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Weierstrass Polynomials for Links

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Abstract. There is a natural way of identifying links in 3-space with polynomial covering spaces over the circle. Thereby any link in 3-space can be defined by a Weierstrass polynomial over the circle. The equivalence relation for covering spaces over the circle is, however, completely different from that for links in 3-space. This paper initiates a study of the connections between polynomial covering spaces over the circle and links in 3-space.

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There is a natural way of identifying links in 3-space with polynomial covering spaces over the circle which goes as follows. According to a well known theorem of Alexander any link L in 3-space can be constructed by closing a braid around an axis. Choose a circle orthogonal to the axis for the closed braid and with its centre on the axis. Now project the closed braid onto this circle by an orthogonal projection onto the plane of the circle followed by a radial projection from the centre of the circle. By this projection the closed braid, and thereby also the link L, gets structure as a finite covering space over the circle with monodromy (characteristic homomorphism) defined by the braid; hence it is a polynomial covering space [5]. In particular, any link L in 3-space can be defined by a Weierstrass polynomial over the circle.

The equivalence relation for covering spaces over the circle is, however, completely different from that for links. For example, the polynomial covering map $\pi : E \to S^1$ corresponding to the Weierstrass polynomial $P(x,z) = z^2 - x$, $x \in S^1$, $z \in \mathbb{C}$, is the nontrivial 2-fold covering map onto the circle while E represents the trivial knot. On the other hand, the polynomial covering map $\pi : E \to S^1$ corresponding to the Weierstrass polynomial $P(x,z) = z^2 - x^2$, $x \in S^1$, $z \in \mathbb{C}$, is equivalent to the trivial 2-fold covering map onto the circle while E represents a nontrivial 2-link (the Hopf link).

In this paper we shall study the connections between polynomial covering spaces over the circle and links in 3-space. In particular, we shall exhibit Weierstrass polynomials for some of the well known knots and links in 3-space.

1. Weierstrass polynomials and polynomial covering maps

Throughout the paper, X denotes a connected topological space with the homotopy type of a CW-complex. In most cases, X will be the unit circle S^1 in the complex plane $\mathbb{C}$.

Recall that a *Weierstrass polynomial* of degree $n \geq 1$ over X is a continuous family of monic, complex polynomials of degree n, or more precisely, a polynomial map $P = P(x,z) : X \times \mathbb{C} \to \mathbb{C}$ of the form

$$\begin{aligned} P(x,z) &= z^n + \sum_{i=1}^{n} a_i(x) z^{n-i} \\ &= z^n + a_1(x) z^{n-1} + \cdots + a_{n-1}(x) z + a_n(x), \end{aligned}$$

where $a_1, \ldots, a_n : X \to \mathbb{C}$ are continuous functions. It is called *separable*, or *simple*, if $P(x,z)$ has no multiple roots for any $x \in X$.

Suppose $P(x,z)$ is a separable Weierstrass polynomial of degree n over X. Then the projection map onto X of the zero set

$$E = \{(x,z) \in X \times \mathbb{C} \mid P(x,z) = 0\}$$

for $P(x,z)$, defines an n-fold covering map $\pi : E \to X$, which is called the *polynomial covering map* associated with $P(x,z)$ [4], [5], [7].

For the description of the polynomial covering map associated with a separable Weierstrass polynomial $P(x,z)$ over X, the braiding along X of the roots in $P(x,z)$ is important [5], [7]. Let $B(n)$ denote the Artin braid group on n strings and $H(n)$ the subgroup of pure braids, i.e. braids with no permutation of strings allowed. Following [3], the pure braid group $H(n)$ can be defined as the fundamental group of the configuration space $F_n(\mathbb{C})$ of configurations of ordered n-tuples of pairwise different points in the complex plane. Correspondingly, the braid group $B(n)$ is the fundamental group of the configuration space $C_n(\mathbb{C})$ constructed from $F_n(\mathbb{C})$ by identifying n-tuples which agree up to permutation, i.e. $C_n(\mathbb{C})$ is the orbit space for the action of the symmetric group on n letters Σ_n on $F_n(\mathbb{C})$ by permutation of coordinates. This construction exhibits $F_n(\mathbb{C})$ as a covering space of $C_n(\mathbb{C})$, and therefore also the pure braid group $H(n)$ as a subgroup of the braid group $B(n)$. The boundary homomorphism in the homotopy sequence for this covering defines the epimorphism $\tau_n : B(n) \to \Sigma_n$, which to an n-braid associates the permutation of the braid.

The polynomial covering map $\pi : E \to X$ associated with the separable Weierstrass polynomial $P(x,z)$ is completely characterized by the *root map* $\alpha : X \to C_n(\mathbb{C})$, which to every $x \in X$ associates the configuration defined by the n complex roots for the complex polynomial $P(x,z)$ corresponding to $x \in X$. The root map induces a homomorphism $\alpha_* : \pi_1(X) \to B(n)$ of the fundamental group of X into the braid group $B(n)$ on n strings, well-defined up to conjugacy in $B(n)$. If we compose this homomorphism with $\tau_n : B(n) \to \Sigma_n$, we get a *characteristic homomorphism* $\chi(\pi) : \pi_1(X) \to \Sigma_n$ for the covering map π describing the monodromy in π [5], ([7], Chapter III, §5).

2. Representing finite covering spaces over the circle by braids

By a classical theorem of Alexander any link L in 3-space can be obtained by closing a braid around an axis [1], [7, Theorem II.2.5]. The following theorem provides a covering space version of this result.

Theorem 2.1. *Every n-fold covering map $\pi : E \to S^1$ onto the circle is a polynomial covering map and it can be constructed by closing an n-braid β around an axis in 3-space and suitably projecting it onto a circle orthogonal to the axis. The characteristic homomorphism for the covering map $\pi : E \to S^1$ is completely described by the permutation $\tau_n(\beta) \in \Sigma_n$ of the braid β.*

Proof. Choose an orthogonal coordinate system in 3-dimensional Euclidean space. The circle S^1 can then be identified with the unit circle in the corresponding Euclidean plane. Using cylindrical coordinates, the total space in the trivial real plane bundle over S^1 can readily be identified with Euclidean 3-space minus the axis for the cylindrical coordinates.

Consider now an arbitrary n-fold covering space E over the circle S^1. According to a general embedding theorem, such a covering space can be (fibrewise) embedded into the trivial real plane bundle over S^1 [6], ([7], Theorem III.7.2). It follows that it is a polynomial covering space [4], ([7], Theorem III.3.6). By the above identifications, E clearly defines a link in 3-space equivalent to a link L obtained by closing a suitable n-braid β around the axis for the cylindrical coordinates. (It is not necessary to appeal to Alexander's theorem in this context.) The braid β completely determines the n-fold covering space under consideration and the remaining part of the theorem follows immediately from properties of the characteristic homomorphism of a finite covering map ([7], Chapter III, §5). $\square$

Remark 2.2. There is a canonical n-fold covering map

$$\bar{p}_n : F_n(\mathbb{C}) \times_{\Sigma_n} \{1, \ldots, n\} \longrightarrow C_n(\mathbb{C})$$

onto $C_n(\mathbb{C})$, ([7], Chapter III, §2), from which any polynomial covering map $\pi : E \to X$ can be induced by pull-back along the root map $\alpha : X \to C_n(\mathbb{C})$ for π.

Given a braid $\beta \in B(n)$, we can represent it by a loop $\alpha : S^1 \to C_n(\mathbb{C})$. If we pull-back $\bar{p}_n$ along α, we get an n-fold polynomial covering map $\pi_\beta : E_\beta \to S^1$. Clearly this covering map is the n-fold covering map obtained by closing β as described in the proof of Theorem 2.1.

Our next result can be considered as a covering space version of the theorem of Markov [2], [7], [8], [9]. As will be clear, the theorem for covering maps is far easier than the theorem for links.

Theorem 2.3. *Two n-braids $\beta_i \in B(n)$, $i = 1, 2$, represent equivalent n-fold covering maps $\pi_i : E_i \to S^1$ if and only if the permutations of the braids $\tau_n(\beta_i) \in \Sigma_n$ are conjugate elements in Σ_n.*

Proof. The characteristic homomorphisms for the two n-fold covering maps $\pi_i : E_i \to S^1$ constructed from the n-braids $\beta_i \in B(n)$ are completely determined by the two permutations $\tau_n(\beta_i) \in \Sigma_n$, $i = 1, 2$. According to ([7], Theorem III.5.1), two n-fold covering maps are equivalent if and only if their characteristic homomorphisms are conjugate in Σ_n. This proves the theorem. □

3. Weierstrass polynomials for torus links

In this section we shall discuss certain specific knots and links from the point of view of polynomial covering spaces and we shall determine Weierstrass polynomials for all torus links.

We start out by a well known example. Consider the Weierstrass polynomial

$$P(x, z) = z^2 - x, \quad x \in S^1, z \in \mathbb{C} ,$$

over the circle S^1. The associated polynomial covering space is the nontrivial 2-fold covering space of the circle, which, when regarded as a knot, is equivalent to a circle. We conclude that the above Weierstrass polynomial represents the *trivial knot.*

From the point of view of braids, the nontrivial 2-fold covering space of a circle can be constructed by closing an elementary braid σ on two strings in which the first string just overcrosses the second string once in a plane projection of the braid.

The 2-fold polynomial covering space associated with the Weierstrass polynomial

$$P(x, z) = z^2 - x^2, \quad x \in S^1, z \in \mathbb{C} ,$$

consists of two interlocking closed curves (deformed circles): the so-called *Hopf link.* The braid σ^2 provides a braid representation of this link. The power 2 of σ in the braid representation of the link shows up as the power 2 of x in the representation by a Weierstrass polynomial. We note that the link is nontrivial while the 2-fold covering map involved is trivial.

Also the *trefoil knots* have simple representations by Weierstrass polynomials. The *left-handed* trefoil knot is described by the 2-fold polynomial covering space associated with the Weierstrass polynomial

$$P(x, z) = z^2 - x^3, \quad x \in S^1, z \in \mathbb{C} .$$

The braid σ^3 provides a braid representation of the knot ([7], Example II.5.1). The power 3 of σ in the braid representation of the knot shows up as the power 3 of x in the representation by a Weierstrass polynomial.

Similarly, the 2-fold polynomial covering space associated with the Weierstrass polynomial

$$P(x, z) = z^2 - x^{-3}, \quad x \in S^1, z \in \mathbb{C} ,$$

represents the *right-handed* trefoil knot. This knot can be represented by the braid σ^{-3}.

For each pair of relatively prime integers p and q, we have more generally the *torus knot* $T_{p,q}$. This knot is represented by a closed curve that traverses a standard torus p times in the longitudinal direction and q times in the meridional direction with orientations taken into account. For each pair of integers n and m, not necessarily relatively prime, we have in a similar way, the *torus link* $T_{n,m}$. The number of components in this link is equal to the greatest common divisor of n and m. If we consider the class of torus links from the point of view of polynomial covering spaces we have the following general result.

Theorem 3.1. *Regarded as a covering space over the circle, the torus link $T_{n,m}$ corresponding to a pair of integers n and m is equivalent to the polynomial covering space associated with the Weierstrass polynomial*

$$P(x,z) = z^n - x^m, \quad x \in S^1, z \in \mathbb{C}.$$

Proof. We represent the standard torus by the product space $S^1 \times S^1$, where the first factor is considered as the longitudinal direction and the second factor as the meridional direction. The first factor is also considered as the base space in the polynomial covering spaces over the circle and the second factor as the unit circle in the fibre of the trivial complex line bundle over the circle. Accordingly, we use the coordinates $(x, z) \in S^1 \times S^1$ to describe the torus. Now divide the torus into m equal pipes (bended cylinders) along the longitudinal direction such that each pipe corresponds to an angle of $\frac{2\pi}{m}$. When x traverses each pipe once in the longitudinal direction, x^m will traverse a complete circle exactly once. Hence it follows immediately that the n roots of unity in the polynomial $P(x,z) = z^n - x^m$ will be cyclically permuted during the variation of x from one end of each pipe to the other. This description shows that when x traverses the complete circle in the longitudinal direction once, then the set of roots of unity for the polynomial $P(x,z) = z^n - x^m$ will trace out the torus link $T_{n,m}$. This proves that the torus link $T_{n,m}$ is described by the Weierstrass polynomial $P(x,z) = z^n - x^m, \quad x \in S^1, z \in \mathbb{C}$, as asserted. □

4. Links from 2-fold polynomial covering spaces over the circle

In the case of 2-fold polynomial covering spaces over the circle there are strong restrictions on the links that can occur within the equivalence class of the covering.

Consider an arbitrary 2-fold polynomial covering map $\pi : E \to S^1$ onto the circle, and let $\bar{D} : S^1 \to S^1$ denote the normalized discriminant map for a Weierstrass polynomial describing the covering. By ([7], Chapter IV, Theorem 5.3), $\pi : E \to S^1$ is then equivalent to the pull-back along $\bar{D} : S^1 \to S^1$ of the nontrivial 2-fold polynomial covering map $\pi_2 : E_2 \to S^1$ associated with the simple Weierstrass polynomial $P_2(x,z) = z^2 - x$, $x \in S^1$, $z \in \mathbb{C}$. Since $\bar{D} : S^1 \to S^1$ is homotopic to the map $x \to x^m$, where m is the degree of $\bar{D}$, we can pull-back along $x \to x^m$ instead of $\bar{D}$ and then it follows easily that $\pi : E \to S^1$ is equivalent to the polynomial covering space over S^1 defined by the Weierstrass polynomial $P(x,z) = z^2 - x^m$, $x \in S^1$, $z \in \mathbb{C}$. As a link this covering space represents the torus link $T_{2,m}$. This proves the following theorem.

Theorem 4.1. *The link defined by an arbitrary 2-fold polynomial covering space over the circle is a torus link of type $T_{2,m}$, where m is the degree of the normalized discriminant map for a Weierstrass polynomial describing the given polynomial covering.*

5. From braid words to Weierstrass polynomials

Let $B^n = \mathbb{C}^n \setminus \triangle$ denote the complement of the discriminant set $\triangle$ in complex n-space $\mathbb{C}^n$ ([7], Chapter III, §2). The separable Weierstrass polynomial over B^n,

$$P^n((a_1,\ldots,a_n),z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n,$$

$(a_1,\ldots,a_n) \in B^n, z \in \mathbb{C}$, defines the so-called *canonical* n-fold polynomial covering map $\pi^n : E^n \to B^n$. It has the property that any n-fold polynomial covering map $\pi : E \to X$ associated with a separable Weierstrass polynomial $P(x,z) = z^n + \sum_{i=1}^n a_i(x) z^{n-i}$ is equivalent to the pull-back of π^n along the *coefficient map* $a = (a_1,\ldots,a_n) : X \to B^n$ [4], ([7], Theorem III.3.1). There is a canonical homeomorphism $\Phi : C_n(\mathbb{C}) \to B^n$, along which the canonical n-fold polynomial covering map $\pi^n : E^n \to B^n$ can be identified with the n-fold covering map mentioned in Remark 2.2 ([7], Chapter III, §2).

In the following we need to specify base points in the spaces involved.

Choose as base point in $B^n = \mathbb{C}^n \setminus \triangle$, the point $a_0 = (0,\ldots,0,-1)$ corresponding to the polynomial $P^n(a_0,z) = z^n - 1$. If the topological space X is equipped with a base point $x_0 \in X$, we shall say that a separable Weierstrass polynomial $P(x,z)$ of degree n over X is *normalized* if $P(x_0,z) = z^n - 1$.

Through the homeomorphism $\Phi : C_n(\mathbb{C}) \to B^n$, the braid group $B(n)$ can, and will in the following, be identified with the fundamental group

$$B(n) = \pi_1(B^n, a_0).$$

Thereby any braid $\beta \in B(n)$ can be represented by the homotopy class of a map

$$a_\beta : (S^1, 1) \to (B^n, a_0),$$

which on the other hand is the coefficient map of an n-fold polynomial covering map

$$\pi_\beta : E_\beta \to S^1,$$

associated with the normalized, separable Weierstrass polynomial of degree n,

$$P_\beta(x,z) = P^n(a_\beta(x), z), \quad x \in S^1, z \in \mathbb{C}.$$

To handle braid words we just need to consider the product of two n-braids $\beta_1, \beta_2 \in B(n)$, represented by the homotopy classes of maps

$$a_{\beta_1}, a_{\beta_2} : (S^1, 1) \to (B^n, a_0).$$

If we parametrize the unit circle by $S^1 = \{x = e^{i\theta} \mid 0 \le \theta \le 2\pi\}$, the product braid $\beta = \beta_1 \cdot \beta_2 \in B(n)$ can be represented by the homotopy class of the map

$$a_{\beta_1} \cdot a_{\beta_2} : (S^1, 1) \to (B^n, a_0),$$

defined by

$$a_{\beta_1} \cdot a_{\beta_2}(e^{i\theta}) = \begin{cases} a_{\beta_1}(e^{i2\theta}), & 0 \le \theta \le \pi \\ a_{\beta_2}(e^{i2\theta}), & \pi \le \theta \le 2\pi \end{cases} .$$

Let $P_{\beta_1}(x, z), P_{\beta_2}(x, z)$ be the corresponding normalized, separable Weierstrass polynomials of degree n, and define the Weierstrass polynomial

$$P_{\beta_1} \vee P_{\beta_2}(x, z) = P^n(a_{\beta_1} \cdot a_{\beta_2}(x), z), \quad x \in S^1, z \in \mathbb{C}.$$

Then we have the following theorem.

Theorem 5.1. *The n-fold polynomial covering map $\pi_\beta : E_\beta \to S^1$ defined by the product braid $\beta = \beta_1 \cdot \beta_2$ of two braids $\beta_1, \beta_2 \in B(n)$, is associated with the normalized, separable Weierstrass polynomial*

$$P_\beta(x, z) = P_{\beta_1} \vee P_{\beta_2}(x, z), \quad x \in S^1, z \in \mathbb{C}.$$

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