique existence of ξ_2 where ${}^{\frac{1}{2}}\varphi_f(c) < 0$ for $c \in (\xi_2, 1)$. The solution structure 1), 2) and 3) results from (A3) and Lemma 2.
If ${}^{\frac{1}{2}}\varphi_f(\frac{1}{2}) \leq 0$ we get $\frac{1}{2}\xi_1 \leq \xi_2$, and if ${}^{\frac{1}{2}}\varphi_f(\frac{1}{2}) > 0$ we get $\frac{1}{2}\xi_1 < \frac{1}{2} \leq \xi_2$ from Lemma 1.
Assuming that $\xi_1 < \xi_2$. For c with $\xi_1 < c \leq \xi_2$ it follows that $x = c$ is a local maximum of F. On $(0,1) \setminus \{c\}$ a local minimum does not exist because of condition (A3). This means that F is monotonously increasing on $[0, c]$ and monotonously decreasing on $[c, 1]$. Otherwise, ${}^{\frac{1}{2}}\varphi_f(c) > 0$ implies $F(c) < F(0)$ for $c \leq \frac{1}{2}$ and $F(c) < F(1)$ for $c \geq \frac{1}{2}$, resp. This contradiction completes the proof. □

2.3. Partition problem for m-tuples. In this section, we want to study the solution branching of the partition problem $({}^m\mathrm{Q}^c)$ under the condition (A3) on f. We denoted by $B_m(c)$ the feasible domain of this problem. In the following, we use the notations

$$ {}^m\mathbf{x}^\sigma = (c, \ldots, c) \in \mathbf{R}^m $$

for the *symmetric solution*,

$$ {}^m\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbf{R}^m $$

for the i-th unit vector. An index set $I_k = \{i_1, \ldots, i_k\}$, $1 \leq i_1 < \cdots < i_k \leq n$, $k = 2, \ldots, m-1$, is denoted as *selection of size k*. Let ${}^m\mathbf{x}$ be a feasible solution of $({}^m\mathrm{Q}^c)$. Then

$$ {}^m\mathbf{x}_{I_k} = (x_{i_1}, \ldots, x_{i_k}) $$

denotes a *k-dimensional partial solution* of ${}^m\mathbf{x}$ according to the selection I_k. This partial solution is a feasible solution of problem $({}^k\mathrm{Q}^{c_{I_k}})$ for

$$ c_{I_k} = \frac{1}{k} \sum_{i \in I_k} x_i. $$

The following necessary optimality conditions result from the hierarchic structure of $({}^m\mathrm{Q}^c)$.

Lemma 4. *If ${}^m\mathbf{x}^*$ is an optimal solution of problem $({}^{\mathrm{m}}\mathrm{Q}^{\mathrm{c}})$ then the following conditions hold:*

(1*) *For each selection $I_k,\ k = 2, \dots, m-1,\ {}^m\mathbf{x}^*_{I_k}$ is an optimal solution of problem $({}^{\mathrm{k}}\mathrm{Q}^{\mathrm{c}_{\mathrm{I}_\mathrm{k}}})$ with $c_{I_k} = \frac{1}{k}\sum_{i\in I_k} x_i^*$.*

(2*) *If $x_i^*,\ x_j^* \in (0,1)$ for some $i,j \in \{1,\dots,m\},\ i<j$, then $x_i^* = x_j^* \in (0,\xi_2]$.*

(3*) *If $x_i^* = 0$ for an $i \in \{1,\dots,m\}$ then $x_j^* \notin (0, 2\xi_2)$ for all $j \in \{1,\dots,m\}$.*

(4*) *If $c < \frac{1}{m}$ and ${}^m\mathbf{x}^* \neq {}^m\mathbf{x}^\sigma$ then ${}^m\mathbf{x}* = mc \cdot {}^m\mathbf{e}_m$.*

(5*) *If $c \geq \frac{1}{m}$ and ${}^m\mathbf{x}^* \neq {}^m\mathbf{x}^\sigma$ then $x_m^* = 1$.*

Proof. Because of

$$\begin{aligned} {}^mF({}^m\mathbf{x}^*) &= \sum_{i\in I_k} f(x_i^*) + \sum_{i\notin I_k} f(x_i^*) \\ &\leq \sum_{i\in I_k} f(x_i) + \sum_{i\notin I_k} f(x_i^*) \end{aligned}$$

for all ${}^m\mathbf{x} \in B_m(c)$ with $\sum_{i\in I_k} x_i = k\cdot c_{I_k}$ and $x_i = x_i^*$ for $i \notin I_k$ we get (1*).
The partial solution ${}^m\mathbf{x}^*_{I_2} = (x_i^*, x_j^*)$ with $I_2 = (i,j)$ has to be an optimal solution of problem $({}^2\mathrm{Q}^{\mathrm{c}_{\mathrm{I}_2}})$. This follows from (1*). Lemma 3 yields (2*) and (3*).
If $c < \frac{1}{m}$ then $\sum_{i=1}^m x_i^* < 1$ and $x_i^* < 1$ for all i. Because ${}^m\mathbf{x}^* \neq {}^m\mathbf{x}^\sigma$ there exists any selection $I_2 = (r,s)$ with $x_r^* \neq x_s^*$. Hence, it follows $x_r^* = 0$ from (2*) and $x_j^* \notin (0, 2\xi_2)$ for all j from (3*). Because of condition (2*) there is no pair (x_i^*, x_j^*) with $x_i^*, x_j^* \in (0,1)$. From $x_i^* < 1$ for all i we get (4*).
If $c \geq \frac{1}{m}$ then $x_m^* = 1$. Assuming that $x_i^* < 1$ for all i then the conditions (2*) and (3*) yield $x_m^* = mc$ which is a contradiction to $x_m^* < 1$. □

To determine solutions of the partition problem we describe the subset of $B_m(c)$ satisfying the necessary optimality conditions of Lemma 4. We denote this set by $B_m^*(c)$. The hierarchic structure of $({}^{\mathrm{m}}\mathrm{Q}^{\mathrm{c}})$ suggests to apply the method of complete induction. If we get just one element then we have found the global optimal solution ${}^m\mathbf{x}^*$. In the other case we take a decision between the elements of $B_m^*(c)$ using the convexity defect of f.

Theorem 1. *Let $\{\xi_k\}$ be the sequence of zeroes defined by the $\frac{1}{k}$-point convexity defects ${}^{\frac{1}{k}}\varphi_f$ of f. Then these zeroes satisfy the recursive conditions*

$$\xi_k \in \left(\frac{k-1}{k}\xi_{k-1}, \xi_{k-1}\right) \quad \text{for } k \geq 2.$$

They define branch points $\frac{1}{m}(k\cdot\xi_k + m - k)$ for the solution structure of the partition problem $({}^{\mathrm{m}}\mathrm{Q}^{\mathrm{c}})$ in the following manner:

1. ${}^m\mathbf{x}^* = {}^m\mathbf{x}^\sigma$
   for $c \in [0, \xi_m]$,

2. ${}^m\mathbf{x}^* = (mc - j) \cdot {}^m\mathbf{e}_{m-j} + \sum_{i=m-j+1}^{m} {}^m\mathbf{e}_i$
   for $c \in \frac{1}{m}[(m-j)\xi_{m-j} + j,\ j+1],\ j \in \{0, \ldots, m-1\}$,

3. ${}^m\mathbf{x}^* = \frac{mc-j+1}{m-(j+1)} \sum_{i=1}^{m-(j+1)} {}^m\mathbf{e}_i + \sum_{i=m-j}^{m} {}^m\mathbf{e}_i$
   for $c \in \frac{1}{m}[\mathrm{Max}\{(m-j)\xi_{m-j}+j,\ j+1\},\ j+1+(m-(j+1))\xi_{m-(j+1)}],\ j \in \{0, \ldots, m-2\}$.

Proof. For $m = 2$ the assertion follows from Lemma 3. Let the assertion hold for all $k \le m-1,\ m > 2$. Then
$$\xi_{i-1} > \xi_i,\ (i-1)\xi_{i-1} < i\xi_i \tag{2.1}$$
holds for $i = 2, \ldots, m-1$.

First step: We study the set $B_m^*(c)$ by distinguishing three cases for the parameter value.

Case 1: $c \in [0, \xi_{m-1}]$.
(2.1) implies $c \in [0, \xi_k]$ for $k = 1, \ldots, m-1$. Hence, ${}^m\mathbf{x}^\sigma \in B_m^*(c)$ by induction hypothesis.

Case 1.1: $c \in [0, \frac{m-1}{m}\xi_{m-1})$.
Let ${}^m\mathbf{x} \in B_m^*(c)$, and let $I_{m-1} \subseteq \{1, \ldots, m\}$ be an arbitrary selection of size $m-1$. Then

$$c_{I_{m-1}} = \frac{1}{m-1} \sum_{i \in I_{m-1}} x_i \le \frac{1}{m-1} \sum_{i=1}^{m} x_i = \frac{m}{m-1} c < \xi_{m-1},$$

and hence we get $x_{i_1} = \cdots = x_{i_{m-1}}$ by (1^*) and induction hypothesis. This means that $B_m^*(c) = \{{}^m\mathbf{x}^\sigma\}$.

Case 1.2: $c \in [\frac{m-1}{m}\xi_{m-1},\ \xi_{m-1}]$.
Let ${}^m\hat{\mathbf{x}} \in B_m^*(c)$ with ${}^m\hat{\mathbf{x}} \ne {}^m\mathbf{x}^\sigma$.

Case 1.2.1: $c < \frac{1}{m}$.
The necessary optimality condition (4^*) implies

$$B_m^*(c) = \{{}^m\mathbf{x}^\sigma,\ {}^m\hat{\mathbf{x}}\} \text{ with } {}^m\hat{\mathbf{x}} = mc \cdot {}^m\mathbf{e}_m.$$

Case 1.2.2: $c \ge \frac{1}{m}$.
This case occurs only for $\xi_{m-1} \ge \frac{1}{m}$. The necessary optimality condition (5^*) implies $\hat{x}_m = 1$. Because of $\xi_{m-1} \in (0, 1)$ we get

$$\sum_{i=1}^{m-1} \hat{x}_i = mc - 1 \le m\xi_{m-1} - 1 < (m-1)\xi_{m-1}$$

and $c_{I_{m-1}} = \frac{mc-1}{m-1} \in [0, \xi_{m-1})$ for the selection $I_{m-1} = \{1, \dots, m-1\}$. By induction hypothesis it follows $\hat{x}_1 = \dots = \hat{x}_{m-1} = c_{I_{m-1}}$ and

$$^m\hat{\mathbf{x}} = \frac{mc-1}{m-1} \sum_{i=1}^{m-1} {}^m\mathbf{e}_i + 1^m\mathbf{e}_m.$$

On the other hand, we apply (1^*) and the induction hypothesis on ${}^m\hat{\mathbf{x}}_{I_{m-1}}$ for the selection $I_{m-1} = \{2, \dots, m\}$ with $c_{I_{m-1}} = \frac{(m-2)mc+1}{(m-1)^2}$. From

$$(m-1)c_{I_{m-1}} = \sum_{i=2}^{m} x_i < (m-2)c + 1 \leq (m-2)\xi_{m-1} + 1$$

and condition (2.1) we get

$$(m-1)c_{I_{m-1}} < (m-2)\xi_{m-2} + 1. \tag{2.2}$$

By induction hypothesis and (1^*), the structure of ${}^m\hat{\mathbf{x}}_{I_{m-1}}$ and (2.2) imply

$$c_{I_{m-1}} \in \frac{1}{m-1} \left[\mathrm{Max}\{(m-1)\xi_{m-1}, 1\}, (m-2)\xi_{m-2} + 1\right].$$

Because of (2.2) and $c \geq \frac{1}{m}$ this condition is equivalent to

$$c \geq \frac{(m-1)^2\xi_{m-1} - 1}{m(m-2)}. \tag{2.3}$$

Summarizing this case we get

$$B_m^*(c) = \begin{cases} \{{}^m\mathbf{x}^\sigma,\ {}^m\hat{\mathbf{x}}\} & \text{for } c \geq \frac{(m-1)^2\xi_{m-1}-1}{m(m-2)} \\ \{{}^m\mathbf{x}^\sigma\} & \text{else} \end{cases}$$

with ${}^m\hat{\mathbf{x}} = \frac{mc-1}{m-1} \sum_{i=1}^{m-1} {}^m\mathbf{e}_i + 1^m\mathbf{e}_m$.

Case 2: $c \in (\xi_{m-1}, \frac{1}{m}]$.
This case occurs for $\xi_{m-1} < \frac{1}{m}$, only. Let ${}^m\mathbf{x} \in B_m^*(c)$. Then ${}^m\mathbf{x} \neq {}^m\mathbf{x}^\sigma$ because the partial solution ${}^m\mathbf{x}^\sigma{}_{I_{m-1}}$ with $c_{I_{m-1}} = c > \xi_{m-1}$ does not satisfy (1^*) and the induction hypothesis. From (4^*) and (5^*) it follows $B_m^*(c) = \{mc \cdot {}^m\mathbf{e}_m\}$.

Case 3: $c \in (\mathrm{Max}\{\xi_{m-1},\ \frac{1}{m}\},\ 1]$.
For ${}^m\mathbf{x} \in B_m^*(c)$ we have ${}^m\mathbf{x} \neq {}^m\mathbf{x}^\sigma$ like in Case 2. This implies $x_m = 1$ from (5^*). We consider the selection $I_{m-1} = \{1, \dots, m-1\}$ with $c_{Im-1} = \frac{mc-1}{m-1}$. Then, the structure of ${}^m\mathbf{x}_{I_{m-1}}$ results from the induction hypothesis.

Second step: In Case 1.2 it remains to compare the objective values of the two solutions coming into question as global optimal solutions. One of these solutions is the symmetric solution ${}^m\mathbf{x}^\sigma$. A decision is taken via the sign of the $\frac{1}{m}$-point convexity defect ${}^{\frac{1}{m}}\varphi_f$ of f. Because of Lemma 1 there exists a unique zero $\xi_m \in (0,1)$ of ${}^{\frac{1}{m}}\varphi_f$ with ${}^{\frac{1}{m}}\varphi_f(c) > 0$ for $c < \xi_m$. This yields the global optimality of ${}^m\mathbf{x}^\sigma$ for parameter values $c \in [0, \xi_m]$. It remains to show the recursive condition (2.1) for m, that is $\xi_m \in (\frac{m-1}{m}\xi_{m-1}, \xi_{m-1})$. For this we verify

$$ {}^{\frac{1}{m}}\varphi_f\left(\frac{m-1}{m}\xi_{m-1}\right) > 0 \tag{2.4}$$

$$ {}^{\frac{1}{m}}\varphi_f\left(\xi_{m-1}\right) < 0 \tag{2.5}$$

Proof of (2.4):

Case I: $c = \frac{m-1}{m}\xi_{m-1} < \frac{1}{m}$.

According to Case 1.2.1 we have $B_m^*(c) = \{{}^m\mathbf{x}^\sigma, {}^m\hat{\mathbf{x}}\}$ with ${}^m\hat{\mathbf{x}} = mc \cdot {}^m\mathbf{e}_m$.
Considering the selection $I_{m-1} = \{2, \ldots, m\}$ we get the optimality of ${}^m\hat{\mathbf{x}}_{I_{m-1}}$ for problem $({}^{m-1}\mathrm{Q}^{c_{I_{m-1}}})$ with $c_{I_{m-1}} = \xi_{m-1}$ from the induction hypothesis. For this parameter value the point ${}^{m-1}\mathbf{x}^\sigma$ is optimal, too. Hence ${}^mF({}^m\hat{\mathbf{x}}) = {}^mF({}^m\tilde{\mathbf{x}})$ where ${}^m\tilde{\mathbf{x}} = \xi_{m-1}\sum_{i=2}^m {}^m\mathbf{e}_i$ and ${}^m\tilde{\mathbf{x}}_{I_{m-1}} = {}^{m-1}\mathbf{x}^\sigma$. But for the selection $I_{m-1} = \{1, \ldots, m-1\}$ the partial solution ${}^m\tilde{\mathbf{x}}_{I_{m-1}}$ is not optimal for problem $({}^{m-1}Q^{\tilde{c}_{I_{m-1}}})$ because $\tilde{c}_{I_{m-1}} < \xi_{m-1}$. Therefore, neither ${}^m\tilde{\mathbf{x}}$ nor ${}^m\hat{\mathbf{x}}$ are optimal solutions of $({}^mQ^c)$. This implies (2.4).

Case II: $c = \frac{m-1}{m}\xi_{m-1} \geq \frac{1}{m}$.

According to Case 1.2.2, $\frac{mc-1}{m-1}\sum_{i=1}^{m-1} {}^m\mathbf{e}_i + 1{}^m\mathbf{e}_m \in B_m^*(c)$ if c satisfies condition (2.3). But

$$(m-1)(m-2)\xi_{m-1} < (m-1)(m-2)\xi_{m-1} + (m-1)\xi_{m-1} - 1$$

is equivalent to

$$c = \frac{m-1}{m}\xi_{m-1} < \frac{(m-1)^2\xi_{m-1} - 1}{m(m-2)}$$

which implies $B_m^*(c) = \{{}^m\mathbf{x}^\sigma\}$. We get ${}^{\frac{1}{m}}\varphi_f(\frac{m-1}{m}\xi_{m-1}) > 0$. Relation (2.4) is shown.

Proof of (2.5):

Case III: $c = \xi_{m-1} < \frac{1}{m}$.

From Case 1.2.1 we have $B_m^*(c) = \{{}^m\mathbf{x}^\sigma, {}^m\hat{\mathbf{x}}\}$ with ${}^m\hat{\mathbf{x}} = m\xi_{m-1}{}^m\mathbf{e}_m$. Because of the induction hypothesis ${}^m\mathbf{x}^\sigma_{I_{m-1}} = {}^{m-1}\mathbf{x}^\sigma$ and ${}^{m-1}\hat{\mathbf{x}} = (m-1)\xi_{m-1}{}^{m-1}\mathbf{e}_{m-1}$ are optimal solutions of problem $({}^{m-1}\mathrm{Q}^{c_{I_{m-1}}})$ with $c_{I_{m-1}} = \xi_{m-1}$. Hence ${}^mF({}^m\mathbf{x}^\sigma) = {}^mF({}^m\tilde{\mathbf{x}})$ holds for ${}^m\tilde{x} = \xi_{m-1} \cdot {}^m\mathbf{e}_{m-1} + (m-1)\xi_{m-1} \cdot {}^m\mathbf{e}_m$. The selection $I_2 = \{m-1, m\}$ generates the partial solution ${}^m\tilde{\mathbf{x}}_{I_2}$ with $c_{I_2} = \frac{m}{2}\xi_{m-1} \in (\xi_2, \frac{1}{2})$. Lemma 3 implies that ${}^m\tilde{\mathbf{x}}_{I_2}$ is not an optimal solution for $({}^2\mathrm{Q}^{c_{I_2}})$ but ${}^2\hat{\mathbf{x}} = m\xi_{m-1}{}^2\mathbf{e}_2$. From ${}^m\hat{\mathbf{x}}_{I_2} = {}^2\hat{\mathbf{x}}$ it follows ${}^mF({}^m\mathbf{x}^\sigma) = {}^mF({}^m\tilde{\mathbf{x}}) > {}^mF({}^m\hat{\mathbf{x}})$ and therefore (2.5).

Case IV: $c = \xi_{m-1} \geq \frac{1}{m}$.
This parameter value satisfies relation (2.3) in Case 1.2.2. Hence $B_m^*(c) = \{{}^m\mathbf{x}^\sigma, {}^m\hat{\mathbf{x}}\}$ with ${}^m\hat{\mathbf{x}} = \frac{mc-1}{m-1}\sum_{i=1}^{m-1} {}^m\mathbf{e}_i + 1{}^m\mathbf{e}_m$. If $\xi_{m-1} \geq \frac{1}{m-1}$ then by induction hypothesis both ${}^{m-1}\mathbf{x}^\sigma_{I_{m-1}} = {}^{m-1}\mathbf{x}^\sigma$ and ${}^{m-1}\hat{\mathbf{x}} = \frac{(m-1)\xi_{m-1}-1}{m-2}\sum_{i=1}^{m-2} {}^{m-1}\mathbf{e}_i + 1 \cdot {}^{m-1}\mathbf{e}_{m-1}$ are optimal solutions of problem $({}^{\mathrm{m-1}}\mathrm{Q}^{\mathrm{c}_{\mathrm{I}_{m-1}}})$ with $c_{I_{m-1}} = \xi_{m-1}$.
This implies ${}^mF({}^m\mathbf{x}^\sigma) = {}^mF({}^m\tilde{\mathbf{x}})$ with

$$
{}^m\tilde{\mathbf{x}} = \frac{(m-1)\xi_{m-1}-1}{m-2}\sum_{i=1}^{m-2} {}^m\mathbf{e}_i + \xi_{m-1}\cdot {}^m\mathbf{e}_{m-1} + 1\cdot {}^m\mathbf{e}_m.
$$

Considering the selection $I_{m-1} = \{1,\ldots,m-1\}$ the partial solution ${}^m\hat{\mathbf{x}}_{I_{m-1}} = {}^{m-1}\mathbf{x}^\sigma$ with $c_{I_{m-1}} = \frac{1}{m-1}(m\xi_{m-1}-1) < \xi_{m-1}$ is the only optimal one for problem $({}^{\mathrm{m-1}}\mathrm{Q}^{\mathrm{c}_{\mathrm{I}_{m-1}}})$. Hence ${}^mF({}^m\mathbf{x}^\sigma) = {}^mF({}^m\tilde{\mathbf{x}}) > {}^mF({}^m\hat{\mathbf{x}})$ implies (2.5). If $\xi_{m-1} < \frac{1}{m-1}$ then the induction hypothesis yields the global optimality of ${}^m\mathbf{x}^\sigma_{I_{m-1}}$ and ${}^{m-1}\hat{\mathbf{x}} = (m-1)\xi_{m-1}{}^{m-1}\mathbf{e}_{m-1}$ for problem $({}^{\mathrm{m-1}}\mathrm{Q}^{\mathrm{c}_{\mathrm{I}_{m-1}}})$ with $c_{I_{m-1}} = \xi_{m-1}$. From this we get ${}^mF({}^m\mathbf{x}^\sigma) = {}^mF({}^m\tilde{\mathbf{x}})$ with ${}^m\tilde{\mathbf{x}} = \xi_{m-1}{}^m\mathbf{e}_{m-1} + (m-1)\xi_{m-1}{}^m\mathbf{e}_m$. With respect to ${}^m\tilde{\mathbf{x}}$ the selection $I_2 = \{m-1, m\}$ leads to problem $({}^2\mathrm{Q}^{\mathrm{c}_{\mathrm{I}_{m-1}}})$ with $c_{I_2} = \frac{m}{2}\xi_{m-1} \in [\mathrm{Max}\{\xi_2, \frac{1}{2}\}, 1)$. Lemma 3 gives the optimal solution ${}^2\hat{\mathbf{x}} = (m\xi_{m-1}-1)^2\mathbf{e}_1 + 1\cdot {}^2\mathbf{e}_2$ differing from ${}^m\tilde{\mathbf{x}}_{I_2}$. According to ${}^m\bar{\mathbf{x}} = (m\xi_{m-1}-1){}^m\mathbf{e}_{m-1} + 1\cdot{}^m\mathbf{e}_m$ with ${}^m\bar{\mathbf{x}}_{I_2} = {}^2\hat{\mathbf{x}}$ we consider the selection $I_{m-1} = \{1,\ldots,m-1\}$ where $c_{I_{m-1}} = \frac{1}{m-1}(m\xi_{m-1}-1) < \xi_{m-1}$. Then ${}^{m-1}\mathbf{x}^\sigma$ is the optimal solution for problem $({}^{\mathrm{m-1}}\mathrm{Q}^{\mathrm{c}_{\mathrm{I}_{m-1}}})$. Because of ${}^m\hat{\mathbf{x}}_{I_{m-1}} = {}^{m-1}\mathbf{x}^\sigma$ we finally get ${}^mF({}^m\mathbf{x}^\sigma) = {}^mF({}^m\tilde{\mathbf{x}}) > {}^mF({}^m\hat{\mathbf{x}})$. Relation (2.5) is shown.

Third step: Collecting the results we verify the solution structure for problem $({}^{\mathrm{m}}\mathrm{Q}^{\mathrm{c}})$ in dependence on the parameter value.
For $c \in [0, \xi_m]$ the global optimal solution is given by ${}^m\mathbf{x}^\sigma$. This yields point 1 of the assertion.
For $c \in [\xi_m, \frac{1}{m}]$, ${}^m\hat{\mathbf{x}} = mc\cdot{}^m\mathbf{e}_m$ is optimal which corresponds to point 2 with $j = 0$ of the assertion.
For $c \in [\mathrm{Max}\{\xi_m, \frac{1}{m}\}, \xi_{m-1}]$ the point ${}^m\hat{\mathbf{x}} = \frac{mc-1}{m-1}\sum_{i=1}^{m-1} {}^m\mathbf{e}_i + 1\cdot{}^m\mathbf{e}_m$ is optimal. This is one part of point 3, $j = 0$ of the assertion.
For $c \in [\mathrm{Max}\{\xi_{m-1}, \frac{1}{m}\}, 1]$ we consider case 3. We have found $x_m = 1$ for the optimal solution ${}^m\hat{\mathbf{x}} \neq {}^m\mathbf{x}^\sigma$. Hence, choosing the selection $I_{m-1} = \{1,\ldots,m-1\}$ the point ${}^m\hat{\mathbf{x}}_{I_{m-1}}$ with corresponding

$$
c_{I_{m-1}} = \frac{mc-1}{m-1} \in \frac{1}{m-1}\left(\mathrm{Max}\{m\xi_{m-1}-1, 0\}, m-1\right] \tag{2.6}
$$

is determined by the induction hypothesis. The zero ξ_{m-1} belongs to the interval used in (2.6), too. All different cases for $c_{I_{m-1}}$ result in conditions on the parameter c and in structural statements on the $(k-1)$-dimensional partial solution ${}^m\hat{\mathbf{x}}_{I_{m-1}}$. Especially, the case $c_{I_{m-1}} \leq \xi_{m-1}$ yields the remaining part of point 3, $j = 0$ of the assertion. This completes the proof. □

The points $\frac{1}{m}((m-j)\xi_{m-j}+j)$, $j \in \{0,\dots,m-1\}$, are branch points for the solution structure. Two different solution structures correspond to these parameter values. For the remaining parameter values the solution is uniquely determined.

3. The m-sector problem

The m-sector problem led to the perimeter partition problem $({}^{\mathrm{m}}\mathrm{P}^{\omega,\Lambda})$. For its solution we use the results of the previous section.

3.1. General solution. Let n be determined from $\Lambda \in (l^\omega_{n-1}, l^\omega_n]$. Because of (1.4) we can transform problem $({}^{\mathrm{m}}\mathrm{P}^{\omega,\Lambda})$ by

$$L(x) = l^\omega_{n-1} + x(l^\omega_n - l^\omega_{n-1}), \quad x \in [0,1] \tag{3.1}$$

into a general partition problem $({}^{\mathrm{m}}\mathrm{Q}^{\mathrm{c}})$. In this we have $f(x) = f_\omega(L(x))$, $L_i = L(x_i)$, $L(c) = \Lambda$ where condition (A3) is valid. The solution structure of the m-sector-problem follows from Theorem 1. For ${}^k\lambda^\omega_n = L(\xi_k)$ we get the recursive conditions

$${}^k\lambda^\omega_n \in \left(\frac{k-1}{k} \cdot {}^{k-1}\lambda^\omega_n + \frac{1}{k} l^\omega_{n-1}, {}^{k-1}\lambda^\omega_n\right). \tag{3.2}$$

The optimal solution ${}^m\mathbf{L}^* = (L^*_1,\dots,L^*_m)$ of problem $({}^{\mathrm{m}}\mathrm{P}^{\omega,\lambda})$, $m \ge 2$, is given via (3.1) by

1.
$${}^m\mathbf{L}^* = (\Lambda,\dots,\Lambda) \tag{3.3}$$
$$\text{for} \qquad \Lambda \in [l^\omega_{n-1}, {}^m\lambda^\omega_n], \tag{3.4}$$

2.
$${}^m\mathbf{L}^* = (\underbrace{l^\omega_{n-1},\dots,l^\omega_{n-1}}_{m-(j+1)}, m\Lambda - jl^\omega_n - (m-j-1)l^\omega_{n-1}, \underbrace{l^\omega_n,\dots,l^\omega_n}_{j}) \tag{3.5}$$

$$\text{for} \qquad \Lambda \in \frac{1}{m}[(m-j) \cdot {}^{m-j}\lambda^\omega_n + jl^\omega_n, (m-(j+1))l^\omega_{n-1} + (j+1)l^\omega_n] \tag{3.6}$$

$$\text{and for} \quad j \in \{0,\dots,m-1\},$$

3.
$${}^m\mathbf{L}^* = \left(\frac{m\Lambda - (j+1)l^\omega_n}{m-(j+1)},\dots,\frac{m\Lambda-(j+1)l^\omega_n}{m-(j+1)}, \underbrace{l^\omega_n,\dots,l^\omega_n}_{j+1}\right) \tag{3.7}$$

$$\begin{aligned} \text{for } \Lambda \in & \frac{1}{m}[\mathrm{Max}\{(m-j)\cdot {}^{m-j}\lambda^\omega_n + jl^\omega_n, (m-(j+1))l^\omega_{n-1} + (j+1)l^\omega_n\}, \\ & (j+1)l^\omega_n + (m-(j+1)) \cdot {}^{m-(j+1)}\lambda^\omega_n] \end{aligned} \tag{3.8}$$

$$\text{and for} \quad j \in \{0,\dots,m-2\}.$$

A parameter interval (3.6) could be empty. But for fixed m and large n we get the asymptotical statement

$$\frac{{}^k\lambda_n^\omega - l_{n-1}^\omega}{l_n^\omega - l_{n-1}^\omega} \le \frac{1}{k} \quad \text{for } n \ge n_0(k)$$

from the asymptotical behavior of ${}^2\lambda_n^\omega$ (see [4]) and condition (3.2). Hence, all parameter intervals are empty for sufficiently large n.

3.2. Numerical treatment. For computing the branch points ${}^k\lambda_n^\omega$, $k = 1, \dots, m$, the Newton method is used and implemented in a Pascal program (see [1]). Numerical results are demonstrated in Table 1 for circles and circular sectors.

	n	λ_n^ω	${}^2\lambda_n^\omega$	${}^3\lambda_n^\omega$
$\omega = 2\pi$	3	4,841229182759	4.781149802531	4,705506391993
	4	5,443310539518	5,418303371589	5,364902005334
	5	5,749730218646	5,744385342917	5,720909572668
	6	5,919419588012	5.917855737553	5,907410698360
	8	6,086130367576	6.085898726378	6,082953567423
	10	6,160823528643	6.160769675755	6,159653317247
	30	6,270946658417	6,270946606643	6,270935708359
$\omega = \frac{\pi}{3}$	3	1,040688511114	1.040450413358	1,040192062429
	4	1,043274926103	1.043114785701	1,042914996012
	5	1,044695465118	1.044661940328	1,044538632024
	6	1,045493875409	1.045485013601	1,045432384556
	8	1,046278280656	1.046277086069	1,046262813669
	10	1,046628166088	1.046627899577	1,046622582538
	30	1,047140858911	1,047140858072	1,047140807906

Table 1

For three circles and $n = 4$ we get four different partition qualities (see Fig. 1). The solution structure of type (3.5) with $j = 0$ does not come into question. This results from

$$\frac{2}{3}\, l_3^\omega + \frac{1}{3}\, l_4^\omega = 5,34972 < 5,36490 = {}^3\lambda_4^\omega,$$

$$\frac{1}{3}\, l_3^\omega + \frac{2}{3}\, l_4^\omega = 5,50329 > 5,49782 = \frac{2}{3}\, {}^2\lambda_4^4 + \frac{1}{3}\, l_4^\omega.$$

The other intervals are nonempty.

For three circular sectors with $n = 3$ and $\omega = \frac{\pi}{3}$ we have

$$\frac{2}{3}\, l_2^\omega + \frac{1}{3}\, l_3^\omega = 1,03748 < 1,04019 = {}^3\lambda_3^\omega,$$

$$\frac{1}{3}\, l_2^\omega + \frac{2}{3}\, l_3^\omega = 1,03968 < 1,04093 = \frac{2}{3}\, {}^2\lambda_3^\omega + \frac{1}{3}\, l_3^\omega.$$

This implies that the intervals of type (3.5) with $j = 0$ and $j = 1$ are empty. Only three different solution structures occur.

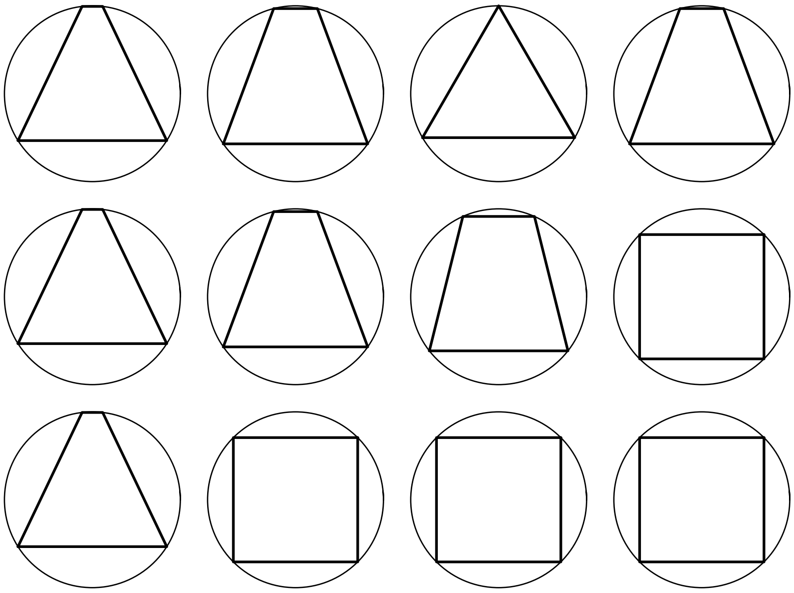

Figure 1

References

[1] Bröcker, C.: *Über ein ungeklärtes Optimierungsproblem zu ebenen konvexen Bereichen.* Diplomarbeit, Universität Leipzig 1990.

[2] Focke, J.; Klötzler, R.: *Flächenminimale Paare von Inpolygonen des Einheitskreises.* Manuskripte 1989.

[3] Kripfganz, A.: *The Generalized Favard Problem.* Beiträge Algebra Geom. **36** (1995) (2), 185–202.

[4] Kripfganz, A.: *An Isoperimetric Partition Problem.* Beiträge Algebra Geom. **38** (1997) (1), 161–181.

[5] März, Ch.: *Ein Optimierungsproblem zu Favard's 'fonction penetrante'.* Diplomarbeit, Universität Leipzig 1990.

Received January 11, 1997, revised version December 11, 1997