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Lagrangian Submanifolds of Quaternion Kaehlerian Manifolds Satisfying Chen's Equality

Dedicated to Professor Qi Keng Lu on the occasion of his seventieth birthday

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Abstract. Lagrangian submanifolds satisfying Chen's equality in a quaternion-space-form are investigated. In particular, we prove that (1) every Lagrangian submanifold of a quaternion-space-form satisfying Chen's equality must be minimal and (2) the only Lagrangian isometric immersion from a Riemannian n-sphere into a quaternionic projective space $P^n(H)$ satisfying Chen's equality is the totally geodesic one.

1. Preliminary

Let $\tilde{M}^m$ be a $4m$-dimensional Riemannian manifold with metric g. $\tilde{M}^m$ is called a *quaternionic Kaehlerian manifold* if there exists a 3-dimensional vector space V of tensors of type $(1,1)$ with local basis of almost Hermitian structure I, J and K such that

(a) $IJ = -JI = K,\ JK = -KJ = I,\ KI = -IK = J,\ I^2 = J^2 = K^2 = -1,$

(b) for any local cross-section φ of V, $\tilde{\nabla}_X\varphi$ is also a cross-section of V, where X is an arbitrary vector field on $\tilde{M}^m$ and $\tilde{\nabla}$ the Riemannian connection on $\tilde{M}^m$.

Condition (b) is equivalent to the following condition:

(b') there exist local 1-forms p, q and r such that

$$\begin{aligned} \tilde{\nabla}_X I &= \qquad\quad r(X)J - q(X)K, \\ \tilde{\nabla}_X J &= -\, r(X)I \qquad\quad + p(X)K, \\ \tilde{\nabla}_X K &= q(x)I - p(X)J. \end{aligned} \tag{1.1}$$

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Let X be a unit vector on $\tilde{M}^m$. Then X, IX, JX and KX form an orthonormal frame on $\tilde{M}^m$. We denote by $Q(X)$ the 4-plane spanned by them. For any two orthonormal vectors X, Y on $\tilde{M}^m$, if $Q(X)$ and $Q(Y)$ are orthogonal, the plane $\pi(X,Y)$ spanned by X, Y is called a *totally real plane*. Any 2-plane in a $Q(X)$ is called a *quaternionic plane.* The sectional curvature of a quaternionic plane π is called the *quaternionic sectional curvature* of π. A quaternionic Kaehlerian manifold is a *quaternion-space-form* if its quaternionic sectional curvatures are equal to a constant, say $4c$. We denote such a $4m$-dimensional quaternion-space-form by $\tilde{M}^m(4c)$. A *quaternionic projective space*, denoted by $P^m(H)$, is a quaternionic Kaehlerian manifold of constant quaternionic sectional curvature.

It is known that a quaternionic Kaehlerian manifold $\tilde{M}^m$ is a quaternion-space-form if and only if its curvature tensor $\tilde{R}$ is of the following form [7]:

$$\begin{aligned}\tilde{R}(X,Y)Z =&c\{g(Y,Z)X - g(X,Z)Y\\ &+ g(IY,Z)IX - g(IX,Z)IY + 2g(X,IY)IZ\\ &+ g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\\ &+ g(KY,Z)KX - g(KX,Z)KY + 2g(X,KY)KZ\}\end{aligned} \tag{1.2}$$

for vectors X, Y, Z tangent to $\tilde{M}^m$.

Let N be an n-dimensional Riemannian manifold and $x : N \to \tilde{M}^m(4c)$ be an isometric immersion of N into a quaternion-space-form $\tilde{M}^m(4c)$. We call N a *totally real submanifold* of $\tilde{M}^m(4c)$ if each 2-plane of N is mapped by x into a totally real plane in $\tilde{M}^m(4c)$. Consequently, if N is a totally real submanifold of $\tilde{M}^m(4c)$, then $\varphi(TN) \subset T^\perp N$ for $\varphi = I$, J or K, where $T^\perp N$ is the normal bundle of N in $\tilde{M}^m(4c)$.

An n-dimensional totally real submanifold of a quaternion-space-form $\tilde{M}^m(4c)$ is called a *Lagrangian submanifold* when $n = m$.

Assume that N is a totally real submanifold of $\tilde{M}^m(4c)$. For any orthonormal vectors X, Y in N, the plane $\pi(X,Y)$ spanned by X and Y is totally real in $\tilde{M}^m(4c)$, $Q(X)$ and $Q(Y)$ are orthogonal and $g(X, \varphi Y) = g(\psi X, Y) = 0$ for $\varphi, \psi = I, J$ or K.

By (1.2) we have

$$\tilde{R}(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \qquad X, Y, Z \in TN.$$

By the Gauss formula the curvature tensor R of N satisfies

$$\begin{aligned}g(R(X,Y)Z,W) =&c\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}\\ &+ g(h(Y,Z),h(X,W)) - g(h(X,Z),h(Y,W)),\end{aligned} \tag{1.3}$$

where h is the second fundamental form of N.

Since $\dim N = n$, and N is totally real in $\tilde{M}^m(4c)$, we have $n \le m$. Let $p = 4m - n$. In $\tilde{M}^m(4c)$ we can choose a local field of orthonormal frames:

$$\begin{aligned}&e_1, \ldots, e_n, e_{n+1}, \ldots, e_m; e_{I(1)} = Ie_1, \ldots, e_{I(m)} = Ie_m;\\ &e_{J(1)} = Je_1, \ldots, e_{J(m)} = Je_m; e_{K(1)} = Ke_1, \ldots, e_{K(m)} = Ke_m,\end{aligned} \tag{1.4}$$

in such a way that, restricting to N, $e_1, e_2, \dots, e_n$ are tangent to N.

We shall use the following convention in the range of indices unless mentioned otherwise:

$$\begin{aligned}
&A, B, C, D, \dots = 1, \dots, m, I(1), \dots, I(m), J(1), \dots, J(m), K(1), \dots, K(m),\\
&i, j, k, l, \dots = 1, \dots, n,\\
&r, s, t, \dots = n+1, \dots, m, I(1), \dots, K(m),\\
&u, v, \dots = n+1, \dots, m,\\
&\varphi, \psi, \dots = I, J, K.
\end{aligned}$$

By (1.3) the Ricci tensor of N is given by

$$\begin{aligned}
S(X,Y) &= \sum_i g(R(e_i, X)Y, e_i)\\
&= (n-1)cg(X,Y) + \sum_r \{(TrA_r)g(A_rX, Y) - (g(A_rX, A_rY))^2\},
\end{aligned}$$

where $A_r = A_{e_r}$, $-A_{e_r}X$ is the tangential component of $\tilde{\nabla}_X e_r$ $(= -A_rX + De_r)$, and it relates to the second fundamental form h by

$$g(h(X,Y), e_r) = g(A_rX, Y).$$

2. Chen's inequality

The *scalar curvature* τ of N is given by

$$\tau = n^2|H|^2 - |h|^2 + n(n-1)c, \tag{2.1}$$

where H denotes the mean curvature vector of N and $|h|^2$ the square length of the second fundamental form. Put

$$\eta = \tau - \frac{n^2(n-2)}{n-1}|H|^2 - (n+1)(n-2)c. \tag{2.2}$$

Substituting (2.1) in (2.2) we have $\eta = \frac{n^2}{n-1}|H|^2 - |h|^2 + 2c$, or equivalently,

$$n^2|H|^2 = (n-1)(|h|^2 + (\eta - 2c)). \tag{2.3}$$

Let

$$\omega^1, \dots, \omega^n, \omega^{n+1}, \dots, \omega^m, \omega^{I(1)}, \dots, \omega^{I(m)}, \dots,\\
\omega^{J(1)}, \dots, \omega^{J(m)}, \omega^{K(1)}, \dots, \omega^{K(m)}$$

be the dual frame of the frame given by (1.4).

Since N is totally real, $Q(e_i)$ and $Q(e_j)$, $i \neq j$, are orthogonal. Thus $g(\varphi(e_i), \psi(e_j)) = 0$ when $i \neq j$. From the structure equations:

$$d\omega^A = -\Sigma \omega^A_B \wedge \omega^B, \;\; \omega^A_B + \omega^B_A = 0$$

and (1.1), we obtain

$$\begin{aligned} &\omega^i_j = \omega^{\varphi(i)}_{\varphi(j)}, \quad \omega^{\varphi(i)}_j = \omega^{\varphi(j)}_i, \quad \omega^i_u = \omega^{\varphi(i)}_{\varphi(u)}, \\ &\omega^{\varphi(i)}_u = \omega^{\varphi(u)}_i, \quad \omega^u_v = \omega^{\varphi(u)}_{\varphi(v)}, \quad \omega^{\varphi(u)}_v = \omega^{\varphi(v)}_u. \end{aligned} \tag{2.4}$$

If we write $\omega^r_i = \sum_j h^r_{ij}\omega^j$ then $h^r_{ij} = h^r_{ji}$, $h^r_{ij} = g(A_r e_i, e_j)$ and the mean curvature vector of N is $H = \frac{1}{n}\sum_{i,r} h^r_{ii} e_r$.

From (2.4) we have

$$h^{\varphi(i)}_{jk} = h^{\varphi(j)}_{ik} = h^{\varphi(k)}_{ji}. \tag{2.5}$$

Let us choose e_{n+1} to be the unit vector in the direction H, $H = |H|e_{n+1}$, then (2.3) can be rewritten as

$$\Big(\sum h^{n+1}_{ii}\Big)^2 = (n-1)\Big\{\sum (h^{n+1}_{ii})^2 + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h^r_{ij})^2 + \eta - 2c\Big\}. \tag{2.6}$$

By (2.6) and using an algebraic lemma given in [2], B.-Y. Chen proved in [2] the following inequality:

$$2h^{n+1}_{11}h^{n+1}_{22} \geq \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{4m}\sum_{i,j=1}^{n}(h^r_{ij})^2 + \eta - 2c \tag{2.7}$$

with the equality holding if and only if

$$h^{n+1}_{11} + h^{n+1}_{22} = h^{n+1}_{33} = \cdots = h^{n+1}_{nn}. \tag{2.8}$$

Let $\pi \subset T_pN$ be a plane section. If we choose orthonormal vectors e_1, e_2 which span π, then, by (1.3), the sectional curvature $K(\pi)$ is given by

$$\begin{aligned} K(\pi) &= c + \sum_r \{g(A_r e_1, e_1)g(A_r e_2, e_2) - (g(A_r e_1, e_2))^2\} \\ &= \sum_{r=n+1}^{4m} h^r_{11}h^r_{22} - \sum_{r=n+1}^{4m} h^r_{12}h^r_{12} + c. \end{aligned}$$

From (2.7), we find

$$
\begin{aligned}
K(\pi) \geq & \frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{r=n+2}^{4m}\sum_{i,j}(h_{ij}^r)^2 + \frac{\eta}{2} + \sum_{r=n+2}^{4m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{4m} h_{12}^r h_{12}^r \\
= & \sum_{r=n+1}^{4m}\sum_{j>2}\{(h_{1j}^r)^2 + (h_{2j}^r)^2\} + \frac{1}{2}\sum_{i,j>2}^{i\neq j}(h_{ij}^r)^2 \\
& + \frac{1}{2}\sum_{r=n+2}^{4m}\sum_{i,j>2}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{4m}(h_{11}^r + h_{22}^r)^2 + \frac{\eta}{2}.
\end{aligned}
$$

Hence we obtain from (2.2) that

$$
K(\pi) \geq \frac{\eta}{2} = \frac{1}{2}\Big\{\tau - \frac{n^2(n-2)}{n-1}|H|^2 - (n+1)(n-2)c\Big\},
$$

with the equality holding if and only if

$$
\begin{aligned}
& h_{1j}^{n+1} = 0,\ h_{2j}^{n+1} = 0,\ j > 2, \\
& h_{ij}^{n+1} = 0,\ i \neq j > 2, \\
& h_{1j}^r = h_{2j}^r = h_{ij}^r = 0,\ r = n+2, \dots, 4m,\ i, j = 3, \dots, n, \\
& h_{11}^r + h_{22}^r = 0,\ r \geq n+2, \\
& h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+3} = \cdots = h_{nn}^{n+1}.
\end{aligned}
$$

Consequently, we have the following.

Proposition 1. *Let N be an n-dimensional $(n \geq 2)$ totally real submanifold of a quaternion-space-form $\tilde{M}^m(4c)$. Then*

$$
\delta \leq \frac{n^2(n-2)}{2(n-1)}|H|^2 + \frac{1}{2}(n+1)(n-2)c, \tag{2.9}
$$

where δ is the Chen invariant defined by $\delta = \frac{1}{2}\tau - \inf K$.

The equality case of (2.9) holds at a point $p \in N$ if and only if, with respect to a suitable orthonormal basis $\{e_1, \dots, e_n,\ e_{n+1}, \dots, e_m\}$ at p, the operators $A_r = A_{e_r}$ take the following forms:

$$
A_{n+1} = \begin{pmatrix} a & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & b & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \mu & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \mu \end{pmatrix},\quad a + b = \mu;
$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdot & 0 \\ h_{12}^r & -h_{11}^r & 0 & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 \end{pmatrix}, \; n+2 \leq r \leq 4m.$$

Corollary. *Let N be an n-dimensional totally real submanifold of a quaternion-space-form $\tilde{M}^m(4c)$. If the equality in (2.9) holds and ξ is a normal vector at a point p of N, then the operator A_ξ has at most 3 eigenspaces and the dimension of one of the eigenspaces is at least $n-2$. Moreover, referring to the entries in the proposition, one of the following cases must occur:*

(1) *$A_\xi = 0$, this occurs when $a = b = 0$ or $h_{11}^r = h_{12}^r = 0$ for some r where $\xi = e_{n+1}$ or e_r.*
(2) *A_ξ has a two-dimensional eigenspace with a non-zero real number α as the eigenvalue and an $(n-2)$-dimensional eigenspace with eigenvalue 2α. This occurs only when $a = b = \alpha$, $\mu = 2\alpha$, $\xi = e_{n+1}$.*
(3) *A_ξ has a one-dimensional eigenspace with eigenvalue zero and an $(n-1)$-dimensional eigenspace with nonzero eigenvalue α. This occurs when $\alpha = \mu$, $b = 0$, $\xi = e_{n+1}$.*
(4) *A_ξ has an eigenspace with eigenvalue, say α, another eigenspace with eigenvalue, say β, both of them are one-dimensional, and an $(n-2)$-dimensional eigenspace with eigenvalue $\alpha+\beta$. This occurs when $\xi = e_{n+1}, \alpha \neq \beta$ or $\xi = e_r$ for some r with $\alpha = -\beta = \pm\sqrt{(h_{11}^r)^2 + (h_{12}^r)^2}$.*

Inequality (2.9) is known as *Chen's inequality.* The submanifold N is said to satisfy *Chen's equality* if the equality case of (2.9) holds identically.

We will prove the following theorems in the next sections.

Theorem 1. *Let N be a Lagrangian submanifold of dimension > 2 in a quaternion-space-form $\tilde{M}^m(4c)$. If N satisfies Chen's equality, then N is minimal.*

There exist many non-totally geodesic Lagrangian submanifolds in quaternionic projective spaces satisfying Chen's equality. In contrast, we have the following uniqueness result for spheres.

Theorem 2. *The only Lagrangian isometric immersion from a Riemannian n-sphere into a quaternionic projective space $P^n(H)$ which satisfies Chen's equality is the totally geodesic one.*

3. Some Lemmas

Let N be a totally real submanifold of $\tilde{M}^m(4c)$, $\{\phi_r, \phi_s, \phi_t\}$ be the set $\{I, J, K\}$ or a set of a circular permutation of the three elements I, J, and K.

Lemma 1. [6] *For any X, Y, Z, U and V in TN, we have*

(a) $\tilde{R}(U, V, \phi_r X, \phi_r Y) = \tilde{R}(U, V, X, Y)$,

(b) $< h(X, Y), \phi_r Z >=< h(Z, Y), \phi_r X >, \; r = 1, 2, 3.$

The following Lemma follows immediately from Lemma 1 (b).

Lemma 2. *If N is a Lagrangian submanifold, then*

$$h(X, Y) = IA_{IX}Y + JA_{JX}Y + KA_{KX}Y.$$

Lemma 3. *For $X \in TN$ and $\xi \in I(TN) \oplus J(TN) \oplus K(TN)$, we have*

$$A_{IX}(I\xi) = A_{JX}(J\xi) = A_{KX}(K\xi) = -A_\xi X, \qquad A_{\phi_r X}(\phi_s \xi) = 0 \quad (r \neq s).$$

Proof. It is clear that $\phi_r \xi \in TN$. For any $Y \in TN$, Lemma 1 (b) yields

$$\begin{aligned} < A_{\phi_r X}(\phi_r \xi), \, Y > &=< h(\phi_r \xi, Y), \, \phi_r X >=< h(X, Y), \, \phi_r^2 \xi > \\ &= - < h(X, Y), \, \xi >= - < A_\xi X, \, Y > . \end{aligned}$$

This proved the first relation. The second relation follows from

$$\begin{aligned} &< A_{\phi_r X}(\phi_s \xi), \, Y >=< h(\phi_s \xi, Y), \, \phi_r X > \\ &\quad =< h(X, Y), \, \phi_r(\phi_s \xi) >=< h(X, Y), \, \phi_t \xi >= 0 \end{aligned}$$

by virtue of Lemma 1 (b).

4. Proof of Theorems

Let N be a Lagrangian submanifold of dimension $n > 2$ in a quaternion-space-form $\tilde{M}^m(4c)$. Assume that N satisfies Chen's equality. Let p be a given point in N and ξ a normal vector to N at p.

Suppose that A_ξ has a one-dimensional eigenspace spanned by an eigenvector v with a non-zero eigenvalue. Then by case 4 in the corollary of Proposition 1, for any normal vector η to N, $\{A_\eta v : \eta \in T^\perp N\}$ is at most two-dimensional and is invariant under any A_η. Furthermore, assume A_ξ has a one-dimensional eigenspace with an eigenvalue α, a one-dimensional eigenspace with eigenvalue β and an $(n-2)$-dimensional eigenspace with eigenvalue $\alpha+\beta$. Let the $(n-2)$-dimensional eigenspace be W and the frame $\{e_1, e_2, \ldots, e_n\}$ so chosen that $W = \{e_3, \ldots, e_n\}$.

For any $w \in W$ if $A_\xi w = \mu w$ for some μ, then $A_\eta w = 0$ for any normal vector orthogonal to ξ. Also for an normal vector η orthogonal to ξ if $A_\eta w = 0$, then $A_\xi w = \mu w$ for some μ. In any case for any $w \in W$ there is a real number μ such that $A_\eta w = \mu w$ for any normal vector to N. We sum up this as the following.

Lemma 4. *Suppose that N is a Lagrangian submanifold of dimension > 2 in $\tilde{M}^n(4c)$ satisfying Chen's equality. Let $p \in N$. Then we have*

(1) *If there exists a normal vector ξ at p such that A_ξ has a one-dimensional eigenspace with nonzero eigenvalue and a unit eigenvector v, then the vector space spanned by vectors $A_\eta v$ is at most two-dimensional, invariant under all A_η and contains v, where η is any normal vector at p.*

(2) *If A_ξ has eigenvalue α, β and $\alpha+\beta$ and W the eigenspace associated with the eigenvalue $\alpha+\beta$ with $\dim W = n-2$. Then, for any normal vector η at p, there is a real number μ such that $A_\eta w = \mu w$ for $w \in W$.*

At a given point $p \in N$ let $UN_p = \{v \in T_pN : < v, v >= 1\}$. Then UN_p is a compact set. Let ϕ be one of the I, J, K and consider the function $f : UN_p \to \mathcal{R}$ associated with ϕ defined by $u \mapsto f(u) =< h(u,u), \phi u >$. Following the same argument as in [4] we can derive the following. When f has a maximum at $v \in T_pN$, then v is an eigenvector of $A_{\phi v}$, $A_{\phi v} v = \lambda_1 v$ with eigenvalue $\lambda_1 > 0$. Choose a frame $\{e_1, e_2, \ldots, e_n\}$ of T_pN so that $e_1 = v$ and $e_i\,(i \geq 2)$ be an eigenvector of $A_{\phi e_1}$ with eigenvalue λ_i. The function $f_i(t) = f(\cos t\, e_1 + \sin t\, e_2)$ has relative maximum at $t = 0$, $f_i''(0) \leq 0$. This will lead to the inequality $\lambda_1 \geq 2\lambda_i$. Since $\lambda_1 > 0$ we have $\lambda_i \neq \lambda_1$ for all $i \geq 2$. By Lemma 1, $A_{\phi e_i} e_1 = A_{\phi e_1} e_i = \lambda_i e_i$.

According to Proposition 1 we conclude that $\lambda_1 + \lambda_2 = 0$, $\lambda_3 = \lambda_4 = \cdots = \lambda_n = 0$ for eigenvalues of $A_{\phi e_1}$ and

$$A_{\phi e_1} e_1 = \lambda_1 e_1, \;\; A_{\phi e_1} e_2 = -\lambda_1 e_2, \;\; A_{\phi e_1} e_i = 0, \;\; i > 2. \tag{4.1}$$

Since e_1 spans a one-dimensional eigenspace of $A_{\phi e_1}$ with non-zero eigenvalue λ_1 and $A_{\phi e_1} e_2 = -\lambda_1 e_2$, the space $\{e_1, e_2\}$ spanned by e_1, e_2 is invariant under A_η for an arbitrary normal vector η by Lemma 4.

For the normal vector ϕe_2, using Lemma 1 we find

$$A_{\phi e_2} e_1 = -\lambda_1 e_2, \quad A_{\phi e_2} e_2 = -\lambda_1 e_1 + \rho e_2.$$

This concludes that $A_{\phi e_2}$ takes the form of the second matrix in Proposition 1 and $\rho = 0$. Consequently, we have

$$A_{\phi e_2} e_1 = -\lambda_1 e_2, \;\; A_{\phi e_2} e_2 = -\lambda_1 e_1, \;\; A_{\phi e_2} e_i = 0, \;\; i \geq 3. \tag{4.2}$$

For $A_{\phi e_k}$, $k \geq 3$, since its eigenspace spanned by e_1, e_2 has eigenvalue 0, $A_{\phi e_k} e_1 = A_{\phi e_1} e_k = 0$, and $A_{\phi e_k} e_2 = A_{\phi e_2} e_k = 0$, we have

$$A_{\phi e_k} = 0, \quad k \geq 3. \tag{4.3}$$

From (4.1), (4.2) and (4.3), $A_{\phi e_i}$ has one of the following forms for $i = 1, \ldots, n$:

$$A_{\phi e_1} = \begin{pmatrix} \lambda_1 & 0 & \cdot & 0 \\ 0 & -\lambda_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix}, \quad A_{\phi e_2} = \begin{pmatrix} 0 & -\lambda_1 & \cdot & 0 \\ -\lambda_1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix}, \quad A_{\phi e_i} = 0 \;\; i \geq 3. \tag{4.4}$$

Similarly, for $\varphi \in \{I, J, K\}$ other than ϕ, there also exists a frame $\{\overline{e}_1, \overline{e}_2, \ldots, \overline{e}_n\}$ on TN such that with respect to this frame we have

$$A_{\varphi\overline{e}_1} = \begin{pmatrix} \mu_1 & 0 & \cdot & 0 \\ 0 & -\mu_1 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix}, \quad A_{\varphi\overline{e}_2} = \begin{pmatrix} 0 & -\mu_1 & \cdot & 0 \\ -\mu_1 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 \end{pmatrix}, \quad A_{\varphi\overline{e}_i} = 0,\ i \geq 3,$$

for some μ_1. Therefore, the submanifold N is minimal in $\tilde{M}^n(4c)$. This proves Theorem 1.

Theorem 2 follows immediately from Theorem 1 and Theorem 9 of [6].

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