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# Sufficient Conditions for the Extendibility of an $n$-th Order Flex of Polyhedra

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**Abstract.** A new approach to describing a high order flex of polyhedra is given. Sufficient conditions are derived under which an infinitesimal flex of polyhedra has an extension to an authentic flex. It is shown that one set of these conditions is fulfilled for Bricard octahedra.

## 1. Introduction

A very long-standing (and still open) problem is to prove that a smooth compact surface in the Euclidean 3-space is rigid, i.e., one cannot find a continuous family of smooth surfaces in the Euclidean 3-space each of which is isometric (in the intrinsic metric) to any other one and is not obtained from the initial surface by a rigid motion [9]. Classical attempts to solve this problem were based on the notion of an infinitesimal flex (see [6] or [7] for further references).

Since a counterexample was constructed in the class of polyhedral surfaces [2], particular attention was given to studying infinitesimal flexes of polyhedra. In particular, the "classical" definition of a high order infinitesimal flex occurs to be improper from the functional point of view [4].

Without revising the "classical" definition of a high order infinitesimal flex, we give a new technique of calculations in Section 2 below. Section 3 contains some auxiliary facts from linear algebra. The main results of this paper are given in Section 4 and provide some sufficient conditions under which an infinitesimal flex of a polyhedron can be extended to an authentic flex. In Section 5, we discuss results of computer experiments demonstrating that one set of our sufficient conditions is fulfilled for Bricard octahedra.

In [8] some formal sufficient conditions were given for extendibility of an infinitesimal flex of a smooth surface vaguely resembling our conditions.

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## 2. Reduction of the flex problem for polyhedra to an algebraic problem

**Definition.** *A closed polyhedron $Q$ in the Euclidean 3-space is called flexible if there exists an analytic (with respect to a parameter $t \in [0,1]$) family of polyhedra $Q_t$ such that*
(i) *$Q$ coincides with $Q_0$;*
(ii) *for each $t \in [0,1]$, the polyhedron $Q_t$ is isometric to the polyhedron $Q_0$ in the intrinsic metrics; moreover, the restriction of this isometry onto each face of $Q_t$ can be extended to an isometry of the whole 3-space;*
(iii) *the polyhedron $Q_1$ cannot be obtained from $Q_0$ by a rigid motion.*

**Remark.** (i) Some other natural definitions of flexible polyhedra are known. In [5] it is shown that all of them are equivalent to the above definition. The family of polyhedra $Q_t$ is called an authentic flex of $Q$.
(ii) Without loss of generality, we will always assume in this paper that all faces of a flexible polyhedron are triangular. Otherwise it suffices to triangulate all faces in advance and consider the new polyhedron that is obviously flexible.

Let $Q$ be a closed flexible polyhedron. Suppose that $Q$ has $v$ vertices $(x_i, y_i, z_i)$, $1 \le i \le v$, $e$ edges, and $f$ faces. If vertices $(x_i, y_i, z_i)$ and $(x_j, y_j, z_j)$ are joined together by an edge of $Q$, then we assign a number $e(i,j)$, $1 \le e(i,j) \le e$, to this edge and denote by $l_{e(i,j)}$ the length of the edge.

The coordinates of the $i$-th vertex of the polyhedron $Q_t$ are analytic functions with respect to $t$, $0 \le t \le 1$. Denote them by $x_i(t), y_i(t)$, and $z_i(t)$.

Since all faces of $Q$ are supposed to be triangular, the statement that the polyhedra $Q_t$ and $Q_0$ are isometric to each other in intrinsic metrics is equivalent to the statement that relevant edges of these polyhedra have equal lengths. In other words, if vertices $i$ and $j$ are joined by an edge of $Q$ then

$$\sqrt{[x_i(t) - x_j(t)]^2 + [y_i(t) - y_j(t)]^2 + [z_i(t) - z_j(t)]^2} =$$

$$= \sqrt{[x_i(0) - x_j(0)]^2 + [y_i(0) - y_j(0)]^2 + [z_i(0) - z_j(0)]^2} = l_{e(i,j)}.$$

Substituting Maclaurin expansions of the functions $x_j$, $y_j$, and $z_j$,

$$x_j(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{d^n}{dt^n} x_j(t) \right] \Bigg|_{t=0} t^n \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} x_{j,n} t^n,$$

$$y_j(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{d^n}{dt^n} y_j(t) \right] \Bigg|_{t=0} t^n \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} y_{j,n} t^n,$$

$$z_j(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \frac{d^n}{dt^n} z_j(t) \right] \Bigg|_{t=0} t^n \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} z_{j,n} t^n,$$

into the expression

$$[x_i(t) - x_j(t)]^2 + [y_i(t) - y_j(t)]^2 + [z_i(t) - z_j(t)]^2 = l^2_{e(i,j)},$$

we obtain

$$\sum_{n=0}^{\infty}\Big\{\sum_{m=0}^{n}\big[(x_{i,m} - x_{j,m})(x_{i,n-m} - x_{j,n-m}) + (y_{i,m} - y_{j,m})(y_{i,n-m} - y_{j,n-m}) + (z_{i,m} - z_{j,m})(z_{i,n-m} - z_{j,n-m})\big]\Big\}t^n = l^2_{e(i,j)}.$$

Therefore, the equation

$$\sum_{m=0}^{n}\big[(x_{i,m} - x_{j,m})(x_{i,n-m} - x_{j,n-m}) + (y_{i,m} - y_{j,m})(y_{i,n-m} - y_{j,n-m}) + (z_{i,m} - z_{j,m})(z_{i,n-m} - z_{j,n-m})\big] = 0 \tag{1}$$

holds for all $n \geq 1$ and all vertices $i$, $j$ joined by an edge.

Formula (1) gives a necessary condition for the polyhedron $Q$ to be flexible. Note that this condition is close to a sufficient one. Indeed, if the set of numbers $x_{i,m}$, $y_{i,m}$, and $z_{i,m}$ satisfies (1) and leads to a convergent Maclaurin series, then the corresponding functions $x_i$, $y_i$, and $z_i$ generate a flex of $Q$.

Let $B : \mathbf{R}^{3v} \times \mathbf{R}^{3v} \to \mathbf{R}^e$ be the bilinear map that takes each couple of vectors

$$X = (x_1, y_1, z_1, x_2, y_2, z_2, \dots, x_v, y_v, z_v)$$

and

$$U = (u_1, v_1, w_1, u_2, v_2, w_2, \dots, u_v, v_v, w_v),$$

into the vector $B(X, U) \in \mathbf{R}^e$ whose $e(i, j)$-th component is equal to

$$(x_i - x_j)(u_i - u_j) + (y_i - y_j)(v_i - v_j) + (z_i - z_j)(w_i - w_j)$$

(recall that $e(i, j)$ stands for the index of the edge joining vertices $i$ and $j$).

Using the operator $B$, we can rewrite (1) in the form

$$\sum_{m=0}^{n} B(X_m, X_{n-m}) = 0, \tag{2}$$

where the following notation is used:

$$X_k = (x_{1,k}, y_{1,k}, z_{1,k}, x_{2,k}, y_{2,k}, z_{2,k}, \dots, x_{v,k}, y_{v,k}, z_{v,k}).$$

**Remark.** Under the customary presentation of the theory of infinitesimal flexes, the vector $X_0 + tX_1 + \cdots + t^k X_k$ is called the $k$-th order infinitesimal flex of the polyhedron $X_0$ (see, for example, [6] or [7]).

We now can reformulate the problem of verifying if a given polyhedron $X_0$ is flexible. It is sufficient to verify that

a) the system (2) has a solution $X_n$ for each $n \geq 1$;

b) the series
$$\sum_{n=0}^{\infty} X_n t^n$$
converges for all numbers $t$ sufficiently close to zero;

c) there exists a number $n \geq 1$ such that the vector $X_0 + tX_1 + \cdots + X_n$ is not generated by any rigid motion of $X_0$.

## 3. Bilinear game

Suppose that $L$ and $K$ are vector spaces, $B : L \times L \to K$ is a bilinear map, and $x_0$ is a vector in $L$.

To make the first step of the bilinear game, one has to find (if possible) a vector $x_1 \in L$ satisfying the equality
$$B(x_0, x_1) + B(x_1, x_0) = 0.$$

Suppose that vectors $x_0, x_1, \dots, x_{n-1}$ are obtained according to the rules of the bilinear game. To make the $n$-th step of the bilinear game, one has to find (if possible) a vector $x_n \in L$ satisfying the equality
$$\sum_{j=0}^{n} B(x_j, x_{n-j}) = 0. \tag{3}$$

**Definition.** *An ordered set of vectors $x_0, x_1, \dots, x_n$ obtained according to the rules of the bilinear game is called a protocol of the bilinear game; the number $n$ is said to be the length of the protocol.*

*We say that a protocol $x_0, x_1, \dots, x_n$ can be extended to a protocol of length $n + p$ if there exist vectors $x_{n+1}, x_{n+2}, \dots, x_{n+p}$ such that each vector $x_{n+k}$, $k = 1, \dots, p$, can be obtained from the vectors with smaller indexes, $x_0, x_1, x_2, \dots, x_{n+k-1}$, according to the rules of the bilinear game, i.e., if, for each $k = 1, \dots, p$, the following equality similar to* (3) *holds:*
$$\sum_{j=0}^{n+k} B(x_j, x_{n+k-j}) = 0.$$

*We say that a protocol $x_0, x_1, \dots, x_n$ can be infinitely extended if there exists an infinite sequence of vectors $x_{n+1}, x_{n+2}, \dots$ such that each vector $x_{n+k}$, $k = 1, 2, \dots,$ can be obtained from the vectors with smaller indexes, $x_0, x_1, x_2, \dots, x_{n+k-1}$, according to the rules of the bilinear game.*

We seek for conditions implying that a protocol $x_0, x_1, \dots, x_n$ of the bilinear game can be infinitely extended.

**Remark.** An important point of our solution to this problem is that one has to return from a protocol of length $n+N$ to the protocol of the smaller length $n$, and only afterwards obtains an infinite extension of the protocol of the smaller length $n$. We demonstrate this point by the following simple example.

**Example.** Suppose that $L$ is a vector space and $e_0, e_1, \dots, e_{n-1}$ is a basis in $L$. Let $B$ be the bilinear map from $L \times L$ into $L$ such that

$$B(e_i, e_j) = \begin{cases} 0, & \text{if } i \neq 1 \text{ or } j \neq 2; \\ e_1, & \text{if } i = 1 \text{ and } j = 2. \end{cases}$$

An easy direct test shows that, for each $N \geq 2$, the protocol $x_0 = e_0$, $x_1 = x_2 = \dots = x_{N-1} = e_1$, $x_N = e_2$ of length $N$ cannot be extended to a protocol of length $N+1$, while the protocol $x_0 = e_0$, $x_1 = x_2 = \dots = x_{N-1} = e_1$ can be infinitely extended.

To each vector $x_k$ belonging to a protocol $x_0, x_1, \dots, x_n$ assign a linear operator $A_k : L \to K$ such that $A_k x = B(x, x_k) + B(x_k, x)$ for all $x \in L$. Denote by $\ker A_0$ the kernel of $A_0$ and by $A_k(\ker A_0)$ the image of $\ker A_0$ under $A_k$. Denote by $\operatorname{im} A_0$ the image of $L$ under $A_0$.

**Theorem 1.** *Let $x_0, x_1, \dots, x_n$ be a protocol of the bilinear game. Suppose that the linear span of the subspaces $\operatorname{im} A_0$ and $A_1(\ker A_0)$ coincides with $K$. In other words $\operatorname{im} A_0 + A_1(\ker A_0) = K$. Then the protocol $x_0, x_1, \dots, x_{n-1}$ can be infinitely extended.*

*Proof.* It suffices to prove that, for every $N \geq n$ and every protocol $x_0, x_1, \dots, x_N$, the protocol $x_0, x_1, \dots, x_{N-1}$ can be extended to a protocol $x_0, x_1, \dots, x_{N-1}, x_N + \xi_N, x_{N+1}$ of length $N+1$.

Let $k = N$ or $k = N+1$. According to the rules of the bilinear game, to obtain a recurrent vector $x_k$, one has to solve the equation

$$\sum_{j=0}^{k} B(x_j, x_{k-j}) = 0 \tag{4}$$

with respect to $x_k$. Using the notation

$$b_{N+1} = -\sum_{j=2}^{N-1} B(x_j, x_{N+1-j}), \quad b_N = -\sum_{j=1}^{N-1} B(x_j, x_{N-j})$$

and grouping terms in (4) by pairs, we obtain the following equations for the redefined $x_{N+1}$ and $x_N$:

$$A_0 x_{N+1} + A_1 x_N = b_{N+1}, \tag{5}$$

$$A_0 x_N = b_N. \tag{6}$$

Note that $A_0$, $A_1$, $b_N$, and $b_{N+1}$ are not functions of $x_N$ and the yet to be defined $x_{N+1}$. By the hypothesis, there are vectors $x_{N+1}$ and $\xi_N$ such that

$$A_0 x_{N+1} + A_1 \xi_N = b_{N+1} - A_1 x_N$$

where $A_0 \xi_N = 0$. Hence the equations

$$A_0 x_{N+1} + A_1(x_N + \xi_N) = b_{N+1},$$

$$A_0(x_N + \xi_N) = b_N$$

follow. Therefore, the ordered set of the vectors $x_0, x_1, \dots, x_N + \xi_N$ is a protocol of the bilinear game that can be extended to the protocol $x_0, x_1, \dots, x_N + \xi_N, x_{N+1}$ of length $N+1$. $\square$

Another set of sufficient conditions for a protocol to be infinitely extendable is given by the following result.

**Theorem 2.** *Let $x_0, x_1, \dots, x_{2n+1}$ be a protocol of the bilinear game and let $L_n$ be the linear span of the vectors $x_1, \dots, x_n$. Suppose that* (i) *$x_j \in L_n$ for each $j = n+1, \dots, 2n+1$;* (ii) *the equation $A_0 x = -B(x_i, x_j) - B(x_j, x_i)$ has a solution $x \in L_n$ for each $1 \le i \le n$, $1 \le j \le n$. Then the protocol $x_0, x_1, \dots, x_{2n+1}$ can be infinitely extended.*

*Proof.* Suppose that, for some $k \ge 0$, there exists an extension $x_0, x_1, \dots, x_{2n+1+k}$ of the protocol $x_0, x_1, \dots, x_{2n+1}$ such that all the vectors $x_{2n+2}, \dots, x_{2n+1+k}$ belong to $L_n$. It suffices to prove that there exists a vector $x_{2n+k+2} \in L_n$ satisfying the equality

$$A_0 x_{2n+k+2} = -\sum_{j=1}^{2n+k+1} B(x_j, x_{2n+k+2-j}). \tag{7}$$

The right-hand side of (7) is a linear combination of the vectors $y = B(x_i, x_j) + B(x_j, x_i)$ $(1 \le i \le n, 1 \le j \le n)$ in the image of $A_0$. Moreover, for each of these vectors $y$, a solution to the equation $A_0 x = y$ can be taken as a linear combination of the vectors $x_1, \dots, x_n$. Therefore, (7) has a solution $x_{2n+k+2} \in L_n$. $\square$

## 4. Sufficient conditions for extendibility of an infinitesimal flex of a polyhedron

Let $Q$ be a closed polyhedron with triangular faces in the Euclidean 3-space. Using $Q$, construct a bilinear mapping $B$ according to Section 2 above. Suppose that the bilinear game associated with $B$ has a protocol $X_0, X_1, \dots, X_n$. Assign a linear operator $A_k$ to each vector $X_k$ according to Section 3 above.

The $k$-th order infinitesimal flex $X_0 + tX_1 + \dots + t^k X_k$ of the polyhedron $Q = X_0 = Q_0$ is said to be *trivial* if there exists a one-parameter motion group of the Euclidean 3-space such that (i) the coordinates $x_i(t)$, $y_i(t)$, and $z_i(t)$ of the $i$-th vertex of the polyhedron $Q_t$ are generated by this group from the coordinates $x_i$, $y_i$, and $z_i$ of the $i$-th vertex of the polyhedron $Q_0$; (ii) each vector $X_j$ $(1 \le j \le k)$ is generated by the functions $x_i(t)$, $y_i(t)$, and $z_i(t)$ according to Section 2 above.

**Lemma.** *Let $X_0, X_1, X_2, \dots, X_n, \dots$ be an infinite sequence of vectors such that* (i) *for each $n$, the vector $X_0 + tX_1 + t^2X_2 + \dots + t^nX_n$ is an $n$-th order infinitesimal flex of the polyhedron $Q = X_0$ and* (ii) *the $n$-th order infinitesimal flex $X_0 + tX_1 + t^2X_2 + \dots + t^nX_n$ is not trivial for some $n$. Then the polyhedron $Q$ is flexible.*

*Proof.* The proof is based on the following algebraic theorem of M. Artin [1]: Given a system of polynomial equations $f(x, y) = 0$, where $f = (f_1, \dots, f_k)$, $x = (x_1, \dots, x_m)$, and $y = (y_1, \dots, y_n)$, there exists an integer $\beta = \beta(m, n, d, \alpha)$ depending on $m$, $n$, on the total degree $d$ of the polynomials $f$, and on a nonnegative integer $\alpha$ such that if $f(x, \overline{y}(x)) = 0 \pmod{x^\beta}$, $\beta = \beta(m, n, d, \alpha)$, for some polynomial $\overline{y}(x)$ (that can be referred to as a *$\beta$-approximate solution*), then the system $f(x, y) = 0$ has a solution $y(x)$ that can be taken in the form of a convergent power series whose initial coefficients coincide with those of the polynomial $\overline{y}(x)$ up to the term $x^\alpha$.

According to Section 2 above, to prove that $Q$ is flexible, we have to solve a system $F(t, Y) = 0$ of algebraic equations of the form

$$[x_i(t) - x_j(t)]^2 + [y_i(t) - y_j(t)]^2 + [z_i(t) - z_j(t)]^2 = l^2_{e(i,j)}$$

with respect to the unknown variables $t$ and $Y(t) = (x_1(t), y_1(t), z_1(t), \dots, x_v(t), y_v(t), z_v(t))$ (the first variable is not explicitly used in the system $F(t, Y) = 0$).

Let $X_0 + tX_1 + t^2X_2 + \dots + t^\alpha X_\alpha$ be a nontrivial flex of $Q$. Choose $\beta$ as in the Artin theorem. Condition (i) of the lemma implies that there exists a $\beta$-approximate solution $\overline{Y}(t) = X_0 + tX_1 + t^2X_2 + \dots + t^\beta X_\beta$ to the system under consideration. According to the Artin theorem, the system has a solution $\widetilde{Y}(t)$ that can be taken in the form of a convergent power series $\widetilde{Y}(t) = \sum\limits_{j=0}^{\infty} c_j t^j X_j$ such that $c_0 = c_1 = \dots = c_\alpha = 1$. Grouping coordinates of $\widetilde{Y}$ by triples, we obtain an authentic flex of $Q$. Thus, $Q$ is flexible. □

**Remark.** As is mentioned in [3] and [7], the idea of using the Artin theorem to obtain the above lemma was suggested by M. Gromov.

**Corollary 1.** *Let a polyhedron $Q = X_0$ have a second-order infinitesimal flex $X_0 + tX_1 + t^2X_2$ such that* (i) *the first-order infinitesimal flex $X_0 + tX_1$ is not trivial*; (ii) *the linear span of the subspaces* $\operatorname{im} A_0$ *and* $A_1(\ker A_0)$ *coincides with* $\mathbf{R}^e$, *in other words* $\operatorname{im} A_0 + A_1(\ker A_0) = \mathbf{R}^e$. *Then $Q$ is flexible.*

*Proof.* According to Theorem 1, the protocol $X_0, X_1$ can be infinitely extended. Then the statement follows from the lemma. □

**Corollary 2.** *Let a polyhedron $Q = X_0$ have a nontrivial infinitesimal flex $X_0 + tX_1 + \dots + t^{2n+1}X_{2n+1}$ and let $L_n \subset \mathbf{R}^{3v}$ be the linear span of the vectors $X_1, \dots, X_n$. Suppose that*

(i) $X_j \in L_n$ *for each* $j = n+1, \dots, 2n+1$; (ii) *the equation* $A_0X = -B(X_i, X_j) - B(X_j, X_i)$ *has a solution* $X \in L_n$ *for each* $1 \le i \le n$, $1 \le j \le n$. *Then* $Q$ *is flexible.*

*Proof.* According to Theorem 2, the protocol $X_0, X_1, \dots X_n$ can be infinitely extended. Then the statement follows from the lemma. □

## 5. Computer experiments

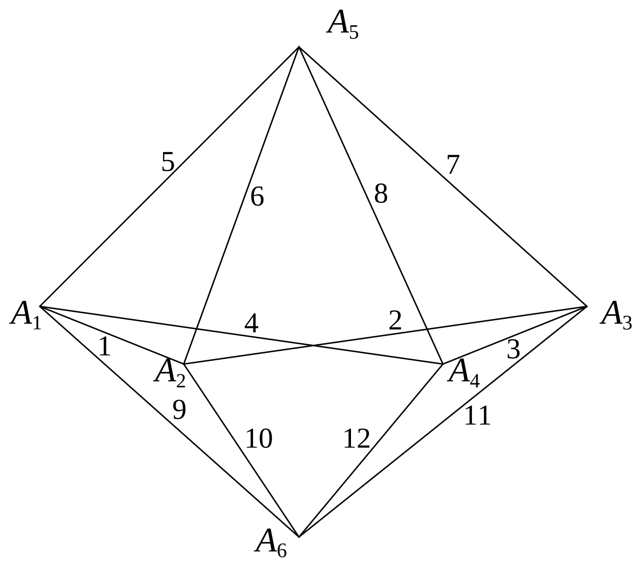
Consider the Bricard octahedron $Q$ of first type with the following vertex coordinates: $A_1 = (2,0,0)$, $A_2 = (1,\sqrt{3},0)$, $A_3 = (-2,0,0)$, $A_4 = (-1,\sqrt{3},0)$, $A_5 = (0,0,1)$, and $A_6 = (0,0,-1)$. It is presupposed that the vertices $A_1$, $A_2$, $A_3$, and $A_4$ lie on the equator of $Q$ in the order named, while $A_5$ and $A_6$ are the north and the south poles of $Q$ (see, e.g., [7] for more details about Bricard octahedra).

Enumerate the edges of $Q$ according to the following table:

| | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| **1** | * | 1 | * | 4 | 5 | 9 |
| **2** | 1 | * | 2 | * | 6 | 10 |
| **3** | * | 2 | * | 3 | 7 | 11 |
| **4** | 4 | * | 3 | * | 8 | 12 |
| **5** | 5 | 6 | 7 | 8 | * | * |
| **6** | 9 | 10 | 11 | 12 | * | * |

Here, a number at the intersection of the column with the top entry $i$ and of the row with the left entry $j$ defines the index $e = e(i,j)$ of the edge of $Q$ that joins the $i$-th and the $j$-th vertices of $Q$; the symbol $*$ is used to indicate that the corresponding vertices are not joined by any edge of $Q$.

The following picture gives an indication of the Bricard octahedron under study.



Numbers given near each edge generate the above introduced edge enumeration.

The bilinear map $B : \mathbf{R}^{18} \times \mathbf{R}^{18} \to \mathbf{R}^{12}$ associated with $Q$ according Section 2 above has the following components:

$$
\begin{aligned}
B_1(X,U) &= (x_1 - x_2)(u_1 - u_2) + (y_1 - y_2)(v_1 - v_2) + (z_1 - z_2)(w_1 - w_2),\\
B_2(X,U) &= (x_2 - x_3)(u_2 - u_3) + (y_2 - y_3)(v_2 - v_3) + (z_2 - z_3)(w_2 - w_3),\\
B_3(X,U) &= (x_3 - x_4)(u_3 - u_4) + (y_3 - y_4)(v_3 - v_4) + (z_3 - z_4)(w_3 - w_4),\\
B_4(X,U) &= (x_4 - x_1)(u_4 - u_1) + (y_4 - y_1)(v_4 - v_1) + (z_4 - z_1)(w_4 - w_1),\\
B_5(X,U) &= (x_5 - x_1)(u_5 - u_1) + (y_5 - y_1)(v_5 - v_1) + (z_5 - z_1)(w_5 - w_1),\\
B_6(X,U) &= (x_5 - x_2)(u_5 - u_2) + (y_5 - y_2)(v_5 - v_2) + (z_5 - z_2)(w_5 - w_2),\\
B_7(X,U) &= (x_5 - x_3)(u_5 - u_3) + (y_5 - y_3)(v_5 - v_3) + (z_5 - z_3)(w_5 - w_3),\\
B_8(X,U) &= (x_5 - x_4)(u_5 - u_4) + (y_5 - y_4)(v_5 - v_4) + (z_5 - z_4)(w_5 - w_4),\\
B_9(X,U) &= (x_6 - x_1)(u_6 - u_1) + (y_6 - y_1)(v_6 - v_1) + (z_6 - z_1)(w_6 - w_1),\\
B_{10}(X,U) &= (x_6 - x_2)(u_6 - u_2) + (y_6 - y_2)(v_6 - v_2) + (z_6 - z_2)(w_6 - w_2),\\
B_{11}(X,U) &= (x_6 - x_3)(u_6 - u_3) + (y_6 - y_3)(v_6 - v_3) + (z_6 - z_3)(w_6 - w_3),\\
B_{12}(X,U) &= (x_6 - x_4)(u_6 - u_4) + (y_6 - y_4)(v_6 - v_4) + (z_6 - z_4)(w_6 - w_4),
\end{aligned}
$$

where

$$X = (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, x_4, y_4, z_4, x_5, y_5, z_5, x_6, y_6, z_6)$$

and

$$U = (u_1, v_1, w_1, u_2, v_2, w_2, u_3, v_3, w_3, u_4, v_4, w_4, u_5, v_5, w_5, u_6, v_6, w_6)$$

are arbitrary vectors in $\mathbf{R}^{18}$.

It is well-known that the Bricard octahedron is flexible. For the sake of convenience, suppose that the vertices $A_5$ and $A_6$ have the following coordinates when the absolute value of the time variable $t$ is small enough: $A_5(t) = (0, 0, 1-t)$ and $A_6(t) = (0, 0, -1+t)$. Moreover, suppose that the edges $A_1A_2$ and $A_3A_4$ are symmetric to each other with respect to the plane spanned by the vectors $(1, 0, 0)$ and $(0, 0, 1)$. Under these conditions, we have the following equations to determine the coordinates of the points $A_1(t) = (x_1(t), y_1(t), 0)$, $A_2(t) = (x_2(t), y_2(t), 0)$, $A_3(t) = (x_1(t), -y_1(t), 0)$, and $A_4(t) = (x_2(t), -y_2(t), 0)$ as functions of the time variable $t$:

$$
\begin{aligned}
&[x_1(t) - x_2(t)]^2 + [y_1(t) - y_2(t)]^2 = 4,\\
&[x_1(t) + x_2(t)]^2 + [y_1(t) - y_2(t)]^2 = 12,\\
&[x_1(t)]^2 + [y_1(t)]^2 + (1-t)^2 = 5,\\
&[x_2(t)]^2 + [y_2(t)]^2 + (-1+t)^2 = 5.
\end{aligned}
$$

Differentiating these equations with respect to $t$ as it was described in Section 2 above, we obtain the following protocol of the bilinear game associated with the bilinear mapping $B$:

$$X_0 = (2, 0, 0, 1, \sqrt{3}, 0, -2, 0, 0, -1, \sqrt{3}, 0, 0, 0, 1, 0, 0, -1),$$

$$X_1 = \Big(\frac{1}{2}, \frac{2}{\sqrt{3}}, 0, -\frac{1}{4}, \frac{5}{4\sqrt{3}}, 0, -\frac{1}{2}, \frac{2}{\sqrt{3}}, 0, \frac{1}{4}, \frac{5}{4\sqrt{3}}, 0, 0, 0, -1, 0, 0, 1\Big),$$

$$X_2 = \Big(-\frac{31}{48}, -\frac{11}{6\sqrt{3}}, 0, \frac{37}{96}, -\frac{113}{96\sqrt{3}}, 0, \frac{31}{48}, -\frac{11}{6\sqrt{3}}, 0, -\frac{37}{96}, -\frac{113}{96\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0\Big),$$

$$X_3 = \Big(\frac{445}{576}, \frac{223}{144\sqrt{3}}, 0, -\frac{649}{1152}, \frac{1325}{1152\sqrt{3}}, 0, -\frac{445}{576}, \frac{223}{144\sqrt{3}}, 0, \frac{649}{1152}, \frac{1325}{1152\sqrt{3}}, \\ 0, 0, 0, 0, 0, 0, 0\Big),$$

$$X_4 = \Big(-\frac{30239}{27648}, -\frac{3683}{1728\sqrt{3}}, 0, \frac{50249}{55296}, -\frac{101413}{55296\sqrt{3}}, 0, \frac{30239}{27648}, -\frac{3683}{1728\sqrt{3}}, 0, -\frac{50249}{55296}, \\ -\frac{101413}{55296\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0\Big),$$

$$X_5 = \Big(\frac{566191}{331776}, \frac{283771}{82944\sqrt{3}}, 0, -\frac{1027159}{663552}, \frac{2128499}{663552\sqrt{3}}, 0, -\frac{566191}{331776}, \frac{283771}{82944\sqrt{3}}, 0, \frac{1027159}{663552}, \\ \frac{2128499}{663552\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0\Big),$$

$$X_6 = \Big(-\frac{7751729}{2654208}, -\frac{225415}{36864\sqrt{3}}, 0, \frac{14770211}{5308416}, -\frac{3512383}{589824\sqrt{3}}, 0, \frac{7751729}{2654208}, -\frac{225415}{36864\sqrt{3}}, 0, \\ -\frac{14770211}{5308416}, -\frac{3512383}{589824\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0\Big),$$

$$X_7 = \Big(\frac{170000039}{31850496}, \frac{92241497}{7962624\sqrt{3}}, 0, -\frac{332342771}{63700992}, \frac{730291327}{63700992\sqrt{3}}, 0, -\frac{170000039}{31850496}, \\ \frac{92241497}{7962624\sqrt{3}}, 0, \frac{332342771}{63700992}, \frac{730291327}{63700992\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0\Big),$$

$$X_8 = \Big(-\frac{31271000117}{33057647616}, -\frac{2176523761}{95551488\sqrt{3}}, 0, \frac{61898822459}{6115295232}, -\frac{138739916623}{6115295232\sqrt{3}}, 0, \\ \frac{31271000117}{3057647616}, -\frac{2176523761}{95551488\sqrt{3}}, 0, -\frac{61898822459}{6115295232}, -\frac{138739916623}{6115295232\sqrt{3}}, 0, 0, 0, 0, 0, 0, 0\Big),$$

$$X_9 = \Big(\frac{2230982344027}{110075314176}, \frac{1264655218003}{4194304\cdot 3^8\sqrt{3}}, 0, -\frac{4441640263531}{220150628352}, \frac{10101894456215}{220150628352\sqrt{3}}, 0, -\frac{2230982344027}{110075314176}, \frac{1264655218003}{4194304\cdot 3^8\sqrt{3}}, 0, \frac{4441640263531}{220150628352}, \frac{10101894456215}{220150628352\sqrt{3}}, 0,0,0,0,0,0,0\Big),$$

$$X_{10} = \Big(-\frac{108770280601489}{2641807540224}, -\frac{31215406114457}{16777216\cdot 3^9\sqrt{3}}, 0, \frac{217110627870619}{5283615080448}, -\frac{499163140516943}{268435456\cdot 3^9\sqrt{3}}, 0, \frac{108770280601489}{2641807540224}, -\frac{31215406114457}{16777216\cdot 3^9\sqrt{3}}, 0, -\frac{217110627870619}{5283615080448}, -\frac{499163140516943}{268435456\cdot 3^9\sqrt{3}}, 0,0,0,0,0,0,0\Big),$$

$$X_{11} = \Big(\frac{2701985126197979}{31701690482688}, \frac{1564814608325921}{134217728\cdot 3^{10}\sqrt{3}}, 0, -\frac{5399405279552831}{63403380965376}, \frac{12515889452241211}{1073741824\cdot 3^{10}\sqrt{3}}, 0, -\frac{2701985126197979}{31701690482688}, \frac{1564814608325921}{134217728\cdot 3^{10}\sqrt{3}}, 0, \frac{5399405279552831}{63403380965376}, \frac{12515889452241211}{1073741824\cdot 3^{10}\sqrt{3}}, 0,0,0,0,0,0,0\Big).$$

Denote by $L_n$ the linear span of the vectors $X_1, \dots, X_n$. One can verify that $L_1 \subset L_2 \subset L_3 \subset L_4 \subset L_5 = L_6 = L_7 = L_8 = L_9 = L_{10} = L_{11}$. Here, the formula $L_i \subset L_j$ means that $L_i$ is a subset of $L_j$, but $L_i \neq L_j$.

Moreover, direct calculations show that $B(X_i, X_j) \in A_0(L_5)$ for all $i, j = 1, \dots, 5$. Hence, Corollary 2 can be applied to the Bricard octahedron under consideration. In other words, we can assert that the Bricard octahedron is flexible according to Corollary 2.

The conditions of Corollary 1 are not fulfilled for the Bricard octahedron under consideration.

## 6. Final remarks and acknowledgments

**Remark.** (i) One can see from the proof of Theorem 1 that some new interesting theorems can be obtained if some additional information about the behavior of the sums

$$\sum_{j=k}^{n-k} B(x_j, x_{n-j})$$

is derived. We mean a statement of the following form: if sufficiently many sums of this kind lie in a subspace, then one can choose consequent vectors $x_j$ in such a way that all the subsequent sums lie in this subspace too.

(ii) One can reduce the problem of bending a smooth surface and a bar framework to a bilinear game by analogy with Section 2.

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