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The Minimum Mean Width Translation Cover for Sets of Diameter One

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Abstract. In this note we prove that the minimum mean width translation cover for all sets of diameter one in E^d is the d-dimensional ball of diameter $\sqrt{\frac{2d}{d+1}}$. This generalizes a well-known theorem of Jung [3].

0. Introduction

Let K be a compact convex set with nonempty interior in d-dimensional Euclidean space E^d. In other words, K is a *convex body in* E^d. The *support function* h_K of K is defined by $h_K(\mathbf{u}) = \sup\{\langle \mathbf{x}, \mathbf{u}\rangle \mid \mathbf{x} \in K\}$ for any vector $\mathbf{u}$ in E^d, where $\langle \mathbf{x}, \mathbf{u}\rangle$ denotes the standard inner product of the vectors $\mathbf{x}$ and $\mathbf{u}$. Let S^{d-1} be the unit sphere in E^d centered at the origin. That is, $S^{d-1} = \{\mathbf{x} \in E^d \mid \langle \mathbf{x}, \mathbf{x}\rangle = 1\}$. The number $w_K(\mathbf{u}) = h_K(\mathbf{u}) + h_K(-\mathbf{u})$ for $\mathbf{u} \in S^{d-1}$ is called the *width* of K in the direction $\mathbf{u}$. Finally, the mean value of the width function over S^{d-1} is called the *mean width* $w(K)$ of K. Thus,

$$w(K) = \frac{2}{\omega_d} \int_{S^{d-1}} h_K(\mathbf{u}) d\mathcal{L}^{d-1}(\mathbf{u}),$$

where $\mathcal{L}^{d-1}$ denotes the $(d-1)$-dimensional spherical Lebesgue measure computed in S^{d-1} and

$$\omega_d = \mathcal{L}^{d-1}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$$

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is the surface area of S^{d-1}, and Γ is the gamma function.

Now, let $\mathcal{F}$ be a family of sets in E^d. We say that the convex body C of E^d is a *translation cover* of $\mathcal{F}$ if each member of $\mathcal{F}$ lies in some translate of C. Recall that Jung's theorem ([3], p.84–85) asserts that any set of diameter one in E^d can be covered by (a translate of) a d-dimensional ball of diameter $\sqrt{\frac{2d}{d+1}}$. In this note we prove the following stronger statement.

Theorem. *The unique minimum mean width translation cover for all sets of diameter one in E^d is the d-dimensional ball of diameter $\sqrt{\frac{2d}{d+1}}$.*

The proof of this Theorem presented in Sections 1, 2 and 3 gives the following characterization of balls. (See also the Lemma in Section 3.)

Remark 1. The d-dimensional ball is the only convex body in E^d such that all circumscribed tangential regular d-dimensional simplices are congruent and meet the convex body at the centers of their facets.

If K is a convex domain in E^2, then the perimeter of K is equal to $\pi \cdot w(K)$. Thus, the following remark is an immediate corollary of the Theorem. (This corollary has been claimed in [1] without proof. A proof based on the proper Fourier expansion of the support function can be found in [2]. The proof in Sections 1 and 2 is an even shorter one.)

Remark 2. The smallest perimeter convex domain that is a translation cover for all planar sets of diameter 1 is the circle of diameter $\frac{2\sqrt{3}}{3}$.

Recall that the Reuleaux triangle of constant width 1 is defined as the intersection of the three disks of radius 1 centered at the vertices of a regular triangle of side length 1.

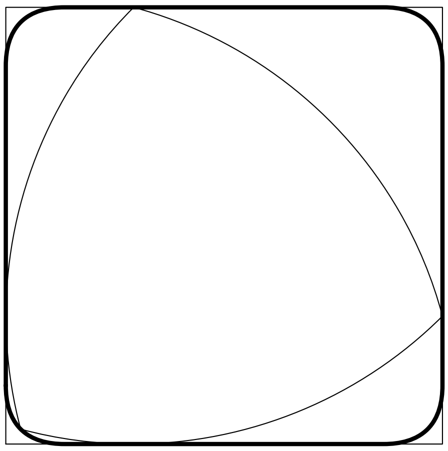

Figure 1

Notice that the Reuleaux triangle of constant width 1 can be placed in a square Q of side length 1, and can then be turned freely within Q while maintaining contact with all four sides of Q. The covered part of Q has area $\frac{\pi}{6} + 2\sqrt{3} - 3 = 0.9877\ldots$. It is not hard to show that the above truncated unit square is a translation cover for all planar sets of diameter one (see for example [1]). Thus, the following problem, that has been raised independently by B. C. Rennie (1977) [4] and by K. Bezdek and R. Connelly (1989) [1] several years ago, is a rather natural question. Surprisingly, it is still open (Figure 1).

Problem. Prove or disprove that the smallest area translation cover for all planar sets of diameter one is the truncated unit square of area $\frac{\pi}{6} + 2\sqrt{3} - 3 = 0.9877\ldots$.

For related results on translation covers see [1].

1. A lower bound for the mean width of a translation cover in E^d

Jung's theorem [3] implies that the lower bound stated in the following claim is in fact a sharp one. (In fact, this is our Theorem without the uniqueness condition.)

Proposition. *If C is an arbitrary translation cover for all sets of diameter 1 in E^d, then the mean width $w(C) \geq \sqrt{\frac{2d}{d+1}}$.*

Proof. Let V be an arbitrary d-dimensional regular simplex of edge length 1 centered at the origin $\mathbf{o}$ in E^d. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ be the vertices of V. Moreover, let $\mathbf{u}_i = \frac{1}{\|\mathbf{v}_i\|}\mathbf{v}_i$ with $\|\mathbf{v}_i\| = \sqrt{\langle \mathbf{v}_i, \mathbf{v}_i \rangle}$ for all $1 \leq i \leq d+1$. Notice that

$$\sum_{i=1}^{d+1} \mathbf{u}_i = \mathbf{0}$$

and $\langle \mathbf{v}_i, \mathbf{u}_i \rangle = \sqrt{\frac{d}{2(d+1)}}$ for all $1 \leq i \leq d+1$. It is easy to see that the Proposition follows from the inequality

$$\sum_{i=1}^{d+1} w_C(\mathbf{u}_i) \geq (d+1)\sqrt{\frac{2d}{d+1}}.$$

Finally, the inequality in question can be proved as follows.

Without loss of generality we may assume that V is a subset of C. Moreover, as C has a translate that contains $-V = \{-\mathbf{x} \in E^d \mid \mathbf{x} \in V\}$, there exists a (translation) vector $\mathbf{t}$ such that the vertices $\mathbf{t} - \mathbf{v}_1$, $\mathbf{t} - \mathbf{v}_2, \ldots, \mathbf{t} - \mathbf{v}_{d+1}$ of the d-dimensional regular simplex $V^* = \mathbf{t} + (-V)$ of edge length 1 all belong to C. Now, let $K = V \cup V^*$. Since $K \subset C$ we get that

$$\begin{aligned}\sum_{i=1}^{d+1} w_C(\mathbf{u}_i) &\geq \sum_{i=1}^{d+1} w_K(\mathbf{u}_i) = \sum_{i=1}^{d+1} h_K(\mathbf{u}_i) + h_K(-\mathbf{u}_i) \geq \sum_{i=1}^{d+1} \langle \mathbf{v}_i, \mathbf{u}_i \rangle + \langle \mathbf{t} - \mathbf{v}_i, -\mathbf{u}_i \rangle \\ &= 2\sum_{i=1}^{d+1} \langle \mathbf{v}_i, \mathbf{u}_i \rangle - \left\langle \mathbf{t}, \sum_{i=1}^{d+1} \mathbf{u}_i \right\rangle = 2(d+1)\sqrt{\frac{d}{2(d+1)}} = (d+1)\sqrt{\frac{2d}{d+1}}.\end{aligned}$$

This completes the proof of the Proposition. □

The above proof leads to the following important observation.

Corollary. *If C is a minimal mean width translation cover for all sets of diameter 1 in E^d, then any d-dimensional regular simplex of edge length 1 in E^d has a translate with vertices say, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ such that each $\mathbf{v}_i$ is a boundary point of C. Moreover, the hyperplane through the vertex $\mathbf{v}_i$ parallel to the facet of the d-simplex opposite to $\mathbf{v}_i$ is a supporting hyperplane of C at $\mathbf{v}_i$, $1 \le i \le d+1$.*

Remark 3. The existence of a minimal mean width translation cover for all sets of diameter 1 in E^d can be proved using standard arguments (see for example [1]).

2. The smallest mean width translation cover in E^2: the proof of the Theorem in E^2

Let C be a smallest mean width translation cover for all sets of diameter 1 in E^2. Without loss of generality we may assume that the origin $\mathbf{o}$ of E^2 is an interior point of the convex domain C. Also, assume that a Cartesian (x, y)-coordinate system is given. Then there is a one-to-one correspondence (modulo 2π) between the vectors $\mathbf{u} \in S^1$ and the angle, say θ, between $\mathbf{u}$ and the positive x-axis; in other words: $\mathbf{u} = \mathbf{u}(\theta) = (\cos\theta, \sin\theta)$. Consequently, we write the restricted support function h_C of C (defined on S^1) as a periodic function $h_C(\theta)$ of the angle θ.

Now, let $\mathbf{v}_1^*$, $\mathbf{v}_2^*$, $\mathbf{v}_3^*$ be the vertices of a positively oriented regular triangle of side length 1 in E^2 such that the side $\mathbf{v}_2^*\mathbf{v}_3^*$ is parallel to the y-axis. Then the Corollary implies that there exists a translate $\Delta\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ of the $\Delta\mathbf{v}_1^*\mathbf{v}_2^*\mathbf{v}_3^*$ such that the vertices $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are all boundary points of C. Moreover, the line ℓ_i parallel to the side $\mathbf{v}_{i-1}\mathbf{v}_{i+1}$ and passing through $\mathbf{v}_i$ is a supporting line of C at $\mathbf{v}_i$, where $i-1$, i, $i+1$ are computed in mod 3, $1 \le i \le 3$ (Figure 2). Finally, let $\mathbf{w}_i = \ell_{i-1} \cap \ell_{i+1}$, $1 \le i \le 3$. Assume that $\mathbf{v}_1$ is a regular boundary point of C that is assume that $h'_C(0)$ exists. Notice that if the coordinates of $\mathbf{w}_1$ are x_1 and y_1, then y_1 is the y-coordinate of $\mathbf{v}_1$ as well and so $y_1 = h'_C(0)$. Moreover, $-\frac{1}{2}x + \frac{\sqrt{3}}{2}y = h_C(\frac{2\pi}{3})$ (resp., $-\frac{1}{2}x - \frac{\sqrt{3}}{2}y = h_C(-\frac{2\pi}{3})$) is the standard distance equation of the line ℓ_2 (resp., ℓ_3). As $\mathbf{w}_1 = \ell_2 \cap \ell_3$ we have that $-\frac{1}{2}x_1 + \frac{\sqrt{3}}{2}y_1 = h_C(\frac{2\pi}{3})$ and $-\frac{1}{2}x_1 - \frac{\sqrt{3}}{2}y_1 = h_C(-\frac{2\pi}{3})$. Hence, $\sqrt{3}y_1 = h_C(\frac{2\pi}{3}) - h_C(-\frac{2\pi}{3})$. Recall that $y_1 = h'_C(0)$. Thus, $h'_C(0) = \frac{1}{\sqrt{3}}[h_C(\frac{2\pi}{3}) - h_C(-\frac{2\pi}{3})]$. Obviously, this implies that $h'_C(\theta) = \frac{1}{\sqrt{3}}[h_C(\theta + \frac{2\pi}{3}) - h_C(\theta - \frac{2\pi}{3})]$ for all θ for which $h'_C(\theta)$ exists. As $h_C(\theta)$ is absolutely continuous we get that $h'_C(\theta)$ is absolutely continuous as well and $h''_C(\theta) = \frac{1}{\sqrt{3}}[h'_C(\theta + \frac{2\pi}{3}) - h'_C(\theta - \frac{2\pi}{3})]$ for almost all θ. Thus, $h''_C(\theta + \frac{2\pi}{3}) + h_C(\theta + \frac{2\pi}{3}) = \frac{1}{\sqrt{3}}[h'_C(\theta + \frac{4\pi}{3}) - h'_C(\theta)] + h_C(\theta + \frac{2\pi}{3}) = \frac{1}{3}[h_C(\theta) - h_C(\theta + \frac{2\pi}{3}) - h_C(\theta + \frac{2\pi}{3}) + h_C(\theta - \frac{2\pi}{3})] + h_C(\theta + \frac{2\pi}{3}) = \frac{1}{3}[h_C(\theta - \frac{2\pi}{3}) + h_C(\theta) + h_C(\theta + \frac{2\pi}{3})] = \frac{1}{3}\sqrt{3} = \frac{1}{\sqrt{3}}$ for almost all θ. Hence, $h''_C(\theta) + h_C(\theta) = \frac{1}{\sqrt{3}}$ for almost all θ. Thus $h_C(\theta)$ must be equal to the general solution of the above second-order nonhomogeneous linear differential equation that is $h_C(\theta) = \frac{1}{\sqrt{3}} + a\cos\theta + b\sin\theta$. This is the support function for a circle whose center is at (a, b) with radius $\frac{1}{\sqrt{3}}$. This completes the proof of the Theorem in E^2.

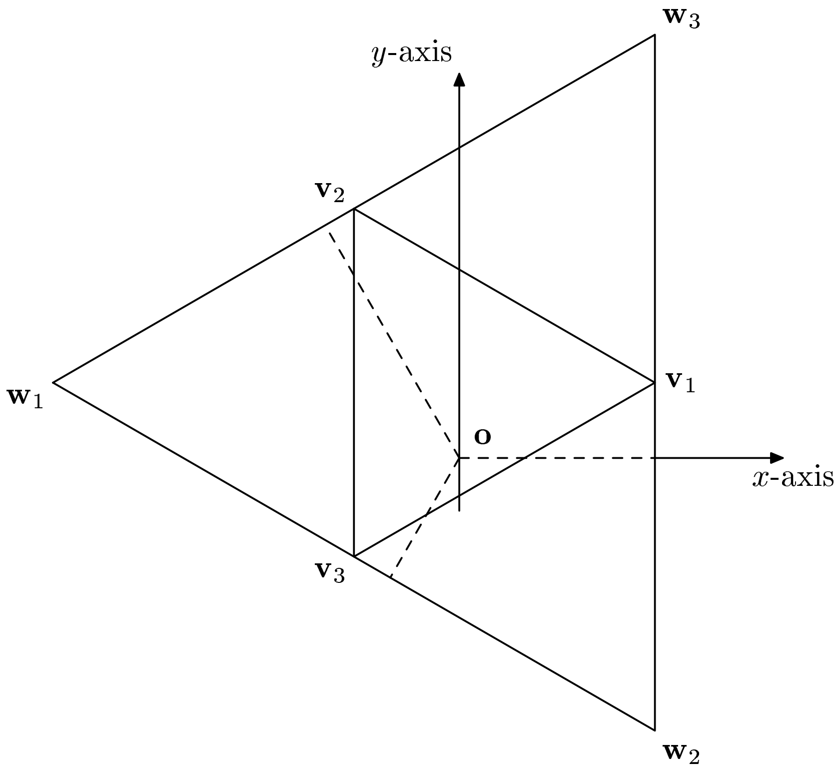

Figure 2

3. The smallest mean width translation cover in E^d: the proof of the Theorem in E^d

The Corollary implies that in order to prove the Theorem it is sufficient to show the following.

Lemma. *If C is a convex body in E^d with the property that any d-dimensional regular simplex of edge length 1 in E^d has a translate with vertices say, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ such that each $\mathbf{v}_i$ is a boundary point of C, and the hyperplane through the vertex $\mathbf{v}_i$ parallel to the facet of the d-simplex opposite to $\mathbf{v}_i$ is a supporting hyperplane of C at $\mathbf{v}_i$, $1 \leq i \leq d+1$, then C is a d-dimensional ball of diameter $\sqrt{\frac{2d}{d+1}}$.*

Proof. We prove the Lemma by induction on the dimension d. Since the claim has been proved for $d = 2$, we assume that it is true in any d'-dimensional Euclidean space with $d' \leq d-1$ and we prove it in E^d.

Let C be a convex body in E^d with the property described in the Lemma. Let $\mathbf{u} \in S^{d-1}$ be a unit vector in E^d and let $H_{\mathbf{u}}$ (resp., $H_{\mathbf{u}}^+$) be the supporting hyperplane (resp., supporting closed halfspace) of C with outer normal vector $\mathbf{u}$. (So $C \cap H_{\mathbf{u}} \neq \emptyset$, $H_{\mathbf{u}} \cap H_{\mathbf{u}}^+ = H_{\mathbf{u}}$ and $C \subset H_{\mathbf{u}}^+$.) Now, take any d-dimensional regular simplex of edge length 1 in E^d with vertices $\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_{d+1}^*$ such that $\mathbf{v}_1^* \in H_{\mathbf{u}}$, $\{\mathbf{v}_2^*, \ldots, \mathbf{v}_{d+1}^*\} \subset H_{\mathbf{u}}^+$ and the facet determined by $\mathbf{v}_2^*, \ldots, v_{d+1}^*$ is parallel to $H_{\mathbf{u}}$. Then this has a translate that

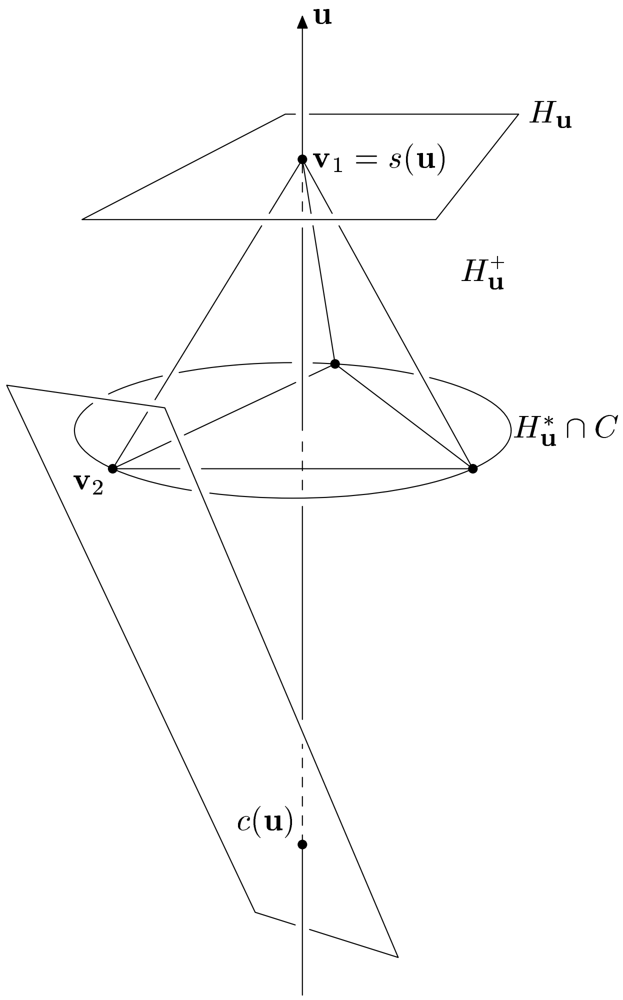

Figure 3

is a regular d-simplex of edge length 1 with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ such that $\mathbf{v}_1 \in H_{\mathbf{u}}$, $\{\mathbf{v}_2, \ldots, \mathbf{v}_{d+1}\} \subset H_{\mathbf{u}}^{+}$ and the hyperplane $H_{\mathbf{u}}^{*}$ spanned by the vertices $\mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ is parallel to $H_{\mathbf{u}}$ and all of the vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$ lie on the boundary of C. Thus, the induction hypothesis implies that $H_{\mathbf{u}}^{*} \cap C$ is a $(d-1)$-dimensional ball of diameter $\sqrt{\frac{2(d-1)}{d}}$. Moreover, $\mathbf{v}_1$ is fixed for a given $\mathbf{u}$ that is we can write $\mathbf{v}_1$ as the corresponding support point $s(\mathbf{u})$ of C. Also, if we take the corresponding support hyperplanes of C through the vertices $\mathbf{v}_2, \ldots, \mathbf{v}_{d+1}$, then they all intersect in a single point that depends on $\mathbf{u}$ only. We call this point $c(\mathbf{u})$ the cone point of C assigned to $\mathbf{u}$. (See Figure 3). Obviously, the line $\ell(\mathbf{u})$ connecting the points $s(\mathbf{u})$ and $c(\mathbf{u})$ is orthogonal to $H_{\mathbf{u}}^{*}$ and passes through the center of the $(d-1)$-ball $H_{\mathbf{u}}^{*} \cap C$.

Finally, notice that if $\mathbf{u}_1 \in S^{d-1}$ and $\mathbf{u}_2 \in S^{d-1}$ are sufficiently close to each other, then the congruent $(d-2)$-dimensional spheres $H_{\mathbf{u}_1}^{*} \cap \operatorname{bd} C$ and $H_{\mathbf{u}_2}^{*} \cap \operatorname{bd} C$ intersect each other, where $\operatorname{bd} C$ denotes the boundary of C. In fact, the points in common are the boundary points of C that lie on supporting hyperplanes of C passing through the cone points $c(\mathbf{u}_1)$ and $c(\mathbf{u}_2)$. Suppose that $\mathbf{v}$ is such a point. Then the simplices with unit edge length, guaranteed by the hypothesis, that are in contact with K at $\mathbf{v}$ all share the same centroid, which must also lie on $\ell(\mathbf{u}_1)$ and $\ell(\mathbf{u}_2)$. Thus, $\ell(\mathbf{u}_1) \cap \ell(\mathbf{u}_2) \neq \emptyset$ for any two sufficiently close unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ in S^{d-1}. Hence, $\cap\{\ell(\mathbf{u}) \mid \mathbf{u} \in S^{d-1}\} \neq \emptyset$ and C is the ball of diameter $\sqrt{\frac{2d}{d+1}}$. This completes the inductive proof of the Lemma. $\square$

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