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# The Angle Defect for Arbitrary Polyhedra

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**Abstract.** The classical polyhedral Gauss-Bonnet Theorem for surfaces uses the angle defect to measure curvature. Using a natural stratification for all polyhedra (not necessarily manifolds), the angle defect is generalized to arbitrary polyhedra in all dimensions. A Gauss-Bonnet type theorem is then proved for arbitrary polyhedra, using a modified Euler characteristic based on this stratification rather than the standard Euler characteristic.

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## 1. Introduction

For a triangulated polyhedral surface $M^2$, the usual notion of curvature at a vertex $v$ is the angle defect $d_v = 2\pi - \sum \alpha_i$, where the $\alpha_i$ are the angles of the triangles containing $v$. This curvature function, which goes back at least as far as Descartes (see [9]), satisfies all the standard properties one would expect a curvature function on polyhedra to satisfy. For example, the angle defect is invariant under local polyhedral isometries (that is, functions that preserve the lengths of edges); it is zero at a vertex that has a flat star; and it satisfies the polyhedral Gauss-Bonnet Theorem, which says $\sum_v d_v = 2\pi\chi(M^2)$, where the summation is over all the vertices of $M^2$, and $\chi(M^2)$ is the Euler characteristic of $M^2$.

How should this notion of angle defect be generalized to arbitrary polyhedra? There is a well studied method of defining curvature for arbitrary polyhedra, though it does not look precisely like an angle defect in most treatments of it; we will refer to this definition as "standard curvature". This measure of curvature at a vertex in a polyhedron can be obtained using sums of exterior angle of simplices containing the vertex; the precise formula will be mentioned in Section 4. A number of authors have studied standard curvature from a differential geometric point of view, including [18], [2], [3], [7], [8], [6] and [20]. Other authors have taken a combinatorial approach to standard curvature applied to convex polyhedra; see [10, chapter 14] and [11, §2.2] for references. A number of interesting formulas for the sums of interior and exterior angles (or their products) are summarized in [14], and also made use of by [6]. The convergence of standard curvature to Lipschitz-Killing curvatures in the smooth category is discussed in [8], [6] and [20]. The reverse convergence in the 2-dimensional case is discussed in [5] (which, moreover, takes the same approach as we do to curvature at the boundary of a polyhedral surface, though the idea is not generalized to higher dimensions or arbitrary polyhedra). A rather different approach to the curvature of polyhedral manifolds is in [19], which follows Chern's original proof of the generalized Gauss-Bonnet Theorem.

In this paper we present a curvature function for arbitrary polyhedra that is different from standard curvature. Our approach is to define an angle defect for arbitrary polyhedra as closely analogous as possible to the classical angle defect for polyhedral surfaces. A detailed comparison of our curvature with standard curvature will be given in Section 4. Both our definition and standard curvature have advantages, and it is hard to say that one is "the curvature" for polyhedra.

An angle defect, in any polyhedron, ought to be a measure of curvature based on finding the difference between an angle sum at a simplex and a model "flat" angle sum (which is the $2\pi$ in the surface case). For higher dimensional manifolds the flat angle sum is clearly the volume of the unit $(n-1)$-sphere in $(n-1)$-measure. For arbitrary polyhedra we define in Section 3 a modified "flat" angle sum $T_n(\eta)$ for each simplex $\eta$, depending on which stratum of the polyhedron contains $\eta$, using a (topological) stratification defined for all polyhedra in Section 2. An angle defect using this modified "flat" angle sum is defined in Section 3. In Section 4 we compare our angle defect with standard curvature, both by examples and by discussing various properties that a curvature function might satisfy; this section contains much of the geometric ideas underlying our approach. As in [2], our methods are completely elementary.

A Gauss-Bonnet type theorem for arbitrary polyhedra using the angle defect will be given in Section 3. Instead of the usual Euler characteristic, however, we obtain in our Gauss-Bonnet type theorem the "stratified Euler Characteristic" $\chi^s(K)$ defined in Section 2. For pseudomanifolds $\chi^s(K)$ reduces to the usual Euler characteristic, though for non-pseudomanifolds $\chi^s(K)$ need not equal the Euler characteristic. The number $\chi^s(K)$ is a homeomorphism type invariant, though not a homotopy type invariant. (In [12, p. 343] it is suggested that "there will be no straightforward generalization to higher dimensions of Descartes' formula for the total angular defect of a polyhedron homeomorphic to $S^2$, since this defect ceases in higher dimensions to be a topological invariant." Hopefully our approach will disprove this claim.)

There are a number of reasons for studying our definition of curvature. From a geometric point of view, our approach provides for a very simple generalization of the classical angle defect, whereas standard curvature does not have as simple an interpretation as an angle defect, except in some special cases. An angle defect interpretation for standard curvature is given for manifolds and pseudomanifolds in certain dimensions in [3, §6]. An angle defect formula for all 2-dimensional simplicial complexes is given in [1]; it can be verified that this formula is indeed equal to standard curvature, though this fact is not shown in that paper. A less intuitive formula for standard curvature, resembling an angle defect, is given in [7, §8], and is discussed further in [8, p. 420] and [6, §3]. (A nice geometric interpretation of standard curvature is the Morse-theoretic approach of [2] and [3], which parallels a similar approach in the smooth case, though this is not an angle defect.)

Second, our study of a curvature function different from standard curvature clarifies our understanding of the latter. In particular, the appearance of $\chi^s(K)$ in our approach highlights the question of how to compute curvature in non-pseudomanifolds. Finally, the method developed in this paper is used in the study of simplexwise linear maps of $n$-balls in [4].

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## 2. Stratified Euler Characteristic

We start with some preliminary constructions before turning to our modification of the Euler characteristic.

**Definition.** *For each non-negative integer $i$, let $T_i$ denote the open cone on $i$ points; alternately, $T_i$ is the space obtained by gluing together $i$ copies of the half open interval $[0,1)$ at the point $\{0\}$ in each. We take $T_0$ to be a single point. See Figure 2.1. Let $P_{n,i}$ denote the space $P_{n,i} = T_i \times \mathbb{R}^{n-1}$. If $*$ denotes the cone point of $T_i$, we call $\{*\} \times \mathbb{R}^{n-1} \subseteq P_{n,i}$ the apex set of $T_i$.*

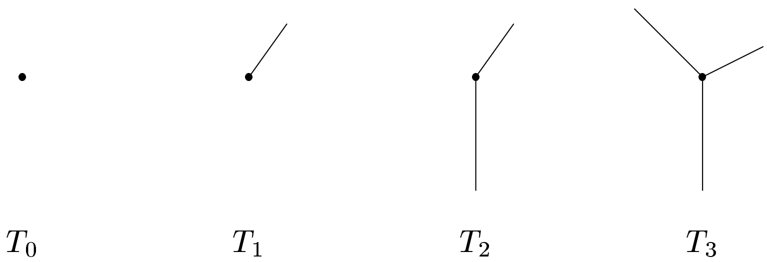


Figure 2.1

Note that $T_1 \approx [0,1)$ and $T_2 \approx (-1,1)$. Further, observe that for $i \neq j$, we have $P_{n,i}$ not homeomorphic to $P_{n,j}$.

We now turn our attention to finite simplicial complexes. All simplicial complexes will be assumed to be finite, and sitting in Euclidean space. We think of simplices as open (and hence disjoint). Throughout this section, let $K$ be a finite $n$-dimensional simplicial complex. We make no additional hypotheses about our $n$-dimensional simplicial complexes; in particular, we do not assume that every simplex in $K$ is a face of some $n$-simplex in $K$. If $\eta$ and $\sigma$ are simplices in $K$, we write $\eta \prec \sigma$ to indicate that $\eta$ is a face of $\sigma$. As usual, we let $|K|$ denote the underlying space of $K$, and we let $K^{(i)}$ denote the $i$-skeleton of $K$.

In an $n$-manifold, each point has a neighborhood homeomorphic to $\mathbb{R}^n \approx (-1,1)\times\mathbb{R}^{n-1} \approx P_{n,2}$; in an $n$-manifold with boundary, each point has a neighborhood homeomorphic to either $P_{n,2}$ or $[0,1)\times\mathbb{R}^{n-1} \approx P_{n,1}$. Certainly, not every point of $|K|$ has a neighborhood of the form $P_{n,2}$ or $P_{n,1}$. However, it is the case that every point in $|K| - |K^{(n-2)}|$ does have a neighborhood of the form $P_{n,r}$ for some $r$ (where the number $r$ varies from point to point). Hence we can define the following collection of subsets of $|K|$.

**Definition.** *Let $K$ be an $n$-dimensional simplicial complex. For each non-negative integer $r$, such that $r \neq 2$, we define the subset $C^n_r(K)$ of $|K|$ by*

$$C^n_r(K) = \{x \in |K| \mid x \textit{ has a neighborhood homeomorphic to } P_{n,r}, \textit{ where the homeomorphism takes } x \textit{ to the apex set of } P_{n,r}\}.$$

*Define*

$$C^n_2(K) = |K| - \bigcup_{r\neq 2} C^n_r(K).$$

It may seem somewhat counterintuitive to combine both the manifold part of $K$ and the set of codimension two or higher singularities into one set $C^n_2(K)$, but this approach will be seen to work out nicely. Though the curvature of polyhedra occurs along the simplices of codimension at least two, in our approach the only place where the "flat" angle sum changes is along codimension one singularities.

**Example 2.1.** Let $K$ be the simplicial complex given in Figure 2.2. Then $Y = \{w\}$. The set $C^n_2$ consists of the interiors of the three triangles together with the vertex $w$. The set $C^n_1$ is the union of the boundaries of the triangles with $w$ removed. $C^n_r = \emptyset$ for $r \neq 1, 2$.

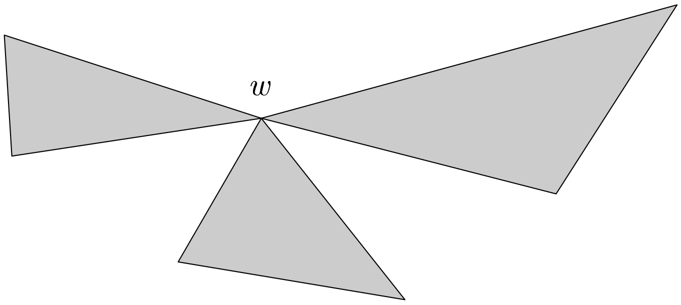


Figure 2.2

**Remark 2.2.** (1) The sets $C^n_r(K)$ are well defined, since each $x \in |K|$ can have a neighborhood homeomorphic to $P_{n,r}$ (where the homeomorphism takes $x$ to the apex set of $P_{n,r}$) for at most one number $r$; this fact can be seen by using local homology around $x$.

(2) Since $K$ is a finite simplicial complex, there is some positive integer $P$ such that $C^n_r(K) = \emptyset$ for all $r > P$. Note that $P \geq 2$. (In principle we should write "$P(K)$" to designate the simplicial complex in question, but since there is never any ambiguity, we will simply write "$P$".)

(3) The sets $C_r^n(K)$ are disjoint, and cover $|K|$. For each $r \neq 2$, the set $C_r^n(K)$ is an $(n-1)$-manifold. Moreover, each set $C_r^n(K)$ is the union of (open) simplices of $K$, since all points in any simplex of $K$ have homeomorphic neighborhoods in $|K|$ (if the neighborhoods are taken small enough). If $\sigma \in K$, then $\sigma \subseteq C_r^n$ for some unique integer $r$.

(4) If $|K|$ is an $n$-manifold, then $C_2^n = |K|$, and $C_r^n = \emptyset$ for $r \neq 2$. If $|K|$ is an $n$-manifold with boundary, then $C_1^n = \partial|K|$, and $C_2^n = |K| - \partial|K|$, and $C_r^n = \emptyset$ for $r \neq 1, 2$.

(5) The sets $C_r^n(K)$ for all $r$ are well defined up to homeomorphism of $|K|$; that is, if $K$ and $L$ are $n$-dimensional simplicial complexes such that $|K| \approx |L|$, then $C_r^n(K) \approx C_r^n(L)$ for all values of $r$.

The above decomposition of a simplicial complex into pieces allows us to define a certain notion of rank for each (open) simplex in the simplicial complex.

**Definition.** *Let $K$ be an $n$-dimensional simplicial complex. Let $\sigma \in K$ be a simplex. The stratification rank of $\sigma$, denoted $T_n(\sigma)$, is defined by $T_n(\sigma) = r/2$, where $\sigma \in C_r^n$ for some unique integer $r$, using Remark* 2.2 (3).

**Remark 2.3.** (1) Let $K$ be an $n$-dimensional simplicial complex. If $\sigma \in K$ is an $n$-simplex, then $T_n(\sigma) = 1$; if $\sigma$ is an $(n-1)$-simplex, then $T_n(\sigma)$ is half the number of vertices in the link of $\sigma$. Moreover, the reader can verify the following two equations, which will be of use later on:

$$(n+1)f_n(K) = 2\sum_{\sigma^{n-1}\in K} T_n(\sigma^{n-1}), \tag{2.1}$$

where the summation is over all $(n-1)$-simplices of $K$, and

$$f_n(K) = \sum_{\sigma^n \in K} T_n(\sigma^n), \tag{2.2}$$

where the summation is over all $n$-simplices of $K$.

(2) The number $T_n(\sigma)$ as defined above is not in general a locally computable invariant, since it's computation requires knowledge of the dimension of $K$, and this dimension is not computable locally. If $K$ is purely $n$-dimensional, that is, if every simplex is the face of an $n$-simplex, then $T_n(\sigma)$ is locally computable.

We now turn to our modification of the Euler characteristic. We let $f_i(K)$ denote the number of $i$-simplices of $K$. Hence the Euler characteristic is $\chi(K) = \sum_{i=0}^{n}(-1)^i f_i(K) = \sum_{\sigma\in K}(-1)^{\dim\sigma}$. Our modified Euler characteristic is now defined as follows.

**Definition.** *Let $K$ be an $n$-dimensional simplicial complex. The stratified Euler Characteristic of $K$, denoted $\chi^s(K)$, is defined by*

$$\chi^s(K) = \sum_{\sigma\in K} T_n(\sigma)\,(-1)^{\dim\sigma}.$$

**Example 2.4.** We continue Example 2.1. This complex has three 2-simplices with stratification rank 1, it has nine 1-simplices with stratification rank 1/2, it has six 0-simplices with stratification rank 1/2, and it has one 0-simplex with stratification rank 1. It is then seen that $\chi^s(K) = 5/2$. By contrast, we have $\chi(K) = 1$.

**Remark 2.5.** Though the stratified Euler Characteristic plays a role with respect to our curvature analogous to the role of the standard Euler with respect to other definitions of curvature, the former does not obey all the properties of the latter. Two relevant observations will be made here; further observations will be made in Remark 2.6.

(1) The number $\chi^s(K)$ is not necessarily an integer, though it is always a rational number with denominator 2.

(2) It is not hard to see that $\chi^s(X)$ is not a homotopy type invariant, in contrast to the standard Euler characteristic. However, as will be seen in Lemma 2.8 (ii) below, the number $\chi^s(K)$ depends only on $|K|$ up to homeomorphism, and in particular, does not depend upon the choice of triangulation of a topological space.

In order both to prove Lemma 2.8 given below, and to develop a more geometric feel for $\chi^s(K)$, we need the following definition.

**Definition.** *Let $K$ be an $n$-dimensional simplicial complex. For each possible $r$, and each $0 \leq i \leq n$, let $f_i(C_r^n)$ denote the number of $i$-simplices of $K$ contained in $C_r^n$ (which is well defined, since $C_r^n$ is the union of simplices of $K$). For each possible value of $r$, define the number $\chi^\circ(C_r^n)$ to be*

$$\chi^\circ(C_r^n) = \sum_{i=0}^{n} (-1)^i f_i(C_r^n).$$

**Remark 2.6.** (1) The numbers $\chi^\circ(C_r^n)$ are not homotopy type invariants of $|K|$, though they are invariant under homeomorphism of $|K|$, as will be seen in Lemma 2.8 (i) given below.

(2) A straightforward calculation shows that

$$\chi^s(K) = \sum_{r=0}^{P} \frac{r}{2}\chi^\circ(C_r^n) = \chi(K) - \sum_{r\neq 2}(1 - \frac{r}{2})\chi^\circ(C_r^n).$$

We thus see that, in essence, the number $\chi^s(K)$ is obtained from $\chi(K)$ by taking into account the non-pseudomanifold parts of $|K|$.

(3) If $|K|$ is an $n$-dimensional pseudomanifold, then $\chi^s(K) = \chi(K)$; if $|K|$ is an $n$-manifold with boundary, then $\chi^s(K) = \chi(K) - \frac{1}{2}\chi(\partial|K|)$.

(4) If $K$ is an $n$-manifold with non-empty boundary, with odd $n$, then $\chi^s(K) = 0$. To see this claim, note that $\chi(\partial|K|) = 2\chi(K)$ for such $K$, and then use part (2) of this remark.

(5) Examples show that $\chi^s(K)$ does not satisfy the usual additivity property of the standard Euler characteristic. However, for the special case of two $n$-manifolds with boundary $M_1$ and $M_2$, such that $M_1 \cap M_2 = \partial M_1 = \partial M_2$, it is seen using part (2) above that $\chi^s(M_1 \cup M_2) = \chi^s(M_1) + \chi^s(M_2)$.

To prove Lemma 2.8, we start with the following result.

**Lemma 2.7.** *Let $Z$ be a topological space. Suppose that there is a $n$-dimensional simplicial complex $K$, and a subcomplex $J$ of $K$ such that $Z \approx |K| - |J|$. Let $f_i(Z)$ denote the number of $i$-simplices of $K$ contained in $Z$ (which is well defined, since $Z$ is the union of (open) simplices of $K$). Then the number $\sum_{i=0}^{n}(-1)^i f_i(Z)$ depends only on $Z$ up to homeomorphism, and not on the choice $K$ and $J$.*

*Proof.* It is easy to see that $\sum_{i=0}^{n}(-1)^i f_i(Z) = \chi(K) - \chi(J)$. By a result of [16, p. 205], we have $\chi(K) - \chi(J) = \chi(K, J)$, where $\chi(K, J)$ is defined by $\sum_{i=1}^{n}(-1)^i b_i(|K|, |J|)$. Because $K$ and $J$ are simplicial complexes, Theorem 2.14 of [17] implies that $H_*(K, J) \approx H_*(|K|/|J|, p)$, where $(|K|/|J|, p)$ is the pair obtained from $(K, J)$ by collapsing $J$ to a point $p$. Observe that since $|K|/|J|$ is compact, and $|K|/|J| - \{p\} \approx Z$, it follows that $|K|/|J| \approx Z^*$, where $Z^*$ is the one point compactification of $Z$ (noting that $Z^* - Z$ is a single point). Thus $\chi(K) - \chi(J) = \chi(K, J) = \chi(|K|/|J|, p) = \chi(Z^*, p) = \chi(Z^*) - \chi(q)$, where $q \in Z^*$ is a point. Since the space $Z^*$ is well defined up to homeomorphism of $Z$, the lemma follows. □

**Lemma 2.8.** *Let $K$ and $L$ be $n$-dimensional simplicial complexes such that $|K| \approx |L|$.*
(i) $\chi^\circ(C_r^n(K)) = \chi^\circ(C_r^n(L))$ *for all possible values of $r$.*
(ii) $\chi^s(K) = \chi^s(L)$.

*Proof.* (i) Let $r \neq 2$. Consider the $n$-dimensional simplicial complex $K$. Let $\overline{C_r^n(K)}$ denote the closure of $C_r^n(K)$. Using Remark 2.2 (3), we note that both $\overline{C_r^n(K)}$ and $\overline{C_r^n(K) - C_r^n(K)}$ are subcomplexes of $K$. Moreover, observe that $C_r^n(K) = \overline{C_r^n(K)} - (\overline{C_r^n(K) - C_r^n(K)})$. It follows from Lemma 2.7 that $\chi^\circ(C_r^n(K)) = \sum_{i=0}^{n}(-1)^i f_i(C_r^n(K))$ only depends upon $C_r^n(K)$ up to homeomorphism. The analogous results hold for $L$. Since $C_r^n(K)$ and $C_r^n(L)$ are homeomorphic, as noted in Remark 2.2 (5), it follows that $\chi^\circ(C_r^n(K)) = \chi^\circ(C_r^n(L))$.

For $r = 2$, we note that $\sum_{r=0}^{P} \chi^\circ(C_r^n(K)) = \chi(K) = \chi(L) = \sum_{r=0}^{P} \chi^\circ(C_r^n(L))$. Since $\chi^\circ(C_r^n(K)) = \chi^\circ(C_r^n(L))$ for each $\neq 2$, it follows that $\chi^\circ(C_2^n(K)) = \chi^\circ(C_2^n(L))$.

(ii) This follows immediately from part (i) of this lemma and Remark 2.7 (2). □

## 3. The Angle Defect

To define the angle defect, we need to measure angles in simplices. For the sake of convenience (and out of necessity when we discuss standard curvature), we adopt the convention that we normalize all angles so that the volume of the unit $(n-1)$-sphere in $(n-1)$-measure is 1 in all dimensions. For any $n$-simplex $\sigma^n$ in Euclidean space, and any $i$-face $\eta^i$ of $\sigma^n$, let $\alpha(\eta^i, \sigma^n)$ denote the solid angle in $\sigma^n$ along $\eta^i$, where by normalization such an angle is always a number in $[0, 1]$. See [8] or [10] for more details. With this normalization, the classical angle defect at a vertex $v$ of a polyhedral surfaces becomes $1 - \sum \alpha(v, \sigma^2)$, where the summation is over all 2-simplices $\sigma^2$ containing $v$.

To prove our Gauss-Bonnet type theorem, we will need the following beautiful (yet not well known) lemma, which generalizes the fact that the angles of a planar triangle add up to

$\pi$ (or 1/2 when normalized); the lemma reduces to that result when $n = 2$. A simple proof of the lemma appears in [13, p. 24]; for historical remarks, see [10, p. 312].

**Lemma 3.1.** *Let $\sigma^n$ be an $n$-simplex in Euclidean space, $n \geq 2$. then*

$$\sum_{\substack{\eta^i \prec \sigma^n \\ 0 \leq i \leq n-2}} (-1)^i \alpha(\eta^i, \sigma^n) = \frac{(-1)^n (n-1)}{2},$$

*where the summation is over all $i$-faces of $\sigma^n$, with $0 \leq i \leq n-2$.*

In order to define the curvature of a polyhedron, we need to be able to measure the angles in its simplices, which makes no sense in an abstract simplicial complex. One possibility is simply to consider simplicial complexes that are embedded in Euclidean space. However, it suffices to consider the following more general situation (which will be used in [4]).

**Definition.** *An $n$-dimensional simplexwise-embedded simplicial complex is a pair $(K, f)$, where $K$ is an $n$-dimensional simplicial complex, and $f$ is a map $f\colon K^{(0)} \to \mathbb{R}^k$ for some $k$ such that if $\{v_0, \ldots, v_n\}$ are the vertices of an $n$-simplex in $K$, then $\{f(v_0), \ldots, f(v_n)\}$ are affinely independent in $\mathbb{R}^k$. Let $f[K]$ denote the collection of simplices in $\mathbb{R}^k$ with vertices $\{f(v_0), \ldots, f(v_p)\}$, for every simplex $\{v_0, \ldots, v_p\}$ in $K$.*

The map $f$ in the above definition need not be an immersion. Nonetheless, it is meaningful to talk about solid angles in the simplices of a simplexwise-embedded simplicial complex. Also, any simplicial complex $K$ in $\mathbb{R}^n$ is a simplexwise-embedded simplicial complex in the obvious way using the identity map.

We can now define our angle defect. In the definition given below, the "flat" angle sum to which all other angle sums are compared is simply the quantity $T_n(\eta^i)$ for each $i$-simplex $\eta^i$.

**Definition.** *Let $(K, f)$ be an $n$-dimensional simplexwise-embedded simplicial complex. The $n$-stratified curvature of $(K, f)$ is the function $D_n\colon K \to \mathbb{R}$ given by*

$$D_n(v, K) = \sum_{i=0}^{n-2} \frac{(-1)^i}{i+1} \sum_{\eta^i \ni v} \{T_n(\eta^i) - \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n)\}, \tag{3.1}$$

*for every vertex $v \in K$. If $(K, f)$ is not clear from the context, we will write $D_n(v, K)$ instead of $D_n(v)$.*

If $n = 2$ and $K$ is a simplicial surface without boundary, then $D_2(v)$ is just the classical angle defect $1 - (1/2\pi) \sum \alpha_i$, where the normalization of solid angles leads to the division by $2\pi$.

Our Gauss-Bonnet type theorem is the following result.

**Theorem 3.2.** *Let $(K, f)$ be an $n$-dimensional simplexwise-embedded simplicial complex. Then*

$$\sum_{v \in K} D_n(v) = \chi^s(K).$$

If $K$ is an $n$-dimensional pseudomanifold, then as remarked above $\chi^s(K) = \chi(K)$, and hence Theorem 3.2 yields $\sum_{v\in K} D_n(v) = \chi(K)$. If $K$ is an $n$-manifold with boundary, then $\sum_{v\in K} D_n(v) = \chi(K) - \frac{1}{2}\chi(\partial|K|)$.

*Proof of Theorem 3.2.* We use $\eta^i$ to denote an $i$-simplex of $K$. First, we compute

$$\begin{aligned}
\sum_{i=0}^{n-2}\sum_{\eta^i\in K}\sum_{\sigma^n\succ\eta^i}(-1)^i\alpha(\eta^i,\sigma^n) &= \sum_{\sigma^n\in K}\sum_{\substack{\eta^i\prec\sigma^n\\ 0\le i\le n-2}}(-1)^i\alpha(\eta^i,\sigma^n)\\
&= \sum_{\sigma^n\in K}\frac{(-1)^n(n-1)}{2} \qquad \text{by Lemma 3.1}\\
&= \frac{(-1)^n(n-1)}{2}f_n(K) = \frac{(-1)^n(n+1)}{2}f_n(K) - (-1)^n f_n(K)\\
&= \frac{(-1)^n}{2}\cdot 2\sum_{\sigma^{n-1}\in K}T_n(\sigma^{n-1}) - (-1)^n\sum_{\sigma^n\in K}T_n(\sigma^n) \qquad \text{by Equations 2.2 and 2.1}\\
&= -\sum_{\sigma^{n-1}\in K}T_n(\sigma^{n-1})(-1)^{\dim\sigma^{n-1}} - \sum_{\sigma^n\in K}T_n(\sigma^n)(-1)^{\dim\sigma^n}. \qquad (3.2)
\end{aligned}$$

We now compute

$$\begin{aligned}
\sum_{v\in K}D_n(v) &= \sum_{v\in K}\sum_{i=0}^{n-2}\frac{(-1)^i}{i+1}\sum_{\eta^i\ni v}\Big\{T_n(\eta^i) - \sum_{\sigma^n\succ\eta^i}\alpha(\eta^i,\sigma^n)\Big\}\\
&= \sum_{i=0}^{n-2}\frac{(-1)^i}{i+1}\sum_{\eta^i\in K}\sum_{v\in\eta^i}\Big\{T_n(\eta^i) - \sum_{\sigma^n\succ\eta^i}\alpha(\eta^i,\sigma^n)\Big\}\\
&= \sum_{i=0}^{n-2}(-1)^i\sum_{\eta^i\in K}\Big\{T_n(\eta^i) - \sum_{\sigma^n\succ\eta^i}\alpha(\eta^i,\sigma^n)\Big\} \qquad \text{since } \eta^i \text{ has } i+1 \text{ vertices}\\
&= \sum_{i=0}^{n-2}\sum_{\eta^i\in K}T_n(\eta^i)(-1)^i - \sum_{i=0}^{n-2}\sum_{\eta^i\in K}\sum_{\sigma^n\succ\eta^i}(-1)^i\alpha(\eta^i,\sigma^n)\\
&= \sum_{i=0}^{n-2}\sum_{\eta^i\in K}T_n(\eta^i)(-1)^{\dim\eta^i} + \sum_{\sigma^{n-1}\in K}T_n(\sigma^{n-1})(-1)^{\dim\sigma^{n-1}}\\
&\qquad + \sum_{\sigma^n\in K}T_n(\sigma^n)(-1)^{\dim\sigma^n} \qquad \text{by Equation 3.2}\\
&= \sum_{\eta\in K}T_n(\eta)(-1)^{\dim\eta} = \chi^s(K),
\end{aligned}$$

where the summation in the last row is over all simplices in $K$. □

## 4. Comparison with Standard Curvature

We start by recalling the definition of stratified curvature, and showing that it is not always equal to standard curvature. Next, we compare the properties that each of these curvature functions satisfies. Finally, we conclude with a few further geometric comments, and pose a few questions.

To define standard curvature, we need the following notation. Let $\alpha^*(v, \sigma^n)$ denote the (normalized) exterior angle of $\sigma^n$ at a vertex $v$; see [8] or [10] for details.

**Definition.** *Let $(K, f)$ be an $n$-dimensional simplexwise-embedded simplicial complex. The standard curvature of $(K, f)$ is the function $D_s\colon K \to \mathbb{R}$ given as follows: For every vertex $v \in K$, let $D_s(v)$ be defined by*

$$D_s(v) = \sum_{i=0}^{n} (-1)^i \sum_{\eta^i \ni v} \alpha^*(v, \eta^i), \tag{4.1}$$

*where the inner summation is over all $i$-simplices of $K$ containing $v$.*

It is simple to show that when $n = 2$, the curvature $D_s(v)$ equals the normalized classical angle defect for polyhedral surfaces. It is quite simple to prove the Gauss-Bonnet theorem for standard curvature, that is $\sum_{v \in K} D_s(v) = \chi(K)$. See [8, p. 419] for a proof. The key ingredient in this proof is the fact that for any $i$-simplex $\eta^i$ of $K$, we have $\sum_{v \in \eta^i} \alpha^*(v, \eta^i) = 1$, where the summation is over all vertices in $\eta^i$. (Contrast this fact with Lemma 3.1, which is the key ingredient in the proof of our Gauss-Bonnet type theorem for stratified curvature.)

We begin our comparison of standard curvature and stratified curvature with the simple observation that in the former exterior angles are used, whereas interior angles are used in the latter. There is an expression for standard curvature in terms of interior angles given in [7, §8], and discussed in [8, p. 420] and [6, §3], though it is not as simple as our formula. Moreover, in the definition of stratified curvature we only added angles of the form $\alpha(\eta^i, \sigma^n)$, where $i$ varies but $n$ is fixed. Hence, although for convenience we normalized all angles so that the volume of the unit $(i-1)$-sphere in $(i-1)$-measure is 1 in all dimensions, we had no need to do so to discuss stratified curvature, since all angles added were measured in the same units. By contrast, in the definition of standard curvature it is necessary to add exterior angles of the form $\alpha^*(v, \eta^i)$, where $i$ varies. In order to add such angles, which are computed using $(i-1)$-measure for differing values of $i$, it is necessary to normalize all angles. (In the formula given in [7, §8] et al., the interior angles used are also in varying dimensions, so normalization is still needed.)

It is easy to see that standard curvature and stratified curvature need not be equal in general, since standard curvature satisfies the Gauss-Bonnet theorem with the standard Euler characteristic, whereas stratified curvature satisfies a Gauss-Bonnet type theorem with the stratified Euler Characteristic (which is in general not equal to the standard Euler characteristic). However, as remarked in Section 4, we know that stratified Euler Characteristic is equal to the standard Euler characteristic for an $n$-dimensional simplicial manifold, and thus the question arises whether stratified curvature and standard curvature are equal for simplicial manifolds. The following example answers this question negatively.

**Example 4.1.** Let $K$ be the 4-dimensional simplicial complex $\mathbb{R}^5$ with vertices $v = (0,0,0,0,0)$, $e_1 = (1,0,0,0,0)$, $e_2 = (0,1,0,0,0)$, $\ldots$, $e_5 = (0,0,0,0,1)$, and with simplices of dimensions $1, \ldots, 4$ all the possible simplices using the given vertices. The underlying space of $K$ is a 4-sphere that is the boundary of a 5-simplex. We can think of $K$ as a 4-dimensional simplexwise-embedded simplicial complex using the identity map $K \to \mathbb{R}^5$. We note that $v$ is contained in one 0-simplex, in five 1-simplices, in ten 2-simplices, in ten 3-simplices and in five 4-simplices.

We start by computing $D_4(v)$. Since all angles at $v$ are right angles, it is easy to see that

$$\alpha(\eta^0, \sigma^4) = \frac{1}{16}, \quad \alpha(\eta^1, \sigma^4) = \frac{1}{8}, \quad \alpha(\eta^2, \sigma^4) = \frac{1}{4},$$

where $\eta^0 = v$, where $\eta^1, \ldots, \eta^4$ are any simplices of the appropriate dimensions containing $v$, and where $\sigma^4$ is any 4-simplex containing a simplex $\eta^i$. Since $|K|$ is a manifold, we have $T_4(\eta^i) = 2/2$ for all $i$. We see that $v$ is contained in five 4-simplices; each $\eta^1$ is contained in four 4-simplices; each $\eta^2$ is contained in three 4-simplices. We then compute that

$$\begin{aligned} D_4(v) &= \sum_{i=0}^{2} \frac{(-1)^i}{i+1} \sum_{\eta^i \ni v} \{T_4(\eta^i) - \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n)\} \\ &= \frac{1}{1} \cdot 1 \cdot (1 - 5 \cdot \frac{1}{16}) - \frac{1}{2} \cdot 5 \cdot (1 - 4 \cdot \frac{1}{8}) + \frac{1}{3} \cdot 10 \cdot (1 - 3 \cdot \frac{1}{4}) = \frac{13}{48}. \end{aligned}$$

We now turn to $D_s(v)$, starting with the fact that

$$\alpha^*(v, \eta^0) = 1, \quad \alpha^*(v, \eta^1) = \frac{1}{2}, \quad \alpha^*(v, \eta^2) = \frac{1}{4}, \quad \alpha^*(v, \eta^3) = \frac{1}{8}, \quad \alpha^*(v, \eta^4) = \frac{1}{16}.$$

Once again using our observation above about the number of simplices of each dimension containing $v$, we make use of Equation 4.1 to see that

$$D_s(v) = \sum_{i=0}^{4} (-1)^i \sum_{\eta^i \ni v} \alpha^*(v, \eta^i) = 1 \cdot 1 - 5 \cdot \frac{1}{2} + 10 \cdot \frac{1}{4} - 10 \cdot \frac{1}{8} + 5 \cdot \frac{1}{16} = \frac{1}{16}.$$

Thus we see that $D_4(v) \neq D_s(v)$.

To have a more complete understanding of the differences between standard curvature and stratified curvature, we can examine the various properties that each of these satisfies. Although we have discussed stratified curvature in the more general setting of simplexwise-embedded simplicial complexes, when we discuss these properties we will do so only for simplicial complexes in Euclidean space to avoid some unnecessary technicalities. Of course, any simplicial complex in Euclidean space is automatically a simplexwise-embedded simplicial complex.

Any curvature function would be a function $\Phi_n$ that takes every $n$-dimensional simplicial complex $K$ and produces a function $\Phi_n(K)\colon K^{(0)} \to \mathbb{R}$. We would wish such a function to satisfy a number of properties, among which would be the following; we state most of the properties informally, since that will suffice for our purposes, though all these properties could be formulated in a completely rigorous fashion. We assume in these properties that $K$ is an $n$-dimensional simplicial complex in Euclidean space, and that $v$ is a vertex of $K$.

(1) Local isometry invariant: $\Phi_n(K)(v)$ is unchanged for each vertex $v$ by a local isometry of $K$.

(2) Locally defined: $\Phi_n(K)(v)$ can be computed from an arbitrarily small neighborhood of $v$ in $K$.

(3) Gauss-Bonnet type theorem: $\sum_{v \in K} \Phi_n(K)(v) = \chi(K)$.

(4) Zero on flat neighborhoods: If some neighborhood of $v$ in $K$ is a flat $n$-ball, then $\Phi_n(K)(v) = 0$.

(5) Subdivision invariant at vertices: $\Phi_n(K)(v)$ is unchanged for each vertex $v$ by a subdivision of $K$.

It can be seen that standard curvature satisfies properties (1) – (5). Properties (1) and (2) follow straightforwardly from the definition given above. Property (3) is proved in [2, p. 253]. Property (4) is discussed in [8, §3]. Property (5) does not appear to be stated explicitly in the literature (at least this author has not found it), but it is true, and can be deduced from the Morse-theoretic definition of standard curvature in [2, p. 251].

As for stratified curvature, properties (1) and (2) again follow straightforwardly from the definition. By Theorem 3.2 we see that property (3) does not hold as stated (though it does hold if we restrict attention to the category of simplicial manifolds, since in that case the stratified Euler Characteristic equals the standard Euler characteristic). However, stratified curvature does satisfy the modified version of property (3) in which $\chi(K)$ is replaced by $\chi^s(K)$. It is straightforward to see that stratified curvature satisfies property (4). However, the following example shows that stratified curvature does not satisfy property (5).

**Example 4.2.** Let $K$ be as in Example 4.1. We form a subdivision $L$ of $K$ by adding one new vertex $w$ inside the 2-simplex $\zeta = \langle v, e_1, e_2 \rangle$, and by joining $w$ to all other simplices of $K$. It is not hard to compute that $D_4(w, L) = \frac{1}{8}$. Using Theorem 3.2 applied to each of $K$ and $L$, which both have the same manifold for underlying space, we have

$$D_4(v, K) + D_4(e_1, K) + \cdots + D_4(e_5, K) = \chi^s(K) = \chi(K) = 2,$$

and

$$D_4(w, L) + D_4(v, L) + D_4(e_1, L) + \cdots + D_4(e_5, L) = \chi^s(L) = \chi(L) = 2.$$

Since $D_4(w, L) \neq 0$, it must be the case that either $D_4(v, L) \neq D_4(v, K)$ or that $D_4(e_i, L) \neq D_4(e_i, K)$ for some $i$. Thus we see that stratified curvature is not subdivision invariant at vertices.

There is a geometric reason why stratified curvature does not satisfy property (5). In our definition of curvature, we have all the curvature occuring at the vertices. We do so because that is the way standard curvature is defined, and we are thus able to compare that curvature with our own. We might question, however, why we should view curvature as happening only at the vertices? In the 2-dimensional case, where the angle defect is the model for both standard curvature and our definition of curvature, it is the certainly the case that curvature only occurs at the vertices. However, rather than viewing the vertices as the essence of where the curvature is happening in all simplicial complexes, we focus on the fact that in a 2-dimensional space, the vertices are the only faces of codimension at least 2, and it is this codimension that is of the essence. As seen in Equation 3.1, our method measures curvature by computing the extent to which each expression of the form $T_n(\eta^i) - \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n)$, which is just an angle defect, is non-zero. In the formula for $D_n(v)$ we simply spread out these expressions to all the vertices, which is why we have the fraction $1/(i+1)$ in Equation 3.1 (since an $i$-simplex has $i+1$ vertices). This spreading out to the vertices is artificial, and is the reason that property (5) does not hold.

The reason we spread out the curvature to the vertices is because standard curvature does, and so we wanted our approach to be comparable to it. It would actually make more geometric sense to say that curvature happens not only at the vertices, but rather at all the simplices of codimension at least 2 (not at each point of each simplex of appropriate dimension, but rather at each simplex itself). For simplicity, we simply assign curvature zero to all simplices of codimension 1 or 0. We could thus replace stratified curvature with the following definition.

**Definition.** *Let $(K, f)$ be an $n$-dimensional simplexwise-embedded simplicial complex. Let $\eta^i \in K$ be an $i$-simplex. The modified stratified curvature at $\eta^i$ is the number $\tilde{D}_n(\eta^i)$ defined as follows: If $0 \le i \le n-2$, then*

$$\tilde{D}_n(\eta^i) = T_n(\eta^i) - \sum_{\sigma^n \succ \eta^i} \alpha(\eta^i, \sigma^n);$$

*if $i = n-1$ or $n$, then $\tilde{D}_n(\eta^i) = 0$.*

Properties (1) – (5) given above can now be restated in this new context. A curvature function is now a function $\Psi_n$ that takes every $n$-dimensional simplicial complex $K$ and produces a function $\Psi_n(K)\colon K \to \mathbb{R}$. The precise statements of most of the analogs of properties (1) – (5) in this new setting are straightforward. Two of these reformulations bear explicit mention; the others are left to the reader.

(3a) Gauss-Bonnet type theorem: $\sum_{\eta^i \in K} (-1)^i \Psi_n(K)(\eta^i) = \chi^s(K)$.

(5a) Subdivision invariant: Let $K$ be an $n$-dimensional simplicial complex, and let $L$ be a subdivision of $K$. Let $\eta \in L$, and let $\sigma \in K$ be the unique simplex of $K$ such that $\eta \subseteq \sigma$. Then $\Psi_n(L)(\eta) = \Psi_n(K)(\sigma)$.

It can be seen that modified stratified curvature satisfies these five reformulated properties.

Finally, the approach of having curvature defined at all simplices not only allows us to have properties analogous to properties (1) – (5) hold using our approach, but we can also explain another difference between our curvature and standard curvature. In [2], we see that standard curvature is discussed in the more general of convex linear cell complexes, rather than simplicial complexes; combinatorial treatments of standard curvature, as cited in [10] for example, while only discussing convex polytopes, also allow convex faces other than simplices. In our treatment of stratified curvature in Section 3, we needed to use simplices to have a canonical way of distributing the curvature among the vertices. If we take the approach of modified stratified curvature then we do not need to distribute the curvature among the vertices, and so our need for simplices vanishes. Indeed, modified stratified curvature can be defined for convex linear cell complexes just as we have already defined it. Our proof of Theorem 3.2 (the Gauss-Bonnet theorem) will work just as well with convex linear cell complexes, provided we replace Lemma 3.1 (which is stated only for simplices) with the generalization of this result to convex polytopes (see [15]), and we replace Equation 3.2 by the corresponding result for convex linear cell complexes.

Finally, we conclude with the following question concerning the possible similarities between standard curvature and stratified curvature. It is shown in [8] that standard curvature converges to smooth curvature in the sense of measures, in a way made precise in that paper. Does stratified curvature also converge to smooth curvature?

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