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Equiform Bundle Motions in E_3 with Spherical Trajectories I

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Abstract. The 3-parameter group of Euclidean bundle transformations moves every point of the 3-space on a ball centered in the bundle vertex. In this paper we investigate motions of the 4-parameter group of equiform bundle transformations with the additional property, that some of the points have spherical orbits, i.e. orbits lying on a plane or a ball. There are classical results for the (6-parameter) group of Euclidean transformations by R. Bricard [2], G. Darboux [5] and E. Duporcq [6]. Here we present a systematical approach to our question, which is based on considerations of E. Borel and R. Bricard. The "principle of transfer" used in this paper was developed in J. Lang [8], [9], [10] and applied for the flag space. A very elegant description and a generalisation is due to H. Vogler. This principle of transfer leads to a number of one- and two-parametric motions not investigated until now. We can also give parametric representations of these motions.

1. Introduction

We consider the 4-parameter group

$$G_4 \quad \ldots \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = s\ T\ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{1}$$

where $T := (a_{ij})$ is an orthogonal (3×3)-matrix and $s \in \Re \setminus \{0\}$. The vectors $(x, y, z)^t$ and $(x', y', z')^t$ describe points of the *moving space* Σ and of the *fixed space* Σ', respectively, where Euclidean coordinate systems are used in both spaces. The group (1) will be called *group of equiform bundle transformations*. The vertex O of the bundle being the origin of both systems is a fixed point for every transformation of (1).

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A k-parameter motion $\gamma = \gamma(t_1, \ldots, t_k)$ $(1 \le k \le 4)$ is given by functions $s = s(t_1, \ldots, t_k)$, $a_{ij} = a_{ij}(t_1, \ldots, t_k)$ $\quad i,j = 1,2,3, \quad (t_1, \ldots, t_k) \in R \subset \Re^k$, where the k partial derivation vectors $(\frac{\partial s(t_1,\ldots,t_k)\ a_{11}(t_1,\ldots,t_k)}{\partial dt_j}, \ \ldots \ , \frac{\partial s(t_1,\ldots,t_k)\ a_{33}(t_1,\ldots,t_k)}{\partial dt_j})^t, \quad j = 1, \ldots k$ are linearly independent on R.

We are interested in motions, where certain points of Σ have orbits on spheres (i. e. on balls or in planes). In the very special case that all of the points of Σ have spherical orbits, we will use the name *Bricard motion.* If all the trajectories are even planar, we call it a *Darboux motion.*

2. A principle of transfer

The condition, that the point $X \ldots (x,y,z)^t \in \Sigma \setminus \{O\}$ is moved on a sphere

$$\Theta' \ \ldots \ A\,(x'^2 + y'^2 + z'^2) \ + \ 2\,B\,x' \ + \ 2\,C\,y' \ + \ 2\,D\,z' \ + \ E \ = \ 0 \tag{2}$$

of the fixed space, gives us the equation

$$\begin{aligned} s^2\,A\,(x^2+y^2+z^2) + \ & 2\,s\,a_{11}\,B\,x \ + 2\,s\,a_{12}\,B\,y \ + 2\,s\,a_{13}\,B\,z \ + \\ + \ & 2\,s\,a_{21}\,C\,x \ + 2\,s\,a_{22}\,C\,y \ + 2\,s\,a_{23}\,C\,z \ + \\ + \ & 2\,s\,a_{31}\,D\,x \ + 2\,s\,a_{32}\,D\,y \ + 2\,s\,a_{33}\,D\,z \ + \quad E \ = \ 0. \end{aligned} \tag{3}$$

If $A = 0$, equation (2) represents a plane with $\mathbf{n} := (B,C,D)^t$ as its normal vector. If $A \neq 0$, Θ' is a ball with centre $M \ldots (l,m,n)^t := (\frac{-B}{A}, \frac{-C}{A}, \frac{-D}{A})^t$ and radius r with the property $l^2 + m^2 + n^2 - r^2 = \frac{E}{A}$. For a pair (X, Θ') equation (3) gives us a condition on the motion parameters a_{ij} $(i,j = 1,2,3)$ and s.

In the following we define two mappings:

First we embed the group G_4 into a 10-dimensional projective space $P(V) = P_{10}(\Re)$ in the following way:

$$\begin{aligned} f(\gamma) \ &= \ (g_0\colon \ldots \colon g_{10})^t \ := \\ &:= \ (s^2\colon 2sa_{11}\colon 2sa_{12}\colon 2sa_{13}\colon 2sa_{21}\colon 2sa_{22}\colon 2sa_{23}\colon 2sa_{31}\colon 2sa_{32}\colon 2sa_{32}\colon 1)^t \end{aligned} \tag{4}$$

The image of G_4 is a 4-dimensional[1] subset (surface) Γ_4 in $P(V)$.

The second mapping b embeds the set of all pairs (X, Θ'), where $X \in \Sigma$ and Θ' is a sphere (2) in Σ', into the 10-dimensional projective space $P(W) = Q_{10}(\Re)$. It is defined in the following way:

$$\begin{aligned} b(X,\Theta') \ &= \ U \ = (\omega_0\colon \ldots \colon \omega_{10})^t \ = \\ &= \ (A\,(x^2+y^2+z^2)\colon B\,x\colon B\,y\colon B\,z\colon C\,x\colon C\,y\colon C\,z\colon D\,x\colon D\,y\colon D\,z\colon E)^t. \end{aligned} \tag{5}$$

In the case $A = 0$, where Θ' is a plane, (5) reads as

$$(\omega_0\colon \ldots \colon \omega_{10})^t \ = \ (0\colon B\,x\colon B\,y\colon B\,z\colon C\,x\colon C\,y\colon C\,z\colon D\,x\colon D\,y\colon D\,z\colon E)^t, \tag{6}$$

[1]It can easily be seen, that the embedding f is injective.

in the case $A \neq 0$ we can set $A := 1$ and write (5) as

$$\begin{aligned}(\omega_0\colon\ldots\colon\omega_{10})^t \ &= ((x^2+y^2+z^2)\colon -l\ x\colon -l\ y\colon -l\ z\colon -m\ x\colon -m\ y\colon -m\ z\colon \\ &\colon -n\ x\colon -n\ y\colon -n\ z\colon l^2+m^2+n^2-r^2)^t.\end{aligned} \tag{7}$$

We exclude the case that the centre M' of the sphere Θ' is equal to the origin O' in Σ'; this exclusion implies, that we do not treat with mere Euclidean bundle motions.
We put:

$$\Omega := \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_4 & \omega_5 & \omega_6 \\ \omega_7 & \omega_8 & \omega_9 \end{pmatrix}. \tag{8}$$

The image set of the mapping b consists of points U (see (5)), for which all the $(2,2)$-subdeterminants of the matrix Ω vanish. These 9 conditions are not independent. They can be deduced from the following four equations:

$$\begin{aligned} \omega_1\ \omega_5 - \omega_2\ \omega_4 &= 0 \\ \omega_5\ \omega_9 - \omega_6\ \omega_8 &= 0 \\ \omega_4\ \omega_8 - \omega_5\ \omega_7 &= 0 \\ \omega_2\ \omega_6 - \omega_3\ \omega_5 &= 0. \end{aligned} \tag{9}$$

The points, for which (9) holds, fill a surface in $P(W)$, which we shall call B_6. Every point $U \in B_6$ is in fact the image of (at least) one pair (X, Θ'): Taking $U \in B_6$ we look for an appropriate pair (X, Θ') with $b(X, \Theta') = U$.

- The direction OX has to be

$$x\colon y\colon z \ = \ \omega_1\colon\omega_2\colon\omega_3 \ = \ \omega_4\colon\omega_5\colon\omega_6 \ = \ \omega_7\colon\omega_8\colon\omega_9. \tag{10}$$

- After having chosen a certain point X on the bundle line (10) the quantities x, y, z and of course the value $x^2+y^2+z^2$ are determined. As can be seen in (5) a suitable sphere is then determined in a unique way[2]:

$$A\colon B\colon C\colon D\colon E \ = \ \frac{\omega_0}{x^2+y^2+z^2}\colon\frac{\omega_1}{x}\colon\frac{\omega_4}{x}\colon\frac{\omega_7}{x}\colon\omega_{10}. \tag{11}$$

Remark 1. The considerations above show, that the mapping b is onto the surface B_6, but it is not at all injective. Two pairs (X, Θ') and (Y, Ψ') are mapped into the same point $U \in B_6$ iff there exists a dilatation σ with its centre in the origin and the properties $\sigma(X) = Y$, $\sigma(\Theta') = \Psi'$. This is a very convenient property: Any motion keeping X on Θ' will always move Y on Ψ' as well. Requiring that one point runs on a sphere will always mean that all the points on the same bundle line are forced to move on appropriate spheres. Taking this into account it is not usefull to distinguish between the demand, that X moves on Θ' and the demand, that Y moves on Ψ'. We shall say, that such demands are "the same sphere condition". This is just what we will – from now on – call a "sphere condition".

[2]One of the three coordinates being $\neq 0$ we can assume $x \neq 0$ without any loss of generality.

Remark 2. If $\omega_0 = 0$ holds, the sphere Θ' is a plane with normal vector

$$B{:}\,C{:}\,D \;=\; \omega_1{:}\,\omega_4{:}\,\omega_7 \;=\; \omega_2{:}\,\omega_5{:}\,\omega_8 \;=\; \omega_3{:}\,\omega_6{:}\,\omega_9. \tag{12}$$

Remark 3. If $\omega_0 \neq 0$ holds, Θ' is a ball with a centre $M' \ldots (l,m,n)^t$:

$$l{:}\,m{:}\,n \;=\; \omega_1{:}\,\omega_4{:}\,\omega_7 \;=\; \omega_2{:}\,\omega_5{:}\,\omega_8 \;=\; \omega_3{:}\,\omega_6{:}\,\omega_9. \tag{13}$$

In this case we can also easily calculate the ratio between the distances $d_{M'} = |OM'|$ and $d_X = |OX|$:

$$\frac{d_{M'}^2}{d_X^2} \;=\; \frac{\sum_{i=1}^{9} \omega_i^2}{\omega_0^2} \tag{14}$$

Remark 4. a) Any point $U \in B_6$ is the image of a sphere condition. According to Remark 1 this condition concerns the points X of a bundle ray $OX \subset \Sigma$ and on the other hand some spheres Θ' with centres M' on another bundle ray $OM' \subset \Sigma'$.
b) The mappings $U \to OX$ (see (10)) and $U \to OM'$ (see (12) and (13)) are collinear mappings between the space $P(W)$ and the bundles with centre O in Σ and Σ', respectively.

Using (4) and (5) the condition, that a point X moves on a sphere Θ' (see (3)) can also be written in the form:

$$\sum_{i=0}^{10} g_i \cdot \omega_i \;=\; 0. \tag{15}$$

Regarding (15) it is suggesting itself to establish a bilinear form

$$h : \begin{cases} V \times W & \longrightarrow \Re \\ ((g_0,\ldots,g_{10}),(\omega_0,\ldots,\omega_{10})) & \longrightarrow \sum_{i=0}^{10} g_i \cdot \omega_i. \end{cases} \tag{16}$$

In the projective spaces $P(V)$ and $P(W)$ this bilinear form induces an *orthogonality relation* between the points of $G \in P(V)$ and $U \in P(W)$.

$$G \ldots (g_0,\ldots,g_{10}) \perp U \ldots (\omega_0,\ldots,\omega_{10}) \quad :\Longleftrightarrow \quad \sum_{i=0}^{10} g_i \cdot \omega_i \;=\; 0. \tag{17}$$

For any subset $S \subset P(V)$ there exists a unique subset $T \subset P(W)$ consisting of all points of $P(W)$ orthogonal to every point $X \in S$. This set T is called the *orthogonal complement of* S. We write $T := S^{\perp}$; it is always a linear subspace (a point, a line, a plane, ...) of the projective space $P(W)$. On the other hand we can define the *orthogonal complement of a given subset* $T \subset P(W)$ as the set of all points in $P(V)$ being orthogonal to every point of T. This construction also yields a linear subspace $T^{\perp}$ of $P(V)$.

As a consequence we have

Theorem 1. 1. *Let $\gamma = \gamma(t_1,\ldots,t_k)$ be a k-parametric motion in the group G_4. With respect to the mapping f we get a submanifold $f(\gamma) \subset \Gamma_4 \subset P(V)$. On the other hand we collect all the pairs (X,Θ') so that $\gamma(X) \subset \Theta'$ for all $(t_1,\ldots,t_k)$. This way we get a "set of sphere conditions" fulfilled by the motion γ. Note that this set might be empty. Applying the mapping b to this set we gain a subset of B_6 which is part of the orthogonal complement $f(\gamma)^{\perp} \subset P(W)$ (see* (17)*).*
2. *If we conversely require that a motion γ fulfills a set β of sphere conditions, then $f(\gamma)$ has to be part of the orthogonal complement $b(\beta)^{\perp}$.*

Our principle of transfer, which will do a good job, works in the following way:

1. The f-image of the *maximal motion* γ satisfying a certain set β of sphere conditions, is simply $b(\beta)^\perp \cap \Gamma_4$.
2. The b-image of the sphere conditions respected by a given motion γ is $f(\gamma)^\perp \cap B_6$.

These considerations show that the maximal motion with respect to any set of sphere conditions is a *complete linear intersection of* Γ_4. For the following sections it is very important to know the second result: *All the b-images of sphere conditions satisfied by any given motion, fill a linear intersection of* B_6.

Taking these results into account it seems to make sense to classify the linear subspaces L of $P(W)$. A k-dimensional subspace $L \subset P(W)$ determines a $(9-k)$-dimensional orthogonal complement $L^\perp$. Its intersection $L^\perp \cap \Gamma_4$ is in general of dimension $3-k$. We are of course interested in an intersection which is at least 1-dimensional. As a consequence we have to put $k \leq 2$. Subspaces $L \subset P(W)$ of dimension $k \leq 2$ won't have any common point with B_6 – except if they are of a very special position with respect to B_6: In order to gain a motion fulfilling a one-parameter set of different sphere conditions (see Remark 1) we have to require, that the intersection $L \cap B_6$ is at least one-parametric. In detail:

- $k = 1$ means that L is a straight line. It has to be part of B_6 (see section 3). The motion to be expected is 2-parametric satisfying a one-parameter set of sphere conditions.
- $k = 2$ means that L is a plane (see section 4). We have to distinguish between the following two cases
  1. $L \subset B_6$. The 1-parameter motion to be expected will satisfy ∞^2 sphere conditions.
  2. $L \cap B_6$ is a curve l. B_6 is the intersection of some quadrics (see (9)). This is why l has to be a second order curve. The motion is 1-parametric with the property that a one-parameter set of sphere conditions hold.

According to Remark 4b we can state:

Remark 5. We regard a straight line L on B_6. Its points are b-images of sphere conditions (X, Θ'). The corresponding points $X \in \Sigma$ have to fill a bundle ray or a bundle plane. The corresponding spheres Θ' will have centres M' which do also fill a linear subspace of the bundle $O(x)$.

3. The straight lines on B_6

Let L be a straight line, which lies on $B_6 \subset P(W)$. Be $U_i \dots (\omega_{i,0}\colon \dots \colon \omega_{i,10})^t$ $(i = 1, 2)$ a pair of different points on L. According to Remark 4a these points are the images of two sphere conditions represented by some pairs (X_i, Θ'_i). We write $X_i \dots (x_i, y_i, z_i)^t$, $\Theta'_i \dots (A_i{:}B_i{:}C_i{:}D_i{:}E_i)^t$. Any point $U \dots (\omega_0\colon \dots \colon \omega_{10})^t = \alpha_1\ (\omega_{1,0}\colon \dots \colon \omega_{1,10})^t + \alpha_2\ (\omega_{2,0}\colon \dots \colon \omega_{2,10})^t \in L$ provides a sphere condition, say (X, Θ') with $X \dots (x, y, z)^t$ and $\Theta' \dots (A\colon B\colon C\colon D\colon E)^t$, by itself. Because of (5) we have

$$
\begin{aligned}
&U \dots (A(x^2+y^2+z^2) : Bx : By : Bz : Cx : Cy : Cz : Dx : Dy : Dz : E)^t = \\
&\alpha_1\,(A_1(x_1^2+y_1^2+z_1^2) : B_1x_1 : B_1y_1 : B_1z_1 : C_1x_1 : C_1y_1 : C_1z_1 : D_1x_1 : D_1y_1 : D_1z_1 : E_1)^t + \\
&\alpha_2\,(A_2(x_2^2+y_2^2+z_2^2) : B_2x_2 : B_2y_2 : B_2z_2 : C_2x_2 : C_2y_2 : C_2z_2 : D_2x_2 : D_2y_2 : D_2z_2 : E_2)^t.
\end{aligned}
\tag{18}
$$

The coordinates of U have to satisfy (9). This is why the equations

$$\begin{array}{rcl} (B_1\,C_2 - C_1\,B_2)\,(x_1\,y_2 - x_2\,y_1) &=& 0,\\ (C_1\,D_2 - D_1\,C_2)\,(y_1\,z_2 - y_2\,z_1) &=& 0,\\ (C_1\,D_2 - D_1\,C_2)\,(x_1\,y_2 - x_2\,y_1) &=& 0,\\ (B_1\,C_2 - C_1\,B_2)\,(y_1\,z_2 - y_2\,z_1) &=& 0. \end{array} \tag{19}$$

We can easily check that this implies either

$$x_1\colon y_1\colon z_1 = x_2\colon y_2\colon z_2 \tag{20}$$

or

$$B_1\colon C_1\colon D_1 = B_2\colon C_2\colon D_2. \tag{21}$$

Thus we have to distinguish the following two types of motions:

Type 1: (20) holds; because of Remark 4a we can assume $X_1 = X_2$. As $U_1 \neq U_2$, the corresponding spheres Θ'_1, Θ'_2 have to be distinct. So the point $X_1 = X_2$ has to move on the intersection of those two spheres. Its trajectory has to be either a circle or (in the case that both spheres are planes) a straight line k'. In fact X moves on every sphere contained in the linear bundle of spheres spanned by Θ'_1, Θ'_2. Of course the different points $U \in L$ provide an infinity of sphere conditions, which are different in our sense.
Thinking of the general point $U \in L$ and the corresponding sphere condition (X, Θ') (see above) we can say: Motions of type 1 will have the property that every point X of a certain bundle ray is moved on a circle or a straight line. If a dilatation with centre O gets point X on this bundle ray into a position Y, the same dilatation transforms the trajectory of X into the trajectory of Y (see also Remark 4a).

Type 2: The points X_1, X_2 are on different bundle rays and as a consequence of (19) equation (21) holds. The two spheres Θ'_1, Θ'_2 are
a) two balls with centres M'_1, M'_2 on a common bundle ray,
b) two parallel planes or
c) a ball with centre M'_1 and a plane perpendicular to OM'_1.
Because of Remark 4b the bundle rays concerned by L build up a pencil with centre O in a certain bundle plane ε. We now discuss the different subcases of type 2 in detail:

Type 2a: The centres M' of the spheres Θ' containing the trajectories fill a bundle ray. Picking out one certain point M' means that on each of the corresponding bundle rays in ε there exists one point X with the property: The orbit of X is on a ball with centre M'. All those points X yield a curve $k \subset \varepsilon$. Without any loss of generality we can demand that ε is the plane $y = 0$ in the moving space Σ, the point X_1 is the unit point of the x-axis of the coordinate system in Σ, and X_2 is a point on the y-axis. In the fixed space we can achieve that M' is the unit point of the x'-axis.
(18) is then of the following shape:

$$U \ldots ((\alpha_1 + \alpha_2 z_2^2)\colon -\alpha_1\colon 0\colon -\alpha_2 z_2\colon 0\colon 0\colon 0\colon 0\colon 0\colon 0\colon \alpha_1(1-r_1^2) + \alpha_2(1-r_2^2)). \tag{22}$$

The value $\alpha_1 : \alpha_2 = -1 : 1$ provides $\omega_0 = 0$. Thus the points of the bundle ray t belonging to this value do have plane trajectories. This is why type 2c is not worth extra consideration. All the other values $\alpha_1 : \alpha_2$ will belong to points X with orbits on balls.
On the lookout for the curve $k \subset \varepsilon$ we set $d_M^2 = 1$ (see equation (14)) and we gain

$$d_X^2 \;=\; (\frac{\alpha_1 + \alpha_2\ z_2^2)^2}{\alpha_1^2 + \alpha_2^2 z_2^2}. \tag{23}$$

By multiplying this scalar with the normed direction vector of OX

$$\frac{1}{\sqrt{\alpha_1^2 + \alpha_2^2 z_2^2}} \cdot \begin{pmatrix} \alpha_1 \\ 0 \\ \alpha_2 z_2 \end{pmatrix} \tag{24}$$

we get a parameter representation of k:

$$k \ \ldots \ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} \;=\; \frac{\alpha_1 + \alpha_2 z_2^2}{\alpha_1^2 + \alpha_2^2 z_2^2} \cdot \begin{pmatrix} \alpha_1 \\ 0 \\ \alpha_2 z_2 \end{pmatrix}. \tag{25}$$

Amazingly this is the representation of a circle in the moving space containing the origin O. The tangent of k in O is just the bundle ray t whose points have plane orbits (see above). Here we have to distinguish the following two cases:

- Either all the (parallel) orbit planes are distinct
- or all those planes are identical and do coincide with a single bundle plane.

For the discussion see section 7.

Type 2b: Because of $A_1 = A_2 = 0$ we conclude $A(\alpha_1 : \alpha_2) = 0$ (see (18)). All further orbit spheres are planes, too. Again we have to consider two cases:

- One of the two planes, e.g. Θ_1', is not a bundle plane. If Θ_2' is a bundle plane, we replace the corresponding point X_2 by some other point $X_3 \in X_1X_2$. It is moved in a plane $\Theta_3' \neq \Theta_1', \Theta_2'$. $\Theta_3' \parallel \Theta_2'$ thus is not a bundle plane. Hence to assume that none of both parallel planes contains O causes no loss of generality in this case. With the help of Remark 4a we can even achieve $\Theta_1' = \Theta_2'$. This case can finally be characterized as follows: The points of a straight line are moved in a non-bundle-plane.
- Both planes Θ_1', Θ_2' are bundle planes and as a consequence are identical. This plane keeps its position during the whole motion and so does the bundle ray n orthogonal to it. This special case is the 2-parameter subgroup of G_4 spanned by all rotations around n and by all dilatations with centre O (rotation-stretchings with axis n and centre O).

Summarizing these results we have:

Theorem 2. *The straight lines on the surface B_6 provide motions which can be described as follows:*

Type 1: *A point of the moving space has a circle (or a straight line) as its trajectory.*
Type 2: *The points of a circle $k \ni O$ in Σ are moved on balls with common centre, or the points of a straight line $g \subset \Sigma$ run in a fixed plane.*

In the first case only the ∞^1 points of a certain bundle ray run on circles (lines).
In the second case all the ∞^2 points of the bundle plane containing k are moved on spheres (see Remark 4a*) and the centres of the orbit spheres are contained on a fixed bundle ray, or all the ∞^2 points of the bundle plane containing g are moved in parallel planes or in a bundle plane.*

Thus our principle of transfer has brought another interesting result:

Theorem 3. *If two points $X_1,\ X_2 \in \Sigma$ (not contained in the same bundle ray) are moved on spheres with centres on the same bundle ray $m' \subset \Sigma'$, then all points of the bundle plane OX_1X_2 have orbits on spheres with centres on m'.*

In the second part of this paper we will give parametric representations of each case.

4. The planes in $P(W)$

We regard three non-collinear points $U_i \ldots (\omega_{i,0}, \ldots \omega_{i,10})^t$ (i=1, 2, 3) on the surface $B_6 \subset P(W)$; be $L := [U_1, U_2, U_3]$ the plane spanned by them. Each of these three points is the b-image of a sphere condition (X_i, Θ_i'); we denote $X_i \ldots (x_i, y_i, z_i)^t \in \Sigma$ and $\Theta_i' \ldots (A_i\colon B_i\colon C_i\colon D_i\colon E_i)^t \subset \Sigma'$. Every point $U \subset L$ has a coordinate column

$$U \ldots (\omega_0, \ldots \omega_{10})^t \ = \ \sum_{i=1}^{3} \alpha_i\ U_i \ldots (\omega_{i,0}, \ldots \omega_{i,10})^t. \tag{26}$$

If the point U is on B_6 it is the b-image of a sphere condition as well. Let it be represented by a pair (X, Θ') with $X \ldots (x, y, z)^t \in \Sigma$ and $\Theta' \ldots (A\colon B\colon C\colon D\colon E)^t$. Then all the $(2,2)$-subdeterminants of the matrix (8) built up by the coordinates (26) of U have to vanish. We denote these 9 conditions by $g_{kl}(\alpha_1, \alpha_2, \alpha_3) = 0$ $(k, l = 1, 2, 3)$. Using the matrices

$$A_X \ := \ \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}, \qquad A_M \ := \ \begin{pmatrix} B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \\ D_1 & D_2 & D_3 \end{pmatrix} \tag{27}$$

the nine conditions occuring above can be gained in the following way: We cancel the k-th line of A_X and the l-th line of A_M ($3 \times 3 \ = \ 9$ possible combinations) and calculate the $(2,2)$-subdeterminants of the resulting matrices. Cancelling line 3 in both matrices will – for example – give us the following subdeterminants:

$$x_i\ y_j \ - \ x_j\ y_i, \qquad B_i\ C_j \ - \ B_j\ C_i \qquad i, j = 1, 2, 3, \quad i < j. \tag{28}$$

The products of the subdeterminants with index (i, j) are just the coefficients of $\alpha_i\alpha_j$ in $g_{kl}(\alpha_1, \alpha_2, \alpha_3)$. The corresponding condition for our example ($k = l = 3$) is:

$$g_{33}(\alpha_1, \alpha_2, \alpha_3) \ = \ \sum_{i,j=1,\ i<j}^{3} (x_i\ y_j \ - \ x_j\ y_i)\ (B_i\ C_j \ - \ B_j\ C_i)\ \alpha_i\ \alpha_j \ = \ 0. \tag{29}$$

We can interpret the parameters $\alpha_1, \alpha_2, \alpha_3$ as homogenous coordinates in the plane L. Each of these nine equations $g_{kl}(\alpha_1, \alpha_2, \alpha_3) \ = \ 0$ describes a second order curve containing the

points U_1, U_2, U_3 in L, if the coefficients of this equation do not vanish simultaneously. We have got two interesting cases:
a) The plane L is a part of B_6, which is characterized by the vanishing of all coefficients of all our nine equations.
b) The intersection of L and B_6 is a second order curve. This case occurs iff the coefficient matrix of our nine equations has got the rank 1.

What are the consequences for the sphere conditions (X, Θ')? The following theorem is a step towards the answer:

Theorem 4. a) *A plane $L \subset P(W)$ is part of B_6 iff either A_X or A_M has rank 1.*
b) *$L \cap B_6$ is a (second order) curve iff A_X and A_M are both of rank 2.*

Proof. a) If A_M is of rank 1 all the coefficients of all the equations $g_{kl}(\alpha_1, \alpha_2, \alpha_3) = 0$ vanish. This is why $L \subset B_6$ holds. If A_X is of rank 1 we conclude in the same way.
If conversely $L \subset B_6$ holds, all the coefficients of $g_{kl}(\alpha_1, \alpha_2, \alpha_3)$ have to be zero. A short computation shows that this can only happen if either A_X or A_M is of rank 1.
b) We have to deal with the case that the ranks of A_X and of A_M are both ≥ 2. The intersection $L \cap B_6$ is a second order curve iff not all coefficients of our nine equations vanish, however all the non-vanishing equations are proportional. This bears the conditions (for example by comparing the coefficients of $g_{33},\ g_{32}$ and $g_{33},\ g_{23}$):

$$(x_1\, y_2 \ -\ x_2\, y_1)\ (x_1\, y_3 \ -\ x_3\, y_1)\ B_1\ \det(A_M) \ =\ 0, \tag{30}$$

$$(B_1\, C_2 \ -\ B_2\, C_1)\ (B_1\, C_3 \ -\ B_3\, C_1)\ x_1\ \det(A_X) \ =\ 0. \tag{31}$$

By choosing suitable coordinate-systems in Σ and Σ' we can get $x_1y_2 - x_2y_1 \neq 0,\ \ x_1y_3 - x_3y_1 \neq 0,\ \ B_1 \neq 0,\ B_1C_2 - B_2C_1 \neq 0,\ \ B_1C_3 - B_3C_1 \neq 0,\ \ x_1 \neq 0$. The conditions (30) and (31) are thus equivalent to $\det A_M = \det A_X = 0$. This is why the rank of these matrices has to be ≤ 2. Because of our assumption ($\mathrm{rank}(A_X), \mathrm{rank}(A_M) \geq 2$) we finally have got the proof for our statement.

Using Theorem 4 we conclude:

a) $L \subset B_6$ means, that either the 3 points $X_1,\ X_2,\ X_3$ or the 3 centres $M_1,\ M_2,\ M_3$ of the orbit spheres are on the same bundle ray.

- If $O,\ X_1,\ X_2,\ X_3$ are collinear it is possible to achieve $X_1 = X_2 = X_3$ according to Remark 4a. This point is simultaneously moved on three different spheres not contained in a linear pencil of spheres. So it has to be fixed during the motion. This case leads to a rotation with axis OX_1 and hence is not of interest here.
- If $M_1,\ M_2,\ M_3$ are on a common bundle ray, all the centres of the spheres concerned lie on this ray[3].
  If not all of the spheres $\Theta_1', \Theta_2', \Theta_3'$ are planes, we can achieve, that (with the help of Remark 4a and eventually by exchanging some of the points $X_1,\ X_2,\ X_3$) the points $X_1,\ X_2,\ X_3$ move on concentric spheres.
  If, however, $\Theta_1', \Theta_2', \Theta_3'$ are (parallel) planes we can obtain $\Theta_1' = \Theta_2' = \Theta_3'$ in the same

[3]If some of the spheres are planes, they have to be orthogonal to this bundle ray.

way[4]. Elementary geometric considerations show, that both of these cases only lead to a rotation with a fixed axis.

So we have got the following result: Planes on B_6 only provide rotations around a fixed bundle ray.

b) If the plane L intersects B_6 in a (second order) curve, the resulting motion keeps the points of a bundle plane (in Σ) on spheres with centres in another bundle plane (fixed in Σ'). This case yields the following

Theorem 5. *If three points of the same bundle plane ε are forced to move on spheres whose centres are also lying in a bundle plane ε', the resulting motion necessarily has the following property: All the points of ε have spherical trajectories and the centres of the orbits are in ε'.*

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[4]The case that the three planes coincide with the same bundle plane implies that X_1, X_2, X_3 themselves are in a common bundle plane in contradiction with the assumption $\det A_X \neq 0$.