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On Holonomicity Criteria in Second Order Geometry

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Abstract. We introduce the intrinsic exterior differential of the contact morphism on the first jet prolongation of a fibered manifold Y. Using this concept, we deduce a holonomicity criterion for the second semi-holonomic prolongation of Y and we clarify that this is closely related to several classical procedures. The same idea is applied to the bundle of all smooth maps between the fibers over the same base point of two fibered manifolds over the same base.

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The contact forms on the first jet prolongation $\pi^1_0 : J^1Y \to Y$ of a fibered manifold $\pi : Y \to M$ represent an important tool for several branches of differential geometry and global analysis. They are generated by the components of a map $\psi_Y : TJ^1Y \to VY$, which is called the contact morphism. It is well known that a section $s : M \to J^1Y$ is holonomic, i.e. $s = j^1(\pi^1_0 s)$ if and only if $s^*\psi_Y = 0$. In the present paper, we first deduce a more subtle holonomicity criterion, in which an original concept of intrinsic exterior differential $D\psi_Y$ of ψ_Y is used. We recall that every element A of the second semi-holonomic prolongation $\overline{J}^2Y$ defines a deviation $\Delta A \in VY \otimes \Lambda^2 T^*M$, which vanishes for the elements of the second jet prolongation $J^2Y \subset \overline{J}^2Y$. Then we prove that the deviation of A coincides with the restriction of $D\psi_Y$ to the horizontal m-plane $\sigma A \subset TJ^1Y$ determined by A. In particular, this implies that A is holonomic if and only if $D\psi_Y|\sigma A = 0$.

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Next we show how this general idea is related with some more concrete procedures from the classical theories of connections and G-structures. In the last section, we develop an infinite dimensional modification of our main idea. (A physical motivation for such a research goes from [3].) We consider two fibered manifolds $Y_1 \to M$, $Y_2 \to M$ over the same base and study the functional bundle $\mathcal{F}(Y_1, Y_2)$ of all smooth maps between the fibers over the same base point. We introduce the contact morphism of $\mathcal{F}(Y_1, Y_2)$ and deduce an analogous holonomicity criterion for the second semi-holonomic prolongation of $\mathcal{F}(Y_1, Y_2)$. All manifolds and maps are assumed to be infinitely differentiable.

1. The intrinsic exterior differential

Given a vector bundle $p : E \to M$, an E-valued 1-form is a base-preserving linear morphism $\varphi : TM \to E$. If γ is a linear connection on E, the exterior differential $d_\gamma\varphi : \Lambda^2 TM \to E$ of φ with respect to γ is defined by

$$(d_\gamma\varphi)(X, Y) = \nabla_X\varphi(Y) - \nabla_Y\varphi(X) - \varphi([X, Y]) \tag{1}$$

for any vector fields X, Y on M, [9]. If $v^p = \varphi^p_i(x)\, dx^i$ is the coordinate expression of φ in some local coordinates x^i on M and some linear fiber coordinates v^p on E, then (1) yields the following coordinate expression of $d_\gamma\varphi$

$$d\varphi^p_i \wedge dx^i + \varphi^q_i dx^i \wedge \Gamma^p_{qj} dx^j \,, \tag{2}$$

where $\Gamma^p_{qi}(x)$ are the Christoffel symbols of γ.

Assume φ has constant rank, so that its kernel K_φ is a vector bundle over M. Then the restriction $D\varphi$ of $d_\gamma\varphi$ to $\Lambda^2 K_\varphi$ does not depend on γ.

Definition 1. *The map $D\varphi : \Lambda^2 K_\varphi \to E$ is called the intrinsic exterior differential of φ.*

By (2), the coordinate form of $D\varphi$ on K_φ is

$$d\varphi^p_i \wedge dx^i \,. \tag{3}$$

Let $\overline{p} : \overline{E} \to \overline{M}$ be another vector bundle, $\overline{\varphi} : T\overline{M} \to \overline{E}$ be an $\overline{E}$-valued 1-form of constant rank and $f : E \to \overline{E}$ be a linear morphism over $f_0 : M \to \overline{M}$. If φ and $\overline{\varphi}$ are f-related, i.e. $f \circ \varphi = \overline{\varphi} \circ Tf_0$, then $Tf_0(K_\varphi) \subset K_{\overline{\varphi}}$ and we denote by $K_f : K_\varphi \to K_{\overline{\varphi}}$ the restriction and corestriction of Tf_0. We have $D\overline{\varphi} : \Lambda^2 K_{\overline{\varphi}} \to \overline{E}$.

Proposition 1. *If φ and $\overline{\varphi}$ are f-related, then $D\varphi$ and $D\overline{\varphi}$ are also f-related, i.e.*

$$f \circ D\varphi = (D\overline{\varphi}) \circ \Lambda^2 K_f \,. \tag{4}$$

Proof. Let z^a be some local coordinates on $\overline{M}$ and w^s be some linear fiber coordinates on $\overline{E}$. Hence the coordinate expression of $\overline{\varphi}$ is $w^s = \overline{\varphi}^s_a(z)\, dz^a$. If $z^a = f^a(x)$, $w^s = f^s_p(x) y^p$ is the coordinate form of f, then the condition for φ and $\overline{\varphi}$ to be f-related reads

$$f^s_p(x)\varphi^p_i(x)\, dx^i = \overline{\varphi}^s_a(f(x))\, df^a \,. \tag{5}$$

The exterior differentiation of (5) yields (4). □

Remark 1. The simpliest situation, in which the intrinsic exterior differential appears, is the case of an arbitrary connection Γ on a fibered manifold $\pi : Y \to M$. Write VY for the vertical tangent bundle of Y. The connection morphism $\omega_\Gamma : TY \to VY$ is a VY-valued 1-form, [4]. Its kernel K_{ω_Γ} is identified with $Y \times_M TM$, so that $D\omega_\Gamma : Y \times_M \Lambda^2 TM \to VY$. If we rewrite the curvature $C\Gamma : Y \times_M \Lambda^2 TM \to VY$ in the form from [7], p. 158, we obtain immediately

$$D\omega_\Gamma = -C\Gamma \,, \tag{6}$$

i.e. the intrinsic exterior differential of the connection morphism coincides with the curvature up to sign.

2. The case of the contact morphism

Consider the first jet prolongation $\pi^1_0 : J^1 Y \to Y$. Every element $A \in J^1_y Y$ is identified with an m-dimensional subspace $\sigma A \subset T_y Y$, $m = \dim M$. The contact morphism $\psi_Y : TJ^1 Y \to VY$ transforms every $B \in T_A J^1 Y$ into the projection of $T\pi^1_0(B) \in T_y Y$ into $V_y Y$ in the direction of σA. Clearly, ψ_Y is a 1-form with values in the pullback bundle $\pi^{1*}_0 VY \to J^1 Y$. Write $K_Y = \operatorname{Ker} \psi_Y \subset TJ^1 Y$. Using Definition 1, we obtain $D\psi_Y : \Lambda^2 K_Y \to \pi^{1*}_0 VY$. If there is no danger of confusion, we shall simplify the notation to

$$D\psi_Y : \Lambda^2 K_Y \to VY \,.$$

If x^i, y^p, y^p_i are the standard coordinates on $J^1 Y$, then the coordinate expression of ψ_Y is

$$dy^p - y^p_i \, dx^i \,. \tag{7}$$

By (3), the coordinate form of $D\psi_Y$ on K_Y is

$$-dy^p_i \wedge dx^i \,. \tag{8}$$

We recall that $\mathcal{FM}_m$ means the category of fibered manifolds with m-dimensional bases and fiber respecting maps with local diffeomorphisms as base maps, [7]. By definition, we have $TJ^1 f(K_Y) \subset K_{\overline{Y}}$ for every $\mathcal{FM}_m$-morphism $f : Y \to \overline{Y}$. Write $K_f : K_Y \to K_{\overline{Y}}$ for the restriction and corestriction of $TJ^1 f$. By Proposition 1, we have

Proposition 2. *For every $\mathcal{FM}_m$-morphism $f : Y \to \overline{Y}$,*

$$Vf \circ D\psi_Y = (D\psi_{\overline{Y}}) \circ \Lambda^2 K_f \,. \tag{9}$$

We are going to describe a direct construction of $D\psi_Y$. We shall need the operation of strong difference, [7], p. 391. Write $p_M : TM \to M$ for the tangent bundle of a manifold M. On the iterated tangent bundle TTM, we have two projections $p_{TM} : TTM \to TM$ and $Tp_M : TTM \to TM$. If W, $Z \in TTM$ satisfy

$$p_{TM}(W) = Tp_M(Z) \,, \quad p_{TM}(Z) = Tp_M(W) \,, \tag{10}$$

they determine a well defined element $W \div Z \in TM$. In coordinates, if $W = (x^i, u^i, v^i, w^i)$, then (10) implies $Z = (x^i, v^i, u^i, z^i)$ and the coordinate expression of $W \div Z$ is

$$(x^i, w^i - z^i)\,. \tag{11}$$

Consider the lifting map $\lambda : J^1Y \times_M TM \to TY$, which transforms every (A, X) into the lift of $X \in TM$ into σA. Hence $T\lambda : TJ^1Y \times_{TM} TTM \to TTY$. For every $B \in TJ^1Y$ we denote by B_0 its projection into TM. Let $\varkappa : TTM \to TTM$ be the canonical involution.

Proposition 3. *Let $B,\ C \in (K_Y)_A$ and $D \in TTM$ be any element satisfying $p_{TM}(D) = B_0$ and $Tp_M(D) = C_0$. Then $T\lambda(B, D)$ and $T\lambda(C, \varkappa D)$ satisfy the condition* (10) *for the existence of the strong difference and*

$$D\psi_Y(B, C) = T\lambda(C, D) \div T\lambda(B, \varkappa D)\,. \tag{12}$$

Proof. We shall use the coordinate expressions $B = (x^i, y^p, y^p_i, b^i, b^p, b^p_i)$, $C = (x^i, y^p, y^p_i, c^i, c^p, c^p_i)$, $D = (x^i, b^i, c^i, d^i)$. By (8), we have

$$D\psi_Y(B, C) = -b^p_i c^i + c^p_i b^i\,. \tag{13}$$

The condition $B, C \in K_Y$ reads

$$b^p - y^p_i b^i = 0, \quad c^p - y^p_i c^i = 0\,. \tag{14}$$

Since $dx^i = dx^i$, $dy^p = y^p_i\, dx^i$ is the coordinate form of λ, we have

$$T\lambda(C, D) = (x^i, y^p, b^i, y^p_i b^i, c^i, c^p, d^i, c^p_i b^i + y^p_i d^i)\,,$$
$$T\lambda(B, \varkappa D) = (x^i, y^p, c^i, y^p_i c^i, b^i, b^p, d^i, b^p_i c^i + y^p_i d^i)\,.$$

By (14), the condition for the existence of the strong difference is fulfilled. Then (11) implies our claim. □

3. The deviation from holonomicity

The second nonholonomic prolongation $\pi^2_1 : \widetilde{J}^2Y \to J^1Y$ is defined by the iteration $\widetilde{J}^2Y = J^1(J^1Y \to M)$, [2]. Every $A \in \widetilde{J}^2Y$ determines an m-dimensional subspace $\sigma A \subset TJ^1Y$. The jet prolongation of $\pi^1_0 : J^1Y \to Y$ is another projection $J^1\pi^1_0 : \widetilde{J}^2Y \to J^1Y$. An element $A \in \widetilde{J}^2Y$ is called semi-holonomic if

$$\pi^2_1(A) = J^1\pi^1_0(A)\,. \tag{15}$$

The subset $\overline{J}^2Y \subset \widetilde{J}^2Y$ of all such elements is called the second semi-holonomic prolongation of Y. The next lemma follows directly from definitions.

Lemma 1. *$A \in \overline{J}^2Y$ if and only if $\sigma A \subset K_Y$.* □

Thus, if x^i, y^p, y^p_i are the standard coordinates on J^1Y, every $A \in \overline{J}^2 Y$ is characterized by $dy^p = y^p_i \, dx^i$ and

$$dy^p_i = y^p_{ij} \, dx^i \tag{16}$$

with arbitrary y^p_{ij}. These y^p_{ij} are the additional coordinates on $\overline{J}^2 Y$. The second holonomic prolongation J^2Y of Y is identified with the subset of all elements of $\overline{J}^2 Y$ satisfying $y^p_{ij} = y^p_{ji}$.

$J^1Y \to Y$ is an affine bundle with associated vector bundle $VY \otimes T^*M$, see e.g. [7], p. 125. By iteration, $\widetilde{J}^2 Y \to J^1 Y$ is also an affine bundle, whose associated vector bundle is $VJ^1Y \otimes T^*M$. The subset $\overline{J}^2 Y$ is an affine subbundle with associated vector bundle $VY \otimes_{J^1Y} \bigotimes^2 T^*M$. Even $J^2Y \subset \overline{J}^2 Y$ is an affine subbundle with associated vector bundle $VY \otimes_{J^1Y} S^2T^*M$.

Taking into account the direct sum decomposition $\bigotimes^2 T^*M = S^2T^*M \oplus \Lambda^2 T^*M$, we can project every $A \in \overline{J}^2 Y$ in the direction of $VY \otimes \Lambda^2 T^*M$ into an element $sA \in J^2Y$.

Definition 2. *The vector $\Delta A = A - sA \in VY \otimes \Lambda^2 T^*M$ is called the deviation of $A \in \overline{J}^2 Y$.*

Obviously, $A \in J^2Y$ if and only if $\Delta A = 0$. In coordinates, if $A = (x^i, y^p, y^p_i, y^p_{ij})$ then

$$\Delta A = (y^p_{[ij]}) \, . \tag{17}$$

where the square bracket means antisymmetrization, [12].

Consider the contact morphism ψ_Y. Clearly, for every $A \in \overline{J}^2 Y$, the restriction $D\psi_Y|\sigma A$ is identified with an element of $VY \otimes \Lambda^2 T^*M$. The following assertion, which follows directly from (8) and (17), represents the "pure idea" for some holonomicity criteria in second order geometry.

Proposition 4. $\Delta A = D\psi_Y|\sigma A$. □

In particular, $A \in J^2Y$ if and only if $D\psi_Y|\sigma A = 0$.

4. Examples

Consider first a connection $\Gamma : Y \to J^1Y$. Clearly, the composition $\psi_Y \circ T\Gamma : TY \to VY$ coincides with the connection morphism ω_Γ of Γ. On the other hand, we can construct the jet prolongation $J^1\Gamma : J^1Y \to \widetilde{J}^2 Y$. The composition $\Gamma' = J^1\Gamma \circ \Gamma$, whose values lie in $\overline{J}^2 Y$, is called the Ehresmann prolongation of Γ. Since the jet prolongation can be interpreted in terms of the tangent map, (6) and Propositions 1 and 4 yield directly a very conceptual proof of the following result, [7], [11], [13].

Corollary 1. *The curvature of Γ differs from the deviation of its Ehresmann prolongation only by sign, i.e. $C\Gamma = -\Delta \circ \Gamma'$.* □

Remark 2. Let Y be a principal fiber bundle $P(M, G)$. Since VP is canonically identified with $P \times \mathfrak{g}$, its pullback over J^1P is $J^1P \times \mathfrak{g}$, so that ψ_P is a $\mathfrak{g}$-valued 1-form on J^1Y. The intrinsic exterior differential $D\psi_P$ is defined in terms of an auxiliary connection. If we use the zero connection of the product bundle $J^1P \times \mathfrak{g} \to J^1P$, we obtain the classical exterior differential $d\psi_P$ of a vector valued 1-form. This leads to a simple holonomicity criterion for the elements of $\overline{J}^2P$, which is somewhat stronger than Proposition 1 of [5].

Remark 3. The solder form $\Theta : TP^1M \to \mathbb{R}^m$ of the first order frame bundle $\pi : P^1M \to M$ of an m-dimensional manifold M is defined analogously to the contact morphism ψ_Y. Every $u \in P^1_xM$ is interpreted as a linear isomorphism $\widetilde{u} : \mathbb{R}^m \to T^1_xM$. For every $X \in T_uP^1M$, one defines

$$\Theta(X) = \widetilde{u}^{-1}(T\pi(X)).$$

Taking into account this analogy, it is interesting to express explicitely the relation of the exterior differential $d\Theta$ of this $\mathbb{R}^m$-valued 1-form on P^1M to the deviation of certain semi-holonomic 2-jets.

Let $\overline{P}^2M$ be the second order semi-holonomic frame bundle of M, i.e. the set of all invertible semi-holonomic 2-jets from $\mathbb{R}^m$ into M with source 0. Every $U \in \overline{P}^2_xM$ determines the deviation $\Delta U \in T_xM \otimes \Lambda^2\mathbb{R}^{m*}$ analogously to Section 3. P. Libermann established an identification $i : J^1P^1M \to \overline{P}^2M$, [10], [8]. On one hand, for every $A \in J^1P^1_xM$, $d\Theta|\sigma A$ is identified with a linear map $\Lambda^2T_xM \to \mathbb{R}^m$. On the other hand, $\Delta(iA) \in T_xM \otimes \Lambda^2\mathbb{R}^{m*}$ can be interpreted as a linear map $\Lambda^2\mathbb{R}^m \to T_xM$. The evaluations from [8], pp. 278 and 281, can be summarized as follows. For every $A \in J^1_uP^1M$, $\pi(u) = x$, the following diagram commutes

(18)
$$\begin{array}{ccc} \Lambda^2\mathbb{R}^m & \xrightarrow{\Delta(iA)} & T_xM \\ {\scriptstyle \Lambda^2\widetilde{u}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{u}^{-1}} \\ \Lambda^2T_xM & \xrightarrow{d\Theta|\sigma A} & \mathbb{R}^m \end{array}$$

We remark that (18) gives an interesting approach to the structure function $\tau : Q \to H^{0,2}(\mathfrak{g})$ of a G-structure Q on M, where $H^{0,2}$ means the Spencer cohomology class of bidegree $(0,2)$. Since $Q \subset P^1M$ is a reduction to $G \subset GL(m,\mathbb{R})$, we have $J^1Q \subset J^1P^1M$. Then we can define

(19)
$$\tau(u) = \{\widetilde{u}^{-1} \circ \Delta(iA);\ A \in J^1_uQ\}$$

for every $u \in Q$, [8]. Such an approach clarifies properly that the structure function is an obstruction for the prolongability of Q, [8], [14].

5. The contact morphism in the functional case

Consider two locally trivial fibered manifolds $Y_1 \to M$, $Y_2 \to M$ over the same base. It was clarified in [1] and [6] that several ideas from the finite dimensional differential geometry can be extended to the functional bundle of all smooth fiber maps

$$\mathcal{F}(Y_1, Y_2) := \bigcup_{x\in M} C^\infty(Y_{1x}, Y_{2x}) \to M.$$

We are going to discuss the concept of contact morphism of $\mathcal{F}(Y_1, Y_2)$.

According to [1], we can construct the jet prolongations of $\mathcal{F}(Y_1, Y_2)$ as well as the tangent and vertical tangent bundles. In particular, every $A \in TJ^1\mathcal{F}(Y_1, Y_2)$ is identified with an m-dimensional subspace $\sigma A \subset T\mathcal{F}(Y_1, Y_2)$, so that we can define the contact morphism

$$\psi : TJ^1\mathcal{F}(Y_1, Y_2) \to V\mathcal{F}(Y_1, Y_2) \tag{20}$$

in the same way as in Section 2. Write K for the kernel of ψ. Even in the functional case it holds that an element $A \in \widetilde{J}^2\mathcal{F}(Y_1, Y_2)$ of the second nonholonomic prolongation is semi-holonomic if and only if $\sigma A \subset K$. Furthermore, every $A \in \overline{J}^2\mathcal{F}(Y_1, Y_2)$ defines the deviation $\Delta A \in V\mathcal{F}(Y_1, Y_2) \otimes \Lambda^2 T^*M$, [1].

On the other hand, it is not clear whether the idea of intrinsic exterior differential can be applied to (20), for the pullback bundle in question has infinite dimensional base. However, the lifting map

$$\lambda : J^1\mathcal{F}(Y_1, Y_2) \times_M TM \to T\mathcal{F}(Y_1, Y_2)$$

is defined in the same way as in Section 2, so that we can use the idea of Proposition 3. Write again B_0 for the canonical projection of $B \in K$ into TM. By [1], the tangent map

$$T\lambda : TJ^1\mathcal{F}(Y_1, Y_2) \times_{TM} TTM \to TT\mathcal{F}(Y_1, Y_2)$$

is well defined and has the following property. If B, $C \in K$ are over the same element of $J^1\mathcal{F}(Y_1, Y_2)$ and $D \in TTM$ is any element satisfying $p_{TM}(D) = B_0$ and $Tp_M(D) = C_0$, then $T\lambda(B, D)$ and $T\lambda(C, \varkappa D)$ satisfy the condition for the existence of the strong difference. Hence we can introduce formally a map Ψ as follows.

Definition 3. *For every $B, C \in K$ over the same element of $J^1\mathcal{F}(Y_1, Y_2)$, we define*

$$\Psi(B, C) = T\lambda(C, D) \div T\lambda(B, \varkappa D) \in V\mathcal{F}(Y_1, Y_2)\,. \tag{21}$$

One verifies directly that (21) is independent of the choice of D. Using the procedures from [1], one deduces easily the following analogy of Proposition 4.

Proposition 5. *For every $A \in \overline{J}^2\mathcal{F}(Y_1, Y_2)$, we have*

$$\Delta A = \Psi | \sigma A\,.$$

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