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The Darboux Mapping of Canal Hypersurfaces

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Abstract. The geometry of canal hypersurfaces of an n-dimensional conformal space C^n is studied. Such hypersurfaces are envelopes of r-parameter families of hyperspheres, $1 \leq r \leq n-2$. In the present paper the conditions that characterize canal hypersurfaces, and which were known earlier, are made more precise. The main attention is given to the study of the Darboux maps of canal hypersurfaces in the de Sitter space M_1^{n+1} and the projective space P^{n+1}. To canal hypersurfaces there correspond r-dimensional spacelike tangentially nondegenerate submanifolds in M_1^{n+1} and tangentially degenerate hypersurfaces of rank r in P^{n+1}. In this connection the problem of existence of singular points on canal hypersurfaces is considered.

0. Introduction

Canal hypersurfaces are envelopes of families of hyperspheres. In a three-dimensional Euclidean space canal surfaces were considered in many classical texts on differential geometry (see, for example, the book [8]). Canal hypersurfaces in a multidimensional Euclidean space were also investigated in many papers. Since the property of a hypersurface to be a canal hypersurface is conformally invariant, it is natural to consider the canal hypersurfaces in a conformal space C^n. Such investigations have been done, for example, in [1], [13], [17] and [19].

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However, in all these works the authors did not note the difference between 1-canal hypersurfaces (i.e., the envelopes of $(n-2)$-parameter families of hyperspheres) and m-canal hypersurfaces, where $m \geq 2$ (i.e., the envelopes of $(n-m-1)$-parameter families of hyperspheres).

In the present paper we discuss this difference, namely the fact that the analytic characterization of 1-canal hypersurfaces is connected with a third-order differential neighborhood of a point x of a hypersurface V^{n-1} while that of m-canal hypersurfaces for $m \geq 2$ is connected with a second-order differential neighborhood of $x \in V^{n-1}$.

After recalling the fundamental equations of the conformal theory of hypersurfaces (Section 1), we find separately an analytic characterization of m-canal hypersurfaces (Section 2) and of 1-canal hypersurfaces (Section 3) and compare them. Note that while Theorem 1 and Theorem 3 were known (see [1], [17] and [6], p. 61, for Theorem 1; [3] and [6], p. 58, for Theorem 3), Theorem 2 in the form as presented in this paper appears for the first time.

In Sections 4 and 5 we study the Darboux mapping of canal hypersurfaces into the de Sitter space M_1^{n+1} and the projective space P^{n+1}, both of dimension $n+1$, and apply this mapping to find a set of their singular points. Theorems 4 and 5 are the core theorems of this paper. They give necessary and sufficient conditions for a submanifold respectively in the de Sitter space M_1^{n+1} and the projective space P^{n+1} to be an image of an m-canal hypersurface of the conformal space C^n under Darboux mapping. Moreover, Theorem 5 establishes the very close connection between m-canal hypersurfaces in C^n and tangentially degenerate submanifolds of P^{n+1}.

Note that in this paper we consider the real geometry of canal hypersurfaces.

1. The fundamental equations and notions of the conformal theory of hypersurfaces

Let V^{n-1} be a hypersurface of a conformal space C^n. With any point $x \in V^{n-1}$ we associate a family of conformal frames consisting of two points A_0 and A_{n+1} and n hyperspheres $A_i, i = 1, \ldots, n-1$, and A_n passing through the points A_0 and A_{n+1}. We assume that the hypersphere A_n is tangent to V^{n-1} at the point x, and the hyperspheres A_i are orthogonal to V^{n-1} at x.

Then the elements of the moving frame satisfy the following equations:

$$\left\{\begin{array}{l} (A_0, A_0) = (A_{n+1}, A_{n+1}) = 0, \;\; (A_{n+1}, A_i) = (A_{n+1}, A_n) = 0, \;\; (A_0, A_i) = 0, \\ (A_i, A_n) = 0, \;\; (A_i, A_j) = g_{ij}, \;\; (A_n, A_n) = 1, \;\; (A_0, A_{n+1}) = -1, \end{array}\right. \tag{1}$$

where $(\ ,\ )$ is the scalar product of the frame elements which is defined by the fundamental quadratic form of the space C^n:

$$(X, X) = g_{ij} x^i x^j + (x^n)^2 - 2x^0 x^{n+1}. \tag{2}$$

Here x^0, x^i, and x^{n+1} are the polyspherical coordinates of elements (points and hyperspheres) of the space C^n. Note that $(X, X) = 0$ if X is a point, $(X, X) > 0$ if X is a hypersphere, and $(X, Y) = 0$ if X is a point and Y is a hypersphere passing through X or X and Y are two orthogonal hyperspheres. The quadratic form $g_{ij} x^i x^j$ occurring in (2) is positive definite, and the signature of (X, X) is $(n+1, 1)$. Note also that the last two relations in (1) are obtained as a result of appropriate normalizations of the hypersphere A_n and the points A_0 and A_{n+1}.

The equations of infinitesimal displacement of the conformal frame $\{A_\xi\}, \xi = 0, 1, \ldots, n+1$, have the form

$$dA_\xi = \omega_\xi^\eta A_\eta, \quad \xi, \eta = 0, 1, \ldots, n+1, \tag{3}$$

where ω_ξ^η are 1-forms satisfying the structure equations of the space C^n:

$$d\omega_\xi^\eta = \omega_\xi^\zeta \wedge \omega_\zeta^\eta. \tag{4}$$

Differentiating (1) by means of (3), we find that the forms ω_ξ^η satisfy the following Pfaffian equations:

$$\begin{cases} \omega_0^{n+1} = 0, \;\; \omega_{n+1}^0 = 0, \;\; \omega_i^n + g_{ij}\omega_n^j = 0, \;\; \omega_i^{n+1} = g_{ij}\omega_0^j, \;\; \omega_0^n - \omega_n^{n+1} = 0, \\ \omega_i^0 = g_{ij}\omega_{n+1}^j, \;\; \omega_n^0 - \omega_{n+1}^n = 0, \;\; dg_{ij} = g_{jk}\omega_i^k + g_{ik}\omega_j^k, \\ \omega_n^n = 0, \;\; \omega_0^0 + \omega_{n+1}^{n+1} = 0, \end{cases} \tag{5}$$

where $\omega^i = \omega_0^i$ are basis forms of the hypersurface V^{n-1}.

Since the hypersphere A_n is tangent to V^{n-1}, we have $(dA_0, A_n) = 0$. By (3), this implies that

$$\omega_0^n = 0. \tag{6}$$

Equations (3) and (6) imply that

$$dA_0 = \omega_0^0 A_0 + \omega^i A_i, \tag{7}$$

where $\omega^i = \omega_0^i$ are basis forms of the hypersurface V^{n-1}. It follows from (7) that

$$g = (dA_0, dA_0) = g_{ij}\omega^i\omega^j \tag{8}$$

is a positive definite quadratic form determining a conformal structure on V^{n-1}. The quantities g_{ij} are the components of a relative $(0,2)$-tensor which is defined in a first-order differential neighborhood of a point $x \in V^{n-1}$.

It follows from (6) and (5) that

$$\omega_n^{n+1} = 0. \tag{9}$$

Taking the exterior derivative of equation (6), we obtain the exterior quadratic equation $\omega_i^n \wedge \omega^i = 0$. Applying Cartan's lemma to this equation, we find that

$$\omega_i^n = \lambda_{ij}\omega^j, \;\; \lambda_{ij} = \lambda_{ji}. \tag{10}$$

Taking exterior derivatives of equation (10) and applying Cartan's lemma to the exterior quadratic equation obtained, we arrive at the equations

$$d\lambda_{ij} - \lambda_{kj}\omega_i^k - \lambda_{ik}\omega_j^k + \lambda_{ij}\omega_0^0 + g_{ij}\omega_n^0 = \lambda_{ijk}\omega^k, \tag{11}$$

where λ_{ijk} are symmetric with respect to all indices.

The quantities $\{g_{ij}\}, \{\lambda_{ij}, g_{ij}\}$, and $\{\lambda_{ijk}, \lambda_{ij}, g_{ij}\}$ form the geometric objects of V^{n-1} which are connected with differential neighborhoods of a point $x \in V^{n-1}$ of orders 1, 2, and 3, respectively.

The object $\{\lambda_{ij}, g_{ij}\}$ allows us to construct the following $(0,2)$-tensor:

$$a_{ij} = \lambda_{ij} - \lambda g_{ij}, \tag{12}$$

where

$$\lambda = \frac{1}{n-1} g^{ij} \lambda_{ij}. \tag{13}$$

The tensor a_{ij} is apolar to the tensor g_{ij}:

$$a_{ij} g^{ij} = 0, \tag{14}$$

where g^{ij} is the inverse tensor of the tensor g_{ij}.

If a_{ij} is a nondegenerate tensor (i.e., if $\det(a_{ij}) \neq 0$), then the object $\{\lambda_{ijk}, \lambda_{ij}, g_{ij}\}$ allows us to construct the following $(0,3)$-tensor associated with a third-order differential neighborhood of a point $x \in V^{n-1}$:

$$a_{ijk} = \lambda_{ijk} + 3(a_{(ij} g_{k)l} - g_{(ij} a_{k)l}) \mu^l, \tag{15}$$

where

$$\mu^i = a^{ij} \lambda_j, \quad \lambda_k = \frac{1}{n-1} g^{ij} \lambda_{ijk}, \tag{16}$$

and a^{ij} is the inverse tensor of the tensor a_{ij}.

It is easy to check that the tensor a_{ijk} is also apolar to the tensor g_{ij}:

$$a_{ijk} g^{ij} = 0. \tag{17}$$

We recall now the definition of the curvature lines on a hypersurface $V^{n-1} \subset C^n$. Consider a symmetric affinor

$$a^i_j = g^{ik} a_{kj}, \tag{18}$$

which is constructed by means of the tensors g_{ij} and a_{ij} and is called the *affinor of Burali-Forti* (see [9]). It is easy to see that the tensor a^i_j is symmetric, and by (14), it is trace-free.

The directions that emanate from the point A_0 and are invariant with respect to the affinor a^i_j (the eigendirections of the affinor a^i_j) are called the *principal directions* of the hypersurface V^{n-1}. They are determined by the equations

$$a^i_j \xi^j = s \xi^i. \tag{19}$$

The system (19) has a nontrivial solution if and only if

$$\det(a^i_j - s\delta^i_j) = 0. \tag{20}$$

Equation (20) is equivalent to the equation

$$\det(a_{ij} - s g_{ij}) = 0. \tag{21}$$

Equation (20) is the characteristic equation of the affinor a^i_j. Its roots are the eigenvalues of the affinor a^i_j. Since the affinor a^i_j is symmetric, its eigenvalues are real, and, by (18) and (14), their sum is equal to 0.

If all eigenvalues of the affinor a^i_j are mutually distinct, then the corresponding principal directions are orthogonal. The curves of V^{n-1} enveloping the principal directions are called the *curvature lines*. If this is the case, we say that *the hypersurface* V^{n-1} *carries a unique net of curvature lines.*

If the affinor a^i_j has an eigenvalue of multiplicity $m > 1$, then it defines an m-dimensional domain L^m of principal directions, and each direction of L^m is a principal direction.

Consider a hypersphere $A_n + sA_0$ that is tangent to the hypersurface V^{n-1} at its point A_0. Let us find those directions on V^{n-1} along which this hypersphere has a second-order tangency with V^{n-1}. Such directions are determined by the equation

$$(d^2A_0, A_n + sA_0) = (\lambda_{ij} - sg_{ij})\omega^i\omega^j = 0.$$

Thus the desired directions constitute a cone of second order with its vertex at the point A_0. In general, the rank of the quadratic form $(a_{ij} - sg_{ij})\omega^i\omega^j$ is equal to $n-1$, since the tensor g_{ij} is nondegenerate. However, for some values of s, this rank can be reduced. These values of s are determined by the equation

$$\det(\lambda_{ij} - sg_{ij}) = 0. \tag{22}$$

Essentially equation (22) coincides with equation (20). Suppose that s_h are the roots of equation (22). Then the hypersphere $B_h = A_n + s_hA_0$ has a second-order tangency with the hypersurface V^{n-1} along the directions defined by the equation

$$(\lambda_{ij} - s_hg_{ij})\omega^i\omega^j = 0. \tag{23}$$

The rank of the quadratic form in the left-hand side of equation (23) is equal to $n - m - 1$, where m is the multiplicity of the root s_h of the characteristic equation (22). Thus equation (23) defines a degenerate cone with an m-dimensional vertex. This vertex is defined by the system of equations

$$(\lambda_{ij} - s_hg_{ij})\omega^j = 0 \tag{24}$$

and coincides with the m-dimensional eigendirection L^m corresponding to the m-multiple root s_h. If s_h is a simple root of equation (22), then the vertex of the corresponding cone is one-dimensional.

2. Analytic characterization of m-canal hypersurfaces for $m \geq 2$

We now consider a hypersurface V^{n-1} whose affinor a^j_i has an eigenvalue of multiplicity m at each point $A_0 \in V^{n-1}$. Suppose, for example, that all eigenvalues of the affinor a^j_i satisfy the conditions

$$a_1 = a_2 = \ldots = a_m = a, \quad m \geq 2, \quad a \neq a_p, \quad p = m+1, \ldots, n-1. \tag{25}$$

Then the rank of the tensor $a_i^j - a\delta_i^j$ is equal to $r = n - m - 1$, and at each point A_0 the system of equations (19) for $s = a$ determines an m-dimensional subspace L^m of principal directions corresponding to this m-multiple root a.

If a hypersurface V^{n-1} is the envelope of an r-parameter family of hyperspheres and $m = n - r - 1$, it is called an *m-canal hypersurface.* Such a hypersurface carries an r-parameter family of m-dimensional spherical characteristics S^m.

We will now prove the following theorem:

Theorem 1. *For $m \geq 2$, a hypersurface V^{n-1} is an m-canal hypersurface if and only if its tensor a_{ij} has an eigenvalue of multiplicity m.*

Proof. First we will prove the *sufficiency.* Let L^m be a subspace of principal directions corresponding to the root a of multiplicity m of the affinor a_j^i of the hypersurface V^{n-1}. With a point $A_0 \in V^{n-1}$ we associate a frame bundle in such a way that its hyperspheres A_a, $a = 1, \ldots, m$, are orthogonal to the subspace L^m, and the hyperspheres A_p, $p = m+1, \ldots, n-1$, are tangent to L^m. Then, in the chosen frame bundle we have

$$(g_{ij}) = \begin{pmatrix} g_{ab} & 0 \\ 0 & g_{pq} \end{pmatrix}, \quad (\lambda_{ij}) = \begin{pmatrix} \widetilde{\lambda} g_{ab} & 0 \\ 0 & \lambda_{pq} \end{pmatrix}, \quad (a_{ij}) = \begin{pmatrix} a g_{ab} & 0 \\ 0 & a_{pq} \end{pmatrix}, \tag{26}$$

where $\widetilde{\lambda} = a + \lambda$ and $a = -\frac{1}{m} a_{ab} g^{ab}$.

The hypersphere $B = A_n + \widetilde{\lambda} A_0$ is a contact hypersphere for the subspace L^m. In fact, we have

$$(d^2 A_0, B) = (\lambda_{ij} - \widetilde{\lambda} g_{ij}) \omega^i \omega^j = (\lambda_{pq} - \widetilde{\lambda} g_{pq}) \omega^p \omega^q,$$

and as a result, the hypersphere B has a second order tangency along the direction L^m defined by the equations $\omega^p = 0$.

We specialize our frame bundle by taking $A_n = B$. Then we have $\widetilde{\lambda} = 0$ and

$$(\lambda_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_{pq} \end{pmatrix}, \tag{27}$$

where (λ_{pq}) is a nondegenerate matrix.

We will now write the differential equations which the quantities g_{ij} and λ_{ij} satisfy in our new specialized frame bundle. It follows from (5) that

$$\left\{ \begin{array}{l} dg_{ab} = g_{ac} \omega_b^c + g_{cb} \omega_a^c, \\ g_{ab} \omega_p^b + g_{pq} \omega_a^q = 0, \\ dg_{pq} = g_{pt} \omega_q^t + g_{tq} \omega_a^p, \end{array} \right. \tag{28}$$

and it follows from (11) that

$$\left\{ \begin{array}{l} g_{ab} \omega_n^0 = \lambda_{abk} \omega^k, \;\; \lambda_{pq} \omega_a^q = -\lambda_{apk} \omega^k, \\ d\lambda_{pq} - \lambda_{pt} \omega_q^t - \lambda_{tq} \omega_p^t + \lambda_{pq} \omega_0^0 + g_{pq} \omega_n^0 = \lambda_{pqk} \omega^k, \end{array} \right. \tag{29}$$

where here and in what follows the index t has the same range as the indices p and q: $t = m + 1, \ldots, n - 1$.

It follows from the first equation of (29) that

$$\omega_n^0 = b_k \omega^k. \tag{30}$$

Substituting this value into the first equation of (29), we find that

$$\lambda_{abk} = g_{ab} b_k. \tag{31}$$

But the quantities λ_{ijk} are symmetric. Thus

$$\lambda_{abc} = g_{ab} b_c = g_{bc} b_a = g_{ca} b_b.$$

Contracting the middle equation with the tensor g^{ab}, we obtain

$$m b_c = b_c, \qquad c = 1, \ldots, m.$$

Since we assume that $m \geq 2$, it follows that

$$b_c = 0, \qquad c = 1, \ldots, m.$$

Thus it follows from equation (31) that

$$\lambda_{abc} = 0, \;\; \lambda_{abp} = g_{ab} b_p, \tag{32}$$

and equation (30) takes the form

$$\omega_n^0 = b_p \omega^p. \tag{33}$$

Note that the above computation is the key step in the proof of Theorem 1. For $m = 1$ a relation similar to the first equation of (32) is not valid.

Now the second equation of system (29) can be written as

$$d\lambda_{pq} - \lambda_{pt}\omega_q^t - \lambda_{tq}\omega_p^t + \lambda_{pq}\omega_0^0 = \lambda_{pqa}\omega^a + (\lambda_{pqt} - g_{pq} b_t)\omega^t. \tag{34}$$

This shows that the quantities λ_{pq} form a tensor with respect to admissible transformations of specialized frames. By (27), this tensor is nondegenerate since the rank of the matrix (λ_{ij}) is equal to r. Hence we can write the second equation of (29) in the form

$$\omega_a^p = -\lambda^{pq}(g_{ab} b_q \omega^b + \lambda_{aqt}\omega^t). \tag{35}$$

Next we consider the family of contact hyperspheres A_n. Differentiating A_n and applying equations (5), we find that

$$dA_n = \omega_n^0 A_0 + \omega_n^a A_a + \omega_n^p A_p.$$

But the form ω_n^0 is expressed by formula (33), and by (27) we have

$$\omega_n^a = 0, \;\; \omega_n^p = -\lambda_q^p \omega^q, \tag{36}$$

where $\lambda_q^p = g^{pt}\lambda_{tq}$ is a nondegenerate affinor. Thus we have

$$dA_n = (b_p A_0 - \lambda_p^q A_q)\omega^p. \tag{37}$$

Hence the hypersphere A_n depends on $r = n - m - 1$ parameters. Since by (35) we have

$$d\omega^p \equiv 0 \pmod{\omega^p},$$

the system of equations

$$\omega^p = 0$$

is completely integrable on the hypersurface V^{n-1}. On V^{n-1} these equations define m-dimensional characteristics of the family of hyperspheres A_n, and along these characteristics the hyperspheres A_n are tangent to the hypersurface V^{n-1}. These characteristics are the intersections of the hypersphere A_n with the linearly independent hyperspheres

$$C_p = b_p A_0 - \lambda_p^q A_q. \tag{38}$$

and they are m-dimensional spherical generators of the hypersurface V^{n-1}.

Thus, if m eigenvalues of the tensor a_{ij} of the hypersurface V^{n-1} coincide, where $m \geq 2$, then this hypersurface is an m-canal hypersurface.

Let us prove the *necessity* of the theorem conditions: if a hypersurface V^{n-1} is an m-canal hypersurface where $m \geq 2$, then its tensor a_{ij} has an eigenvalue of multiplicity m. In fact, an m-canal hypersurface is the envelope of an r-parameter family of contact hyperspheres B, where $r = n - m - 1$. By specializing our moving frame, we make its hypersphere A_n coincide with the hypersphere B. Since

$$dA_n = \omega_n^0 A_0 + \omega_n^i A_i, \tag{39}$$

the forms ω_n^0 and ω_n^i must be expressed in terms of r linearly independent forms ω^p. But we have

$$\omega_n^i = -g^{ij}\lambda_{jk}\omega^k.$$

Thus the rank of the matrix (λ_{ij}) is equal to r, and by (12), the tensor a_{ij} has precisely m equal eigenvalues. This concludes the proof of Theorem 1.

Note that if $r = 0$, then all eigenvalues of the tensor a_{ij} coincide, and by the apolarity condition (14), all these eigenvalues are equal to 0. It follows that $a_{ij} = 0$ and $\omega_n^i = 0$. Exterior differentiation of the last equations leads to the exterior quadratic equations

$$\omega_n^0 \wedge \omega_0^i = 0,$$

where $i = 1, \ldots, n-1$ and $n - 1 \geq 2$. Thus the last exterior equations imply that $\omega_n^0 = 0$. By (39), this means that the hypersphere is fixed, and the hypersurface V^{n-1} coincides with the hypersphere A_n or with its open part.

3. Analytic characterization of 1-canal hypersurfaces

As we saw in Section 2, for $m \geq 2$, the analytic characterization of m-canal hypersurfaces is expressed in terms of the quantities connected with a second-order neighborhood of a point $A_0 \in V^{n-1}$. In contrast, for $m = 1$, the analytic characterization of 1-canal hypersurfaces is expressed in terms of the quantities connected with a third-order neighborhood of a point $A_0 \in V^{n-1}$.

Namely the following theorem is valid:

Theorem 2. *Suppose that s_1 is a simple eigenvalue of the affinor a^i_j of the hypersurface V^{n-1} and that V^{n-1} is referred to a frame whose hypersphere A_1 is orthogonal to the corresponding eigendirection of this affinor and whose hyperspheres $A_p, p \neq 1$ are tangent to this direction. Then the hypersurface V^{n-1} is a 1-canal hypersurface for this eigendirection if and only if in the above frame the diagonal component a_{111} of the tensor a_{ijk} vanishes.*

Proof. On the hypersurface V^{n-1} we consider a family of contact hyperspheres $B_1 = A_n + s_1 A_0$, where s_1 is a simple root of characteristic equation (22). We specialize a family of frames associated with a point $A_0 \in V^{n-1}$ in such a way that the tangent hypersphere A_n coincides with the contact hypersphere B_1. Then $s_1 = 0$, and the hypersphere A_n has a second-order tangency with the hypersurface V^{n-1} along the direction determined by the equations $\omega^p = 0,\ p = 2, \ldots, n-1$. If in addition we specialize our moving frame as indicated in the theorem condition, then the matrices $(g_{ij}), (\lambda_{ij})$, and (a_{ij}) take the form

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & g_{pq} \end{pmatrix}, \quad (\lambda_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_{pq} \end{pmatrix}, \quad (a_{ij}) = \begin{pmatrix} a & 0 \\ 0 & a_{pq} \end{pmatrix}, \tag{40}$$

where $p, q = 2, \ldots, n-1$; g_{pq}, λ_{pq}, and a_{pq} are nondegenerate tensors, and $a = -g^{pq} a_{pq}$.

Now those of equations (5), which the components of the tensor g_{ij} satisfy, and also equations (11) take the form

$$\begin{cases} \omega^1_1 = 0, \quad \omega^1_p + g_{pq}\omega^q_1 = 0, \\ dg_{pq} = g_{pt}\omega^t_q + g_{tq}\omega^t_p, \end{cases} \tag{41}$$

and

$$\begin{cases} \omega^0_n = \lambda_{11k}\omega^k, \quad \lambda_{pq}\omega^q_1 = -\lambda_{1pk}\omega^k, \\ d\lambda_{pq} - \lambda_{pt}\omega^t_q - \lambda_{tq}\omega^t_p + \lambda_{pq}\omega^0_0 + g_{pq}\omega^0_n = \lambda_{pqk}\omega^k, \end{cases} \tag{42}$$

where $p, q, t = 2, \ldots, n-1$.

Next we find the differential of the contact hypersphere A_n when it is moving along the hypersurface V^{n-1}:

$$dA_n = \omega^0_n A_0 + \omega^1_n A_1 + \omega^p_n A_p.$$

The form ω^0_n occurring in the last equation is expressed by formula (42), and by (40), we have

$$\omega^1_n = 0, \quad \omega^p_n = -\lambda^p_q \omega^q,$$

where $\lambda^p_q = g^{pt}\lambda_{tq}$ is a nondegenerate affinor. This implies that

$$dA_n = \lambda_{111}\omega^1 A_0 + (\lambda_{11p}A_0 - \lambda^q_p A_q)\omega^p.$$

By (40), it follows from (15) that $\lambda_{111} = a_{111}$. Thus the last equation can be written in the form

$$dA_n = a_{111}\omega^1 A_0 + (\lambda_{11p}A_0 - \lambda^q_p A_q)\omega^p. \tag{43}$$

This shows that the hypersphere A_n depends on $n-2$ parameters if and only if $a_{111} = 0$. In this case the hypersurface V^{n-1} is the envelope of an $(n-2)$-parameter family of hyperspheres

A_n, that is, the hypersurface V^{n-1} is a 1-canal hypersurface. This completes the proof of Theorem 2.

Characteristics of the $(n-2)$-parameter family of contact hyperspheres A_n are circles S^1 which are generators of the hypersurface V^{n-1}. These circles form the first family of curvature lines of the hypersurface V^{n-1}.

Note that relation (43) differs from relation (37) which is valid for $m \geq 2$ by the presence of the first term containing the coefficient a_{111}. Namely this distinguishes the cases $m = 1$ and $m \geq 2$. For a 1-canal hypersurface we have $a_{111} = 0$. If in addition to this condition we set $\lambda_{11p} = b_p$ in equation (43), then equation (43) completely coincides with equation (37). We will use equation (37) for any $m = 1, 2, \ldots, n-2$. This allows us to consider the cases $m = 1$ and $m \geq 2$ simultaneously.

Theorem 2 implies the following method for determining whether a hypersurface V^{n-1}, whose affinor a^i_j has a simple eigenvalue s_i, is a 1-canal hypersurface. To this end, we must choose a frame subbundle associated with V^{n-1} in such a way that its hypersphere A_i is orthogonal to the corresponding eigendirection and compute the tensor a_{ijk} in this frame. If its diagonal component a_{iii} vanishes on V^{n-1}, then the hypersurface V^{n-1} is a 1-canal hypersurface for the direction $\omega^k = 0, k \neq i$.

If on a hypersurface V^{n-1} the conditions $a_{iii} = 0$ hold for all values of i, then this hypersurface is the envelope of $n-1$ families of hyperspheres, each of which depends on $n-2$ parameters. Such a hypersurface is a multidimensional analog of the well-known Dupin cyclide (see [15], [18], and the book [10] where one can find a detailed bibliography on Dupin's cyclide).

Since the tensor a_{ijk} is apolar to the tensor g_{ij}, in our moving frame we find that if $n = 3$, then the condition $a_{iii} = 0$ implies that $a_{iij} = 0$; that is, the conditions $a_{ijk} = 0$ hold for any values of the indices i, j, and k. Since a_{ijk} is a tensor, it vanishes not only in the special frame under consideration but also in any first-order frame.

Thus we have proved the following theorem:

Theorem 3. *A two-dimensional surface V^2 of a three-dimensional conformal space C^3 is a Dupin cyclide if and only if its tensor a_{ijk}, determined by a third-order differential neighborhood, vanishes.*

This differential geometric characterization of the Dupin cyclide is of invariant nature; that is, it does not depend on either the choice of a coordinate system on the surface V^2 or the choice of a conformal frame associated with the surface V^2. Since the tensor a_{ijk} can also be calculated for a surface V^2 of a three-dimensional Euclidean space, *the invariant characterization of the Dupin cyclides we have obtained is also valid for the Dupin cyclides in the Euclidean space R^3.*

4. Submanifolds in the de Sitter space

In Sections 4 and 5 we will not distinguish the cases $m = 1$ and $m \geq 2$. Consider the Darboux mapping of the conformal space C^n (see [12] or [14], §§28–29, or [6], Sections 1.1 and 1.3). Under this mapping to a point of C^n there corresponds a point of an oval hyperquadric Q^n of the projective space P^{n+1}. The hyperquadric Q^n is called the *Darboux hyperquadric.*

To a hypersphere of C^n there corresponds a point of P^{n+1} lying outside of the Darboux hyperquadric $Q^n \subset P^{n+1}$. To the group of conformal transformations of the space C^n there corresponds the group of projective transformations of the space P^{n+1} whose transformations map the hyperquadric Q^n onto itself. To any geometric property of the preimages of the space C^n there corresponds a certain property of the images of the space P^{n+1} connected with the hyperquadric Q^n.

A domain outside of the Darboux hyperquadric $Q^n \subset P^{n+1}$, onto which the hyperspheres of the space C^n are mapped, is a pseudo-Riemannian manifold of index 1 and of constant positive curvature. This manifold is called the *de Sitter space* and is denoted by M_1^{n+1} (see, for example, [20]). The hyperquadric Q^n is the absolute of the space M_1^{n+1}.

Let x be an arbitrary point of the space M_1^{n+1}. The tangent lines to the hyperquadric Q^n passing through the point x form a cone C_x with vertex at x. This cone is called the *isotropic cone*. The cone C_x separates all straight lines passing through the point x into spacelike straight lines that do not have points in common with the hyperquadric Q^n, timelike straight lines intersecting Q^n at two different points, and lightlike straight lines that are tangent to the hyperquadric Q^n. The lightlike straight lines are generators of the cone C_x. This is the reason that the cone C_x is also called the *light cone*.

To a spacelike straight line $l \subset M_1^{n+1}$ there corresponds an elliptic pencil of hyperspheres in the conformal space C^n. All hyperspheres of this pencil pass through a common $(n-2)$-sphere S^{n-2} (the center of this pencil). The sphere S^{n-2} is the intersection of the hyperquadric Q^n and an $(n-1)$-dimensional subspace of the space P^{n+1} which is polar conjugate to the line l with respect to the hyperquadric Q^n. To a timelike straight line $l \subset M_1^{n+1}$ there corresponds a hyperbolic pencil of hyperspheres in the space C^n. Two arbitrary hyperspheres of this pencil do not have common points, and the pencil contains two hyperspheres of zero radius which correspond to the points of intersection of the straight line l and the hyperquadric Q^n. Finally, to a lightlike straight line $l \subset M_1^{n+1}$ there corresponds a parabolic pencil of hyperspheres in the space C^n consisting of hyperspheres tangent one to another at a point that is a unique hypersphere of zero radius belonging to this pencil.

The curves in the space M_1^{n+1} are also divided into three classes: spacelike, timelike, and lightlike. It is not difficult to prove (see [7]) that lightlike curves are always straight lines, and that to these lines there corresponds a parabolic pencil of hyperspheres in C^n. To a spacelike curve $x = x(t)$ of the space M_1^{n+1}, for which $x'(t) \neq 0$ and $x''(t) \neq 0$, there corresponds a one-parameter family of hyperspheres in C^n having an $(n-2)$-canal hypersurface as its envelope. Characteristics of this canal hypersurface are $(n-2)$-dimensional spheres that correspond to spacelike tangents to the curve $x(t)$. Finally, to a lightlike curve $x = x(t)$ of the space M_1^{n+1}, there corresponds a one-parameter family of hyperspheres in C^n not having an envelope. In fact, neighboring hyperspheres of this family that are determined by timelike tangents to the curve $x(t)$ do not have common points, and as a result, the family does not possess characteristics.

In the de Sitter space M_1^{n+1} we consider a submanifold U^r of dimension $r,\ 1 \leq r \leq n-2$. Such submanifolds are also divided into three classes: spacelike, timelike, and lightlike. The tangent subspaces to spacelike submanifolds do not have common points with the Darboux hyperquadric Q^n, the tangent subspaces to timelike submanifolds intersect Q^n along an $(r-1)$-dimensional sphere, and the tangent subspaces to lightlike submanifolds are tangent to Q^n.

Consider now an m-canal hypersurface in the conformal space C^n. By definition, such a hypersurface is the envelope of an r-parameter differentiable family of hyperspheres S^{n-1}, where $r = n - m - 1$. To such a family there corresponds an r-dimensional differentiable submanifold U^r in the space M_1^{n+1}. However, this submanifold is not arbitrary. Namely, the following theorem is valid:

Theorem 4. *A differentiable submanifold U^r of dimension r in the space M_1^{n+1} corresponds to an m-canal hypersurface V^{n-1}, $m = n - r - 1$, in the conformal space C^n if and only if U^r is tangentially nondegenerate and spacelike.*

Proof. In fact, let V^{n-1} be an m-canal hypersurface in $C^n, m = 1, \ldots, n-2$. As we showed in Sections 2 and 3, for a set of contact hyperspheres of V^{n-1} whose family is enveloped by the hypersurface V^{n-1} itself, condition (37) is satisfied. This condition can be written in the form

$$dA_n = \omega^p C_p,$$

where the hyperspheres C_p are defined by formula (38). In the de Sitter space M_1^{n+1}, to an m-canal hypersurface V^{n-1} there corresponds a submanifold U^r described by the point A_n corresponding to the hypersphere $A_n \subset C^n$ whose tangent subspace is determined by the points A_n and C_p. An arbitrary point z of this tangent subspace can be written in the form

$$z = z^n A_n + z^p C_p = z^n A_n + z^p (b_p A_0 - \lambda_p^q A_q).$$

Substituting the coordinates of this point into equation (2), we find that

$$(z, z) = g_{st} \lambda_i^s \lambda_q^t z^p z^q + (z^n)^2 > 0.$$

It follows that the tangent subspace to the submanifold U^r at the point A_n have no common points with the hyperquadric Q^n, that is, the tangent subspace to U^r at A_n is spacelike. Since this is true for any point $A_n \in U^r$, the submanifold U^r corresponding to the family of hyperspheres A_n in the space M_1^{n+1} is spacelike.

The tangential nondegeneracy of the submanifold U^r follows from the fact that the tangent subspaces $T_{A_n}(U^r)$ of such a submanifold correspond to the characteristic spheres S^m of the hypersurface $V^{n-1} \subset C^n$ that depend on r parameters, and as a result, the tangent subspaces $T_{A_n}(U^r)$ depend on the same number of parameters.

If a submanifold U^r in the space M_1^{n+1} is timelike or lightlike, then it contains curves which correspond to one-parameter families of hyperspheres in C^n not possessing envelopes. Thus an r-parameter family of hyperspheres in C^n, whose image in M_1^{n+1} is the submanifold U^r, also does not possess an envelope, and hence such a family does not define a canal hypersurface.

5. The Darboux mapping of m-canal hypersurfaces $V^{n-1} \subset C^n$

In this section we will consider the same Darboux mapping which was considered in Section 4. However, since here we will be interested in projective properties of this mapping and its images, we will consider the mapping of the conformal space C^n onto a projective space P^{n+1} instead of the mapping of C^n onto the de Sitter space M_1^{n+1}.

A canal hypersurface V^{n-1} is the envelope of an r-parameter family of contact hyperspheres B, where $r = n - m - 1, r \leq n - 2$. As we showed in Sections 2 and 3, by specializing moving frames we can make the hypersphere A_n coincide with the hypersphere B.

Under the Darboux mapping, to the hypersphere A_n there corresponds a point x of the projective space P^{n+1} lying outside of the Darboux hyperquadric $Q^n \subset P^{n+1}$ onto which the points of the conformal space C^n are mapped bijectively. The polar hyperplane of the point x with respect to Q^n is a hyperplane ξ intersecting Q^n at points which are images of points of the hypersphere $A_n \subset C^n$. The point x describes in P^{n+1} a tangentially nondegenerate spacelike submanifold U^r, and its polar hyperplane ξ envelopes a tangentially degenerate hypersurface U^n of rank r (see [2] or [5], Ch. 4). The hypersurface U^n carries timelike plane generators α^{m+1} of dimension $m+1$ which are polar conjugate to the r-dimensional tangent subspaces $T_x(U^r)$ of the submanifold U^r. At each point of a generator α^{m+1} the tangent hyperplane to the hypersurface U^n coincides with the hyperplane ξ. Thus we prove the following result:

Theorem 5. *The hypersurface U^n of the projective space P^{n+1}, that is the image of an m-canal hypersurface V^{n-1} under the Darboux mapping, is a tangentially degenerate hypersurface of rank $r = n - m - 1$ with $(m+1)$-dimensional plane generators.*

A plane generator α^{m+1} of a tangentially degenerate hypersurface U^n carries a focus surface $\mathcal{F}$, $\dim \mathcal{F} = m$, formed by singular points of this generator (see [2] and [4]). The submanifold $\mathcal{F}$ is an algebraic hypersurface of degree r in the generator α^{m+1} of the hypersurface U^n that corresponds to a canal hypersurface $V^{n-1} \subset C^n$ under the Darboux mapping.

We will find the equation of the focus surface $\mathcal{F}$ of the generator α^{m+1} of a hypersurface $U^n \subset P^{n+1}$ that corresponds to an m-canal hypersurface of the conformal space C^n under the Darboux mapping. As we already noted above, for canal hypersurfaces V^{n-1} the differential (37) and (43) of contact hyperspheres will have the same form if we set $\lambda_{11p} = b_p$ in equations (43). This is the reason that we can consider these two cases simultaneously.

Relations (37) are preserved under the Darboux mapping. Since the point $x \in U^r$ coincides with the point A_n, then by (37), the tangent subspace $T_x(U^r)$ to this submanifold U^r is determined by the points A_n and $C_p = b_p A_0 - \lambda_p^q A_q$ of the space P^{n+1}. If $x = x^0 A_0 + x^a A_a + x^p A_p + x^n A_n + x^{n+1} A_{n+1}$ is an arbitrary point of the plane generator $\alpha^{m+1} \subset U^n$ that is polar conjugate to the subspace $T_x(U^r)$, then the generator α^{m+1} is defined by the equations

$$(x, A_n) = x^n = 0, \;\; (x, C_p) = -b_p x^{n+1} - \lambda_{pq} x^q = 0.$$

Thus an arbitrary point x of the α^{m+1} can be expressed as follows:

$$x = x^0 A_0 + x^a A_a + x^{n+1}(A_{n+1} - \lambda^{pq} b_q A_p),$$

where λ^{pq} is the inverse tensor of the tensor λ_{pq}.

We can further specialize the frame bundle associated with the submanifold $U^n \subset P^{n+1}$ if we make the point A_{n+1} coincide with the point $A_{n+1} - \lambda^{pq} b_q A_p$ located in the generator α^{m+1}. Then we obtain

$$b_p = 0,$$

and the points A_0, A_a and A_{n+1} become the basis points of the generator α^{m+1}. Now it follows from equation (32) that

$$\lambda_{abk} = 0,$$

and as a result, the first equation of (29) takes the form

$$\omega_n^0 = 0. \tag{44}$$

Now the second equation of (29) gives

$$\lambda_{pq}\omega_a^q = -\lambda_{apq}\omega^q,$$

and this implies that

$$\omega_a^p = -\lambda^{pt}\lambda_{atq}\omega^q = \lambda_{aq}^p\omega^q, \tag{45}$$

where $\lambda_{aq}^p = -\lambda^{pt}\lambda_{atq}$.

Exterior differentiation of equation (44) and application of (27) gives

$$\lambda_p^q\omega_q^0 \wedge \omega_0^p = 0.$$

The last equation implies that

$$\omega_p^0 = c_{pq}\omega^q, \tag{46}$$

and the coefficients c_{pq} satisfy the conditions:

$$\lambda_p^t c_{tq} = \lambda_q^t c_{tp}, \tag{47}$$

which can be obtained if one substitutes decompositions (46) into preceding exterior equations.

Differentiating the basis points, the points A_0, A_p and A_{n+1} of the generator α^{m+1} and applying equations (5), (6), and (44), we find that

$$\begin{aligned} dA_0 &= \omega_0^0 A_0 + \omega_0^a A_a + \omega_0^p A_p, \\ dA_a &= \omega_a^0 A_0 + \omega_a^b A_b + \omega_a^p A_p + \omega_a^{n+1} A_{n+1}, \\ dA_{n+1} &= \omega_{n+1}^a A_a + \omega_{n+1}^p A_p - \omega_0^0 A_{n+1}. \end{aligned} \tag{48}$$

First it follows from equations (48) that the differential of an arbitrary point $x = x^0 A_0 + x^a A_a + x^{n+1} A_{n+1}$ of the generator $\alpha^{m+1} \subset U^n$ belongs to the tangent hyperplane ξ of the hypersurface U^n defined by the equation $x^n = 0$, and this confirms one more time the result of Theorem 5.

Note that equations (48) allow us to find the equations of the focus surface $\mathcal{F}$ of the generator α^{m+1}. In fact, a *singular point* x of the generator α^{m+1} is a point at which the dimension of the tangent subspace $T_x(U^n)$ is less than n. By (48), we can write this analytically as follows:

$$dx \equiv (x^0\omega_0^p + x^a\omega_a^p + x^{n+1}\omega_{n+1}^p)A_p \pmod{\alpha^{m+1}}. \tag{49}$$

Since the forms ω_a^p are expressed in terms of the basis forms by formulas (45) and since by (5) and (46) the forms ω_{n+1}^p are expressed by the formulas

$$\omega_{n+1}^p = g^{pq}\omega_q^0 = g^{pt}c_{tq}\omega^q = c_q^p\omega^q,$$

where $c^p_q = g^{pt} c_{tq}$, relation (49) takes the form

$$dx \equiv (x^0 \delta^p_q + x^a \lambda^p_{aq} + x^{n+1} c^p_q) \omega^q A_p \pmod{\alpha^{m+1}}. \tag{50}$$

The dimension of the tangent subspace to the hypersurface U^n is lowered at those points in which the determinant of the matrix of coefficients of relation (50) vanishes. Therefore the equation of the focus surface $\mathcal{F}$ in the generator α^{m+1} has the form

$$\det(x^0 \delta^p_q + x^a \lambda^p_{aq} + x^{n+1} c^p_q) = 0, \tag{51}$$

and is an algebraic equation of degree $r = n - m - 1$. Thus the hypersurface $\mathcal{F} \subset \alpha^{m+1}$ is an algebraic hypersurface of order r.

The plane α^{m+1} intersects the hyperquadric Q^n at the image of an m-dimensional sphere $S^m \subset C^n$ (a generator of the m-canal hypersurface $V^{n-1} \subset C^n$). By (2), the equation of this generator in α^{m+1} has the form

$$(x, x) = g_{ab} x^a x^b - 2x^0 x^{n+1} = 0. \tag{52}$$

The intersection $\mathcal{F} \cap S^m$ defines singular points on the generator S^m of the hypersurface V^{n-1}. Since we consider only the real geometry of canal hypersurfaces, we are interested in real singular points only. We denote the set of such points by Σ, $\Sigma = \mathrm{Re}(\mathcal{F} \cap S^m)$.

The classification of canal hypersurfaces V^{n-1} is connected with the structure of the sets Σ on the generators S^m. In general, such a set is an algebraic submanifold in α^{m+1} which has codimension 2 and order $2(n - m - 1)$. However, in real geometry this set can be the empty set; then the generator $S^m \subset V^{n-1}$ does not have singularities.

A detailed investigation of the structure of the set Σ defined by equations (51) and (52) is a part of real algebraic geometry and falls outside of the scope of this paper.

As an example, we consider a two-dimensional canal surface V^2 in the conformal space C^3 (see [16]). In this case we have $n = 3, m = 1$, and $r = 1$. The formulas of Sections 3 and 5 are still valid for this case but we should set $p, q, t = 2$ in these formulas. In addition, we can assume that $g_{22} = 1$. The formulas (41) take the form

$$\omega^1_1 = \omega^2_2 = 0, \;\; \omega^1_2 + \omega^2_1 = 0.$$

Next it follows from (39) that

$$\omega^1_3 = 0, \;\; \omega^2_3 = -\lambda^2_2 \omega^2,$$

and formula (44) becomes

$$\omega^0_3 = 0.$$

Thus we have

$$dA_3 = -\lambda^2_2 \omega^2 A_2$$

since $(dA_3, dA_3) = (\lambda^2_2 \omega^2)^2 > 0$, and as a result, the point A_3 describes a spacelike curve l in the space M^3_1.

Finally, formulas (45) and (46) take the form

$$\omega^3_1 = \lambda^2_{12} \omega^2, \;\; \omega^0_2 = c_{22} \omega^2.$$

A two-dimensional plane generator α^2 of a tangentially degenerate surface $U^3 \subset P^4$ that corresponds to a canal surface $V^2 \subset C^3$ under the Darboux mapping is defined by the equations $x^2 = x^3 = 0$. This and equation (52) imply that the equation of the generator S^1 of the canal surface V^2 has the form

$$(x^1)^2 - 2x^0 x^3 = 0. \tag{53}$$

The focus hypersurface $\mathcal{F}$ in the generator α^2 becomes a straight line that, by (51), is defined in α^2 by the equation

$$x^0 + \lambda^2_{12} x^1 + c_{22} x^4 = 0. \tag{54}$$

The intersection of the focus straight line (54) with the generator (53) defines singular points of the generator. Excluding x^0 from equations (53) and (54), we arrive at the quadratic equation

$$(x^1)^2 + \lambda^2_{12} x^1 x^4 + c_{22}(x^4)^2 = 0.$$

The solution of this equation is

$$\frac{x^4}{x^1} = -\frac{1}{2}\lambda^2_{12} \pm \frac{1}{2}\sqrt{D},$$

where $D = (\lambda^2_{12})^2 - 4c_{22}$. It follows that for $D > 0$, the generator S^1 carries two real singular points; for $D = 0$, it carries one double singular point; and for $D < 0$, it carries no real singular points.

Note also that if a space curve l, described by the point $A_3 \subset P^4$ lying outside of the Darboux hyperquadric Q^3, is a closed curve, then a canal surface $V^2 \subset C^3$, whose image in P^4 is the curve l, is a tube (see [11]).

Consider, for example, a plane α^2 in the space P^4. If this plane meets the hyperquadric Q^3 along a circle σ, then any spacelike closed curve l lying in this plane defines in the space C^3 a smooth tube without selfintersections. In this case the curve l is homotopic to the circle σ. If the plane α has no common points with the hyperquadric Q^3, then any smooth closed spacelike curve l lying in this plane is spacelike and defines in the space C^3 a smooth tube, but this tube has selfintersections. Finally, if the plane α is tangent to the hyperquadric Q^3 at a point z, then any smooth closed spacelike curve l lying in this plane defines in the space C^3 a canal surface with the unique singular point z. In this case the point z is located inside of the area bounded by the curve l.

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