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Regular Hexagons Associated to Centroid Sharing Triangles

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1. Introduction

In the Problem 1493 from the February 1996 issue of Mathematics Magazine and in the Problem 10514 from the March 1996 issue of American Mathematical Monthly, Jiro Fukuta describes two methods of generating regular hexagons associated to any triangle ABC. More precisely, Fukuta observed that for certain real numbers s, after performing certain constructions we always get regular hexagons $F_1^s F_2^s \cdots F_6^s$ and $G_1^s G_2^s \cdots G_6^s$.

The purpose of this note is to give an analogous procedure which will provide families of regular hexagons that depend on two parameters s and t associated to three triangles with the same centroid.

In order to describe our result we shall use the following notation. For different points A and B in the plane, let AB denote the line passing through A and B. For lines m and n that are not parallel, we shall use $m \wedge n$ to denote their intersection.

Let s denote a real number different from -1. For points A and B, let s_A^B be the point A when $A = B$ and the unique point P on the line AB such that $|AP|/|PB| = s$ when $A \neq B$.

Theorem. *Let s and t be real numbers different from -1 such that neither $st + s + 1 = 0$ nor $st + t + 1 = 0$. Let ABC, PQR, and XYZ be triangles whose centroids coincide such that neither A, P, X, nor B, Q, Y, nor C, R, Z are collinear. Let*

$$P_1 = As_P^X \wedge Pt_X^A, \qquad P_2 = As_X^P \wedge Pt_A^X, \quad P_3 = Bs_Q^Y \wedge Qt_Y^B,$$

$$P_4 = Bs_Y^Q \wedge Qt_B^Y, \qquad P_5 = Cs_R^Z \wedge Rt_Z^C, \qquad P_6 = Cs_Z^R \wedge Rt_C^Z.$$

Let $P_7 = P_1$. For $i = 1, \ldots, 6$, let $Q_i P_i P_{i+1}$, be the equilateral triangles built on the sides of the hexagon $P_1 P_2 \cdots P_6$ all with the same orientation. Let $Q_0 = Q_6$ and $Q_7 = Q_1$. Let F_i

be the centroid of the triangle $Q_{i-1}Q_iQ_{i+1}$, $i = 1, \ldots, 6$. Let $R_iQ_{i-1}Q_{i+1}$, $i = 1, \ldots, 6$, be the equilateral triangles built on the small diagonals of the hexagon $Q_1Q_2 \cdots Q_6$ all with the same orientation as equilateral triangles used in the construction of points Q_1, $Q_2, \ldots$, Q_6. Let $R_0 = R_6$ and $R_7 = R_1$. Let G_i be the centroid of the triangle $R_{i-1}R_iR_{i+1}$, $i = 1, \ldots, 6$. Then $F_1F_2 \cdots F_6$ and $G_1G_2 \cdots G_6$ are regular hexagons whose centers coincide with the centroid of the triangle ABC. Moreover, the segments Q_1Q_4, Q_2Q_5, and Q_3Q_6 have equal length ℓ, the segments R_1R_4, R_2R_5, and R_3R_6 have equal length 2ℓ, $|F_1F_2| = \ell/3$, and the hexagon $G_1G_2 \cdots G_6$ is the image of the hexagon $F_1F_2 \cdots F_6$ under the homothety with the center at the centroid of ABC and the stretching factor 2.

Observe that the above theorem includes two groups of related hexagons since there are two orientation classes. Recall that a triangle ABC is positively oriented provided we move counterclockwise when going from A to B and then to C and that it is negatively oriented when this journey is in the clockwise direction.

There are many triangles associated to the triangle ABC which share with it the centroid. For example, the complementary triangle $A_mB_mC_m$ whose vertices are midpoints of sides, the anticomplementary triangle $A_aB_aC_a$ whose vertices are intersections of parallels through vertices to opposite sides, the first Brocard triangle $A_bB_bC_b$ of ABC, the triangles whose vertices are vertices (centers) of equilateral triangles built on sides either outwards or inwards, and the degenerate triangle GGG, where G is the centroid of ABC.

2. Proof of the Theorem

In the proof we shall use complex numbers. The point and its affix will be denoted by the same capital roman letter. The corresponding small roman letter denotes the complex conjugate of its affix.

For an expression Ex that depends on the set of variables S and a permutation b of S, let $b(Ex)$ denote the result of substitution of variables given by b. For example, if $Ex = 3uv^5w^3$ and $b(u) = v$, $b(v) = w$, and $b(w) = u$, then $b(Ex) = 3u^3vw^5$.

Let j, σ, and τ denote substitutions

$$\{s \to t,\ t \to s,\ A \to P,\ B \to Q,\ C \to R,\ P \to A,\ Q \to B,\ R \to C\},$$

$$\{A \to B,\ B \to C,\ C \to A,\ P \to Q,\ Q \to R,\ R \to P,\ X \to Y,\ Y \to Z,\ Z \to X\},$$

and

$$\{A \to C,\ B \to A,\ C \to B,\ P \to R,\ Q \to P,\ R \to Q,\ X \to Z,\ Y \to X,\ Z \to Y\}.$$

Then P_1 is $(Ast + Xs + P)m$, where $m = 1/(st + s + 1)$. Moreover, we have

$$P_2 = j(P_1),\ \ P_3 = \sigma(P_1),\ \ P_4 = \sigma(P_2),\ \ P_5 = \tau(P_1),\ \text{ and }\ P_6 = \tau(P_2).$$

Let η denote the complex number $(-1 + i\sqrt{3})/2$ (the cubic root of unit), where i denotes the imaginary unit (square root from -1).

Let $n = 1/(st + t + 1)$. Let us assume that we are building negatively oriented equilateral triangles. The expressions for the positively oriented case are similar. The affixes of Q_1 and Q_2 are

$$[(A-P)s^2t^2\eta + As^2t^2 + (X-P)s^2t\eta + (A-X)st^2\eta + Xs^2t + Ast^2 + \\ (A+P+X)st + (X-A)s\eta + (P-X)t\eta + Xs + Pt + (P-A)\eta + P]mn,$$

and

$$[(P-B)s^2t^2\eta + Ps^2t^2 + (P-Y)s^2t\eta + (X-B)st^2\eta + Ps^2t + \\ Xst^2 + (A-B+P-Q+X-Y)st\eta + (A+P+X)st + \\ (A-Y)s\eta + (X-Q)t\eta + As + Xt + (A-Q)\eta + A]mn,$$

while $Q_3 = \sigma(Q_1)$, $Q_4 = \sigma(Q_2)$, $Q_5 = \tau(Q_1)$, and $Q_6 = \tau(Q_2)$. Notice that Q_1 and Q_2 are in fact $(1+\eta)P_1 - \eta P_2$ and $(1+\eta)P_2 - \eta P_3$.

Similarly, the affixes of R_1 and R_2 are

$$[(R-B)s^2t^2\eta + (A-B+P)s^2t^2 + (R-Y)s^2t\eta + (Z-B)st^2\eta + \\ (P+X-Y)s^2t + (A-B+X)st^2 + (C-B+R-Q+Z-Y)st\eta + \\ (2A-B+2P-Q+2X-Y)st + (C-Y)s\eta + (Z-Q)t\eta + \\ (A+X-Y)s + (P-Q+X)t + (C-Q)\eta + A + P - Q]mn,$$

and

$$[(A-Q)s^2t^2\eta + (B+P-Q)s^2t^2 + (X-Q)s^2t\eta + (A-Y)st^2\eta + \\ (P-Q+Y)s^2t + (B+X-Y)st^2 + (A-B+P-Q+X-Y)st\eta + \\ (A+P+X)st + (X-B)s\eta + (P-Y)t\eta + (A-B+Y)s + \\ (Q+X-Y)t + (P-B)\eta + A - B + Q]mn,$$

while $R_3 = \sigma(R_1)$, $R_4 = \sigma(R_2)$, $R_5 = \tau(R_1)$, and $R_6 = \tau(R_2)$. Notice that R_1 and R_2 are in fact $(1+\eta)Q_6 - \eta Q_2$ and $(1+\eta)Q_1 - \eta Q_3$.

The affixes of F_1 and F_2 are

$$[(R-B)s^2t^2\eta + (A+P+R)s^2t^2 + (R-Y)s^2t\eta + (Z-B)st^2\eta + \\ (P+R+X)s^2t + (A+X+Z)st^2 + (C-B+R-Q+Z-Y)st\eta + \\ (2A+C+2P+R+2X+Z)st + (C-Y)s\eta + (Z-Q)t\eta + \\ (A+C+X)s + (P+X+Z)t + (C-Q)\eta + A + C + P]mn/3,$$

and

$$[(A-Q)s^2t^2\eta + (A+B+P)s^2t^2 + (X-Q)s^2t\eta + (A-Y)st^2\eta + \\ (P+X+Y)s^2t + (A+B+X)st^2 + (A-B+P-Q+X-Y)st\eta + \\ (2A+B+2P+Q+2X+Y)st + (X-B)s\eta + (P-Y)t\eta + (A+X+Y)s \\ + (P+Q+X)t + (P-B)\eta + A + P + Q]mn/3,$$

while $F_3 = \sigma(F_1)$, $F_4 = \sigma(F_2)$, $F_5 = \tau(F_1)$, and $F_6 = \tau(F_2)$. Notice that F_1 and F_2 are in fact equal to $(Q_6 + Q_1 + Q_2)/3$ and $(Q_1 + Q_2 + Q_3)/3$.

Finally, the affixes of G_1 and G_2 are

$$\begin{aligned}
[(A - B + C + R - P - Q)s^2t^2\eta + (2A + P - Q + R)s^2t^2 + \\
(R - P - Q + X - Y + Z)s^2t\eta + (A - B + C - X - Y + Z)st^2\eta + \\
(P - Q + R + 2X)s^2t + (2A + X - Y + Z)st^2 + \\
2(C - B + R - Q + Z - Y)st\eta + (3A - B + C + 3P - Q + R + 3X - Y + Z)st \\
+ (C - A - B + X - Y + Z)s\eta + (P - Q + R - X - Y + Z)t\eta + \\
(A - B + C + 2X)s + (2P + X - Y + Z)t + \\
(C - A - B + P - Q + R)\eta + A - B + C + 2P]mn/3,
\end{aligned}$$

and

$$\begin{aligned}
[(A - B - C + P - Q + R)s^2t^2\eta + (A + B - C + 2P)s^2t^2 + \\
(P - Q + R + X - Y - Z)s^2t\eta + (A - B - C + X - Y + Z)st^2\eta + \\
(2P + X + Y - Z)s^2t + (A + B - C + 2X)st^2 + \\
2(A - B + P - Q + X - Y)st\eta + (3A + B - C + 3P + Q - R + 3X + Y - Z)st \\
+ (A - B + C + X - Y - Z)s\eta + (P - Q - R + X - Y + Z)t\eta + \\
(2A + X + Y - Z)s + (P + Q - R + 2X)t + \\
(A - B + C + P - Q - R)\eta + 2A + P + Q - R]mn/3,
\end{aligned}$$

while $G_3 = \sigma(G_1)$, $G_4 = \sigma(G_2)$, $G_5 = \tau(G_1)$, and $G_6 = \tau(G_2)$. Again, observe that G_1 and G_2 are in fact equal to $(R_6 + R_1 + R_2)/3$ and $(R_1 + R_2 + R_3)/3$.

Since the affix of the centroid G of ABC is $(A + B + C)/3$, the affix of the centroid G_p of PQR is $(P + Q + R)/3$, and the affix of the centroid G_x of XYZ is $(X + Y + Z)/3$, we see that the assumption that G, G_p, and G_x coincide is equivalent to the assumption that $P = A + B + C - Q - R$ and $X = A + B + C - Y - Z$. Let π denote the substitution

$$\begin{aligned}
\{P \to A + B + C - Q - R,\ p \to a + b + c - q - r, \\
X \to A + B + C - Y - Z,\ x \to a + b + c - y - z\}.
\end{aligned}$$

One can show easily by hand or by programs in symbolic computation (like Mathematica and Maple) that $\pi(|Q_1Q_4|)$, $\pi(|Q_2Q_5|)$, and $\pi(|Q_3Q_6|)$ have equal value ℓ, that $\pi(|GF_i|)$ and $\pi(|F_iF_{i+1}|)$ all have the same value $\ell/3$ and that G is the midpoint of segments $\pi(F_1F_4)$, $\pi(F_2F_5)$, and $\pi(F_3F_6)$. Moreover, $\pi(|R_1R_4|)$, $\pi(|R_2R_5|)$, and $\pi(|R_3R_6|)$ have equal value 2ℓ, $\pi(|GG_i|)$ and $\pi(|G_iG_{i+1}|)$ $(i = 1, \ldots, 6)$ all have the same value $2\ell/3$ and G is the midpoint of segments $\pi(G_1G_4)$, $\pi(G_2G_5)$, and $\pi(G_3G_6)$, the points G, $\pi(F_i)$, and $\pi(G_i)$ $(i = 1, \ldots, 6)$ are collinear, and the lines $\pi(F_iF_{i+1})$ and $\pi(G_iG_{i+1})$ are parallel.

3. Appendix

In this appendix we shall provide details for the proof of the statements:

(a) $\pi(|Q_2Q_5|^2) = \ell^2$, where $\ell = \pi(|Q_1Q_4|)$,
(b) $\pi(|R_1R_4|^2) = 4\ell^2$,
(c) $\pi(|GG_1|^2) = 4\ell^2/9$, and
(d) $\pi(|G_1G_2|^2) = 4\ell^2/9$.

In order to prove (a), note that $|AX|^2 = (A - X)(a - x)$. It follows that $|Q_1Q_4|^2 = kK$, where K is

$$[(A + C - P - Q)s^2t^2\eta + (A - Q)s^2t^2 + (X + Z - P - Q)s^2t\eta + (A + C - X - Y)st^2\eta + (X - Q)s^2t + (A - Y)st^2 + (C - B + R - Q + Z - Y)st\eta + (A - B + P - Q + X - Y)st + (X + Z - A - B)s\eta + (P + R - X - Y)t\eta + (X - B)s + (P - Y)t + (P + R - A - B)\eta + P - B]mn,$$

Similarly, $|Q_1Q_4|^2 = uU$, where U is

$$[(P + R - B - C)s^2t^2\eta + (P - C)s^2t^2 + (P + R - Y - Z)s^2t\eta + (X + Z - B - C)st^2\eta + (P - Z)s^2t + (X - C)st^2 + (A - B + P - Q + X - Y)st\eta + (A - C + P - R + X - Z)st + (A + C - Y - Z)s\eta + (X + Z - Q - R)t\eta + (A - Z)s + (X - R)t + (A + C - Q - R)\eta + A - R]mn,$$

If we now expand the products kK and uU, use the fact that $\eta^2 = -1 - \eta$, and apply the subsitution π, we get that $\pi(|Q_1Q_4|^2 - |Q_2Q_5|^2) = 0$.

The proofs for statements (b), (c), and (d) are similar. For example, we first compute $|R_1R_4|^2 = vV$, where V is

$$[2(B - R)s^2t^2\eta + (B - A + C - P + Q - R)s^2t^2 + 2(Y - R)s^2t\eta + 2(B - Z)st^2\eta + (Q - P - R - X + Y + Z)s^2t + (B - A + C - X + Y - Z)st^2 + 2(B - C + Q - R + Y - Z)st\eta + 2(B - A + Q - P + Y - X)st + 2(Y - C)s\eta + 2(Q - Z)t\eta + (B - A - C - X + Y + Z)s + (Q - P + R - X + Y - Z)t + 2(Q - C)\eta + B - A - C - P + Q + R]mn,$$

If we now expand the products kK and vV, use the fact that $\eta^2 = -1 - \eta$, and apply the subsitution π, we get that $\pi(4|Q_1Q_4|^2 - |R_1R_4|^2) = 0$.

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