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On the Exponential Function of Real Splittable and Real Semisimple Lie Groups

Michael Wüstner

*Fachbereich Mathematik, Technische Hochschule Darmstadt,
Schloßgartenstr. 7, D-64289 Darmstadt, Germany
e-mail: wuestner@mathematik.th-darmstadt.de*

Abstract. A Lie group G is called exponential if its exponential function is surjective. This article contains a characterization of real analytic splittable groups and real semisimple Lie groups with surjective exponential function. It is shown that they are exponential if and only if for each nilpotent element x of the Lie algebra $\mathbf{L}(G)$ the centralizer $Z_G(x) = \{g \in G | \operatorname{Ad}(g)(x) = x\}$ is weakly exponential, i.e., has dense exponential image.

Until today the problem of the surjectivity of the exponential function of Lie groups is not solved in general. J. Dixmier gave a characterization for simply connected exponential Lie groups [3], H. L. Lai characterized complex simple Lie groups [11], [12], [13] while D. Ž. Djoković characterized complex algebraic reductive exponential Lie groups [5] and together with T. Q. Nguyen reductive real exponential Lie groups [6]. They gave also a table of all real linear simple exponential Lie groups. The author characterized real solvable exponential Lie groups [17],[19] and complex splittable Lie groups [17],[18]. This note generalizes the results of [6] and [18]. If there is no danger of confusion we denote the Lie algebra of a Lie group G with $\mathfrak{g}$. A Lie group G is called *exponential* if $G = \exp\mathfrak{g}$. It is called *weakly exponential* if $G = \overline{\exp\mathfrak{g}}$. We shall prove that a real analytic splittable group is exponential if and only if the centralizers $Z_G(x)$ of all nilpotent $x \in \mathfrak{g}$ are weakly exponential. Furthermore, we show that this characterization is even true for exponential covering groups of real analytic splittable groups. This is a modification of the analogous statement for real reductive Lie groups [6, Theorem 2.2]. Our present result also generalizes our earlier one in [18, Theorem 4.22], where it is shown that for complex splittable Lie groups the connectedness of

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the centralizers $Z_G(x) = \{g \in G | \operatorname{Ad}(g)(x) = x\}$ and the surjectivity of the exponential map are equivalent conditions.

1. Preliminaries

In this section we summarize some material on splittable Lie groups and on regular elements. In the whole article V denotes a real finite-dimensional vector space. Let $\operatorname{End}(V)$ be the space of all endomorphisms on V. An element $A \subseteq \operatorname{End}(V)$ is called *semisimple*, if for each invariant subspace $U \subseteq V$ there is an invariant subspace $W \subseteq V$ such that $V = U \oplus W$. This is equivalent to the condition that A is diagonalizable with respect to the complexification $V_{\mathbb{C}}$. The endomorphism A is called *nilpotent* if $A^n = 0$. It is called *unipotent*, if $A - \mathrm{id}$ is nilpotent. If λ is an eigenvalue of A then $V_\lambda(A)$ denotes the eigenspace with respect to λ. Let G be a Lie group. An element $g \in G$ is called Ad-*unipotent* if $\operatorname{Ad}(g)$ is unipotent. It is called Ad-*semisimple* if $\operatorname{Ad}(g)$ is semisimple. If we consider $\operatorname{Gl}(V)$, there are elements which are semisimple and unipotent, respectively, in $\operatorname{Gl}(V)$. We call them *semisimple* and *unipotent*, respectively. A semisimple element is Ad-semisimple and a unipotent element is Ad-unipotent but, in general, the converse is wrong. So we have to distinguish between both definitions. Consider the multiplicative Jordan decomposition $g = g_s g_u = g_u g_s$ with semisimple g_s and unipotent g_u, both in $\operatorname{Gl}(V)$. Note that the Jordan decomposition is unique. A subgroup $G \subseteq \operatorname{Gl}(V)$ is called *splittable* if for all $g \in G$ we have $g_s, g_u \in G$. If, in addition, G is analytic, it is called a splittable analytic group. Analogously, let $\mathfrak{g}$ be a Lie algebra. An element $x \in \mathfrak{g}$ is called ad-*semisimple* if $\operatorname{ad} x$ is semisimple, and ad-*nilpotent* if $\operatorname{ad} x$ is nilpotent. Now assume that $\mathfrak{g} \subseteq \mathrm{gl}(V)$ and $x \in \mathfrak{g}$. We call x *semisimple* if it is semisimple in $\mathrm{gl}(V)$ and *nilpotent* if it is nilpotent in $\mathrm{gl}(V)$. If x is semisimple (nilpotent) then it is ad-semisimple (ad-nilpotent) but the converse is wrong in general. But note that in the case of semisimple Lie algebras, ad-semisimple (ad-nilpotent) elements are semisimple (nilpotent) (c.f. [10, Theorem 6.4]). Consider the additive Jordan decomposition $x = x_s + x_n$ with semisimple x_s and nilpotent x_n, both in $\mathrm{gl}(V)$, with $[x_s, x_n] = 0$. This decomposition is also unique. A Lie algebra $\mathfrak{g}$ is called *splittable* if for all $x \in \mathfrak{g}$ we have $x_s, x_n \in \mathfrak{g}$. An analytic subgroup $G \subseteq \operatorname{Gl}(V)$ is splittable if and only if $\mathfrak{g}$ is splittable. This is proved in [18, Proposition 4.17]. Furthermore, in [1, VII, §5, n°5, Corollaire 3] it is shown that a Lie algebra $\mathfrak{g}$ is splittable if and only if the complexification $\mathfrak{g}_{\mathbb{C}}$ is splittable. In particular, an analytic group G is splittable if and only if its complexification $G_{\mathbb{C}}$ is splittable. By the splittability of complex semisimple Lie groups this implies that real semisimple linear Lie groups are splittable. Let G be a subgroup of $\operatorname{Gl}(V)$. The *splittable hull* $E(G)$ of G is defined as the smallest splittable subgroup in $\operatorname{Gl}(V)$ which contains G. If, in addition, G is analytic, its Lie algebra is denoted by $\mathfrak{e}(\mathfrak{g})$. Note that this is the smallest splittable Lie algebra which contains $\mathfrak{g}$. So it is called the *splittable hull* of $\mathfrak{g}$. The *algebraic hull* $A(G)$ of G is defined as the smallest algebraic subgroup in $\operatorname{Gl}(V)$ which contains G.

Now we review some standard facts on regular elements of Lie groups and Lie algebras. An element g of a Lie group G is called *regular* if and only if for each element g' of a neighborhood of g the nilspace of $\operatorname{Ad}(g') - \mathrm{id}$ has the same dimension as the nilspace of $\operatorname{Ad}(g) - \mathrm{id}$. If, in addition, G is connected, g is regular if and only if the dimension of the nilspace is minimal. This dimension is called the *rank* of $\mathfrak{g}$. We will denote by $\operatorname{Reg}(G)$ the set of all regular elements of a Lie group G, by $\operatorname{reg}\mathfrak{g}$ the set of the regular elements of $\mathfrak{g}$ and by $\operatorname{reg}\exp$ the set of the elements of $\mathfrak{g}$ where the exponential function is regular. By [9, Lemma 1.4], we know that $\exp^{-1}(\operatorname{Reg}(G)) = \operatorname{reg}\mathfrak{g} \cap \operatorname{reg}\exp$ holds. Moreover, a regular element g of a connected Lie group G is contained in one and only one Cartan subgroup. The next proposition is a corollary of [9, Lemma 1.4].

Proposition 1.1. *If $g \in \operatorname{Reg}(G) \cap \exp\mathfrak{g}$, i.e, if there is a $y \in \operatorname{reg}\mathfrak{g} \cap \operatorname{reg}\exp$ with $g = \exp y$, then $e^{\operatorname{ad} y}x = x$ implies $[y, x] = 0$.*

We note that, in general, this y is not unique (cf. the circle group). Next we investigate the preimage of unipotent and of semisimple elements of $\operatorname{Gl}(V)$. For this, we need a standard proposition. A proof can be found in [7, Proposition I.2.7].

Proposition 1.2. *Consider the linear Lie group $\operatorname{Gl}(V)$ and its Lie algebra $\operatorname{gl}(V)$. We define $\mathfrak{n}_p = \{x \in \operatorname{gl}(V) | x^p = 0\}$ and $N_p = \{g \in \operatorname{Gl}(V) | g = \mathbf{1} + x, x \in \mathfrak{n}_p\}$. Then $\exp|_{\mathfrak{n}_p} \colon \mathfrak{n}_p \to N_p$ is a diffeomorphism with the inverse function*

$$\log\colon g \mapsto \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(g-\mathbf{1})^k}{k}.$$

In the whole article we make use of the following statement. It ensures that a unipotent g in a splittable analytic group $G \subseteq \operatorname{Gl}(V)$ has a preimage in $\mathfrak{g}$.

Lemma 1.3. *Let $G \subseteq \operatorname{Gl}(V)$ be an analytic group and assume that for nilpotent $x, y \in \mathfrak{g}$ we have $\exp x = \exp y$. Then we get $x = y$. In particular, 0 is the only nilpotent element of $\mathfrak{g}$ which is in $\exp^{-1}(1)$. If, in addition, G is splittable, then for each unipotent $g \in G$ there is a nilpotent $x \in \mathfrak{g}$ with $g = \exp x$.*

Proof. The first part is a consequence of Lemma 1.2 and the second part is shown in [18, Lemma 4.16]. □

Now we turn to semisimple elements. For this, we denote by S the set of all semisimple elements of G and by $\mathfrak{s}$ the set of all semisimple elements of $\mathfrak{g}$.

Proposition 1.4. *Let $G \subseteq \operatorname{Gl}(V)$ be an analytic group. Then we have $\exp^{-1}(S) = \mathfrak{s}$.*

Proof. Since the exponential map of a semisimple endomorphism is semisimple itself, we have $\exp\mathfrak{s} \subseteq S$, hence $\mathfrak{s} \subseteq \exp^{-1}(S)$. Now assume that $y \in \exp^{-1}(S)$ is not semisimple. We consider the splittable hull $E(G)$ of G in $\operatorname{Gl}(\mathfrak{g})$. In $\mathfrak{e}(\mathfrak{g})$, we have the Jordan decomposition $y = y_s + y_n$ with $[y_s, y_n] = 0$ where $y_s \in \mathfrak{e}(\mathfrak{g})$ is semisimple and $y_n \in \mathfrak{e}(\mathfrak{g})$ is nilpotent. Then

$$\exp_{E(G)} y = \exp_{E(G)} y_s \exp_{E(G)} y_n$$

is a semisimple element. By the uniqueness of the Jordan decomposition we get $\exp_{E(G)} y_n = 1$. By Lemma 1.3, this implies $y_n = 0$. Thus y is semisimple. □

Corollary 1.5. *Let G be a real connected Lie group. Then $\mathfrak{s}^{\operatorname{ad}} = \exp^{-1}(S^{\operatorname{Ad}})$.*

2. Cartan subgroups

Cartan subgroups are a very useful tool in Lie theory. We shall need them, too. Let $\mathfrak{h}$ be a Cartan algebra and $N_G(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in G. Furthermore, $\Lambda_{\mathfrak{h}} \subseteq \mathfrak{h}_{\mathbb{C}}^*$ denotes the set of the roots belonging to $\mathfrak{h}$. Let $\mathfrak{g}_{\mathbb{C}}^{\lambda}$ be the root space with respect to λ. We define

$$C(\mathfrak{h}) = \{g \in N_G(\mathfrak{h}) | \lambda \circ \mathrm{Ad}(g)|_{\mathfrak{h}_{\mathbb{C}}} = \lambda \,\text{for all}\ \lambda \in \Lambda\}.$$

A subgroup H of G is called *Cartan subgroup* if and only if $\mathbf{L}(H)$ is a Cartan subalgebra and $H = C(\mathbf{L}(H))$. Moreover, since $\mathrm{Ad}(g^{-1}) \cdot \mathfrak{g}_{\mathbb{C}}^{\lambda} = \mathfrak{g}_{\mathbb{C}}^{\lambda \circ \mathrm{Ad}(g)}$ we have $C(\mathfrak{h}) = \{g \in G | \mathrm{Ad}(g) \cdot \mathfrak{g}_{\mathbb{C}}^{\lambda} = \mathfrak{g}_{\mathbb{C}}^{\lambda}$ for all $\lambda \in \Lambda \cup \{0\}\}$. In [2, IV.5.1], C. Chevalley has given another definition of Cartan subgroups. In [14, Theorem A.4], K. H. Neeb has proven that these definitions are equivalent for connected Lie groups. We observe that the center of G is contained in each Cartan subgroup of G. We need the following characterization of weakly exponential Lie groups ([14, Theorem 12] and [8, Corollary 18]).

Theorem 2.1. *Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Then the following conditions are equivalent:*

(i) *G is weakly exponential.*

(ii) *All Cartan subgroups of G are connected.*

(iii) *Each regular element of G is contained in* $\exp \mathfrak{g}$.

Proposition 2.2. *If $G \subseteq \mathrm{Gl}(n, \mathbb{C})$ is a connected complex analytic subgroup, then each semisimple element g is contained in the center of a Cartan subgroup.*

Proof. We consider the algebraic hull $A(G)$. By [16, p. 69], each semisimple element g is contained in a $\mathbb{C}$-torus of $A(G)$. Hence it is contained in the center of a Cartan subgroup C of $A(G)$. Thus, g is contained in the center of $C \cap G$. By [2, IV.5.1. Proposition 1], $C \cap G$ is a Cartan subgroup of G. □

Lemma 2.3. *Let $G \subseteq \mathrm{Gl}(n, \mathbb{R})$ an analytic subgroup and $G_{\mathbb{C}}$ its complexification, i.e., the smallest complex analytic subgroup of* $\mathrm{Gl}(\mathfrak{g}_{\mathbb{C}})$ *which contains G cf.* [15, p. 35]. *Let $g \in G$ and denote by $Z_G(g)$ the centralizer of g in G. Then $\mathbf{L}(Z_{G_{\mathbb{C}}}(g)) = \mathbf{L}(Z_G(g))_{\mathbb{C}}$.*

Proof. We note that $\mathbf{L}(Z_{G_{\mathbb{C}}}(g)) \subseteq \mathfrak{g}_{\mathbb{C}}$. Thus we have $\mathbf{L}(Z_{G_{\mathbb{C}}}(g)) = \{x + iy \mid x, y \in \mathfrak{g}$ and $\exp \mathbb{C}(x + iy) \subseteq Z_{G_{\mathbb{C}}}(g)\}$. So, $x + iy \in \mathbf{L}(Z_{G_{\mathbb{C}}}(g))$ is equivalent to $\exp c(x + iy) = g \exp(c(x + iy))g^{-1} = \exp \mathrm{Ad}(g)(c(x + iy))$ for all $c \in \mathbb{C}$. Since exp maps a neighborhood of 0 homeomorphically onto a neighborhood of 1, this is equivalent to $\mathrm{Ad}(g)(x + iy) = x + iy$. Since $\mathrm{Ad}(g)$ acts linear on $\mathfrak{g}_{\mathbb{C}}$, we have $\mathrm{Ad}(g)x + i\,\mathrm{Ad}(g)y = \mathrm{Ad}(g)(x + iy) = x + iy$. Considering the real and the imaginary part, this is equivalent to $\mathrm{Ad}(g)x = x$ and $\mathrm{Ad}(g)y = y$, i.e., $x, y \in \mathbf{L}(Z_G(g))$. This implies the claim. □

Proposition 2.4. *If $G \subseteq \mathrm{Gl}(n, \mathbb{R})$ is an analytic subgroup, then each semisimple element g is contained in a Cartan subgroup.*

Proof. Denote by $Z_G(g)$ the centralizer of g. We set $\mathfrak{w} := \mathbf{L}(Z_G(g))$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{w}$. Thus $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{w}_{\mathbb{C}}$. By Proposition 2.2, g is contained in the center of a Cartan subgroup C of $G_{\mathbb{C}}$. Thus, C is contained in $Z_{G_{\mathbb{C}}}(g)$, hence $\mathfrak{c} \subseteq \mathbf{L}(Z_{G_{\mathbb{C}}}(g)) = \mathfrak{w}_{\mathbb{C}}$ by Lemma 2.3. But $\mathfrak{c}$ is maximal nilpotent and the normalizer $\mathfrak{n}_{\mathfrak{w}_{\mathbb{C}}}(\mathfrak{c})$ of $\mathfrak{c}$ in $\mathfrak{w}_{\mathbb{C}}$ is contained in $\mathfrak{n}_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{c}) \cap \mathfrak{w}_{\mathbb{C}} = \mathfrak{c} \cap \mathfrak{w}_{\mathbb{C}} = \mathfrak{c} \subseteq \mathfrak{n}_{\mathfrak{w}_{\mathbb{C}}}(\mathfrak{c})$. Thus, $\mathfrak{c}$ is a Cartan subalgebra of $\mathfrak{w}_{\mathbb{C}}$. Since in a complex Lie algebra all Cartan subalgebras are conjugate, $\mathfrak{h}_{\mathbb{C}}$ is conjugate to $\mathfrak{c}$, hence a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Since a subalgebra of $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if its complexification is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $g \in C(\mathfrak{h})$. □

Proposition 2.5. *Let G be a connected real Lie group. Then each* Ad*-semisimple element of G is contained in some Cartan subgroup.*

Proof. We note that $\mathrm{Ad}(G)$ is an analytic subgroup of $\mathrm{Gl}(\mathfrak{g})$. Consider the Lie group topology of $\mathrm{Ad}(G)$, i.e., the topology which makes $\mathrm{Ad}(G)$ to a Lie group. (Note that this, in general, is not the induced topology.) Together with this topology $\mathrm{Ad}(G)$ is isomorphic to G/Z by the canonical decomposition of functions. Since the center of G is contained in each Cartan subgroup H of G, by [9, Proposition 1.13] the Cartan subgroups of G/Z are exactly of the form H/Z with Cartan subgroup H in G. Thus the Cartan subgroups of $\mathrm{Ad}(G)$ are exactly of the form $\mathrm{Ad}(H)$ with Cartan subgroup H in G. Thus, there is a Cartan subgroup H of G with $g \in H$ if and only if there is a Cartan subgroup $W \subseteq \mathrm{Ad}(G)$ with $\mathrm{Ad}(g) \in W$. Thus, we have reduced our problem to the linear case. Proposition 2.4 implies the claim. □

We get the following result:

Proposition 2.6. *Let G be a real Lie group and $x \in \mathfrak{g}$ nilpotent. Assume that $g \in Z_G(x)$ is* Ad*-semisimple in G. Then it is* Ad*-semisimple in $Z_G(x)$, too. If, in addition, $Z_G(x)$ is connected, then g is contained in a Cartan subgroup of $Z_G(x)$.*

Proof. We observe that $\mathrm{Ad}(g)$ and $\mathrm{ad}\, x$ commute with each other. Thus they leave their eigenspaces invariant. In particular, we have $\mathrm{Ad}(g).\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(x) \subseteq \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(x)$ and hence

$$\begin{aligned} \mathrm{Ad}(g).((\mathfrak{g}_{\mathbb{C}})_\lambda(\mathrm{Ad}(g)) \cap \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(x)) &\subseteq (\mathrm{Ad}(g).(\mathfrak{g}_{\mathbb{C}})_\lambda(\mathrm{Ad}(g))) \cap (\mathrm{Ad}(g).\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(x)) \\ &\subseteq ((\mathfrak{g}_{\mathbb{C}})_\lambda(\mathrm{Ad}(g))) \cap \mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}(x). \end{aligned}$$

Thus $\mathrm{Ad}(g)|_{\mathfrak{z}_{\mathfrak{g}_{\mathbb{C}}}}$ is diagonalizable. Proposition 2.5 implies the last statement. □

3. Real analytic exponential splittable groups

We develop the main result of this article. For the following lemma we do not need that the group is splittable.

Lemma 3.1. *Let $G \subseteq \mathrm{Gl}(V)$ be a real analytic exponential group. Then for each nilpotent $x \in \mathfrak{g}$ and each semisimple $g_s \in Z_G(x)$ there is a semisimple $x_s \in \mathfrak{z}_{\mathfrak{g}}(x)$ such that $g_s = \exp x_s$.*

Proof. Set $g = g_s \exp x$. Since G is exponential, there is a $y \in \mathfrak{g}$ with $g = \exp y$. Now we turn to $E(G)$. We find a semisimple $x_s \in \mathfrak{e}(\mathfrak{g})$ and a nilpotent $x_n \in \mathfrak{e}(\mathfrak{g})$ with $y = x_s + x_n$ and $[x_s, x_n] = 0$. Thus we get $g_s \exp_{E(G)} x = \exp_{E(G)} x_s \exp_{E(G)} x_n$. Moreover, x is also nilpotent and g_s is semisimple. By the uniqueness of the multiplicative Jordan decomposition we have $g_s = \exp_{E(G)} x_s$ and $\exp_{E(G)} x = \exp_{E(G)} x_n$. Lemma 1.3 implies $x = x_n \in \mathfrak{g}$. This means that $x_s = y - x_n \in \mathfrak{g} \cap \mathfrak{z}_{\mathfrak{e}(\mathfrak{g})}(x) = \mathfrak{z}_{\mathfrak{g}}(x)$ and hence $g_s = \exp x_s \in \exp \mathfrak{z}_{\mathfrak{g}}(x)$. □

Proposition 3.2. *If $G \subseteq \mathrm{Gl}(V)$ is a splittable analytic subgroup then the centralizer $Z_G(y)$ of each element $y \in \mathfrak{g}$ is splittable, too.*

Proof. Suppose that $g \in Z_G(y)$. Then we find a semisimple $g_s \in G$ and a unipotent $g_u \in G$ with $g = g_s g_u = g_u g_s$. Note that the semisimple Jordan component is a polynomial in g. Thus $g_s, g_u \in Z_G(y)$. □

Lemma 3.3. *Let $G \subseteq \mathrm{Gl}(V)$ be a real splittable analytic group and assume that for each nilpotent $x \in \mathfrak{g}$ and each semisimple $g_s \in Z_G(x)$ there is a semisimple $x_s \in \mathfrak{z}_{\mathfrak{g}}(x)$ such that $g_s = \exp x_s$. Then the centralizer $Z_G(x)$ is weakly exponential.*

Proof. Let $x \in \mathfrak{g}$ be nilpotent and $g \in Z_G(x)$. Since $Z_G(x)$ is splittable, we find a semisimple $g_s \in Z_G(x)$ and a unipotent $g_u \in Z_G(x)$ with $g = g_s g_u = g_u g_s$. By Lemma 1.3, there is a unique nilpotent $x_n \in \mathfrak{g}$ with $g_u = \exp x_n$. Furthermore, since x_n is exp-regular we have $x_n \in \mathfrak{z}_{\mathfrak{g}}(x)$ because of Proposition 1.1. By assumption, there is a semisimple $x_s \in \mathfrak{z}_{\mathfrak{g}}(x)$ with $g_s = \exp x_s$. Thus $g \in Z_G(x)_0$, i.e., $Z_G(x)$ is connected.

Now assume that, in addition, g is regular in $Z_G(x)$. We note that $g_s g_u = g \in Z_G(x)$ is regular in $Z_G(x)$ if and only if g_s is regular in $Z_G(x)$. So g_s is regular in $Z_G(x)$, hence x_s is exp-regular in $\mathfrak{z}_{\mathfrak{g}}(x)$. But then $e^{\operatorname{ad} x_s} x_n = \mathrm{Ad}(g_s) x_n = x_n$ and Proposition 1.1 implies $[x_s, x_n] = 0$. Thus $[x_s + x_n, x] = 0$ and $g = \exp x_s \exp x_n = \exp(x_s + x_n)$. This means that $g \in \exp \mathfrak{z}_{\mathfrak{g}}(x)$. By Theorem 2.1, the centralizer is weakly exponential. □

Lemma 3.4. *If $G \subseteq \mathrm{Gl}(V)$ is a splittable analytic group and if for each nilpotent $x \in \mathfrak{g}$ the centralizer $Z_G(x)$ is weakly exponential then G is exponential.*

Proof. Let $g \in G$. Since G is splittable, there is a semisimple $g_s \in G$ and a unipotent $g_u \in G$ with $g = g_s g_u = g_u g_s$. By Lemma 1.3, we find a nilpotent $x \in \mathfrak{g}$ with $g_u = \exp x$. Since g_s is semisimple it is Ad-semisimple. We observe that $Z_G(x)$ is connected. By Proposition 2.6, it is even Ad-semisimple in $Z_G(x)$ and contained in a Cartan subgroup of $Z_G(x)$. By Theorem 2.1, the Cartan subgroups of $Z_G(x)$ are connected. Thus there is a $y_s \in \mathfrak{z}_{\mathfrak{g}}(x)$ with $g_s = \exp y_s$. This implies $g = \exp y_s \exp x = \exp(y_s + x)$. □

We summarize our results in the following theorem:

Theorem 3.5. *Let $G \subseteq \mathrm{Gl}(V)$ be a splittable analytic group. Then the following statements are equivalent:*

(i) *G is exponential.*

(ii) *For each nilpotent $x \in \mathfrak{g}$ and each semisimple $g_s \in Z_G(x)$ there is a semisimple $x_s \in \mathfrak{z}_{\mathfrak{g}}(x)$ with $g_s = \exp x_s$.*

(iii) *For each nilpotent $x \in \mathfrak{g}$ the centralizer $Z_G(x)$ is weakly exponential.*

Proof. (i)$\Rightarrow$(ii) is Lemma 3.1 and (ii)$\Rightarrow$(iii) Lemma 3.3. The implication (iii)$\Rightarrow$(i) is Lemma 3.4. □

4. Example

To illustrate our result we consider the analytic subgroup G of $\mathrm{Gl}(\mathbb{R},4)$ defined as follows:

$$G=\left\{\begin{pmatrix} e^t\cos t & e^t\sin t & a & b\\ -e^t\sin t & e^t\cos t & c & d\\ 0 & 0 & e^t\cos t & e^t\sin t\\ 0 & 0 & -e^t\sin t & e^t\cos t\end{pmatrix}\,\middle|\, t,a,b,c,d\in\mathbb{R}\right\}.$$

A standard computation shows that G is splittable but not algebraic. Thus G is a splittable analytic group. Its Lie algebra is given by

$$\mathfrak{g}=\left\{\begin{pmatrix} t & t & a & b\\ -t & t & c & d\\ 0 & 0 & t & t\\ 0 & 0 & -t & t\end{pmatrix}\,\middle|\, t,a,b,c,d\in\mathbb{R}\right\}.$$

The gl(4)-nilpotent elements of $\mathfrak{g}$ are exactly those with $t=0$. Now we consider the centralizer of an element x where only a is not equal to 0. We get

$$\mathrm{Ad}\begin{pmatrix} e^t\cos t & e^t\sin t & a & b\\ -e^t\sin t & e^t\cos t & c & d\\ 0 & 0 & e^t\cos t & e^t\sin t\\ 0 & 0 & -e^t\sin t & e^t\cos t\end{pmatrix}\begin{pmatrix} 0&0&a&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}=$$
$$\begin{pmatrix} 0 & 0 & a\cos^2 t & -a\cos t\sin t\\ 0 & 0 & -a\cos t\sin t & a\sin^2 t\\ 0&0&0&0\\0&0&0&0\end{pmatrix}.$$

So, for each element of $Z_G(x)$ we have $\cos^2 t=1$ and $\sin t=0$. We get

$$Z_G(x)=\left\{\begin{pmatrix} e^t\cos t & e^t\sin t & a & b\\ -e^t\sin t & e^t\cos t & c & d\\ 0 & 0 & e^t\cos t & e^t\sin t\\ 0 & 0 & -e^t\sin t & e^t\cos t\end{pmatrix}\,\middle|\, t\in\pi\mathbb{Z}\right\}.$$

Hence, $Z_G(x)$ is not connected, in particular not weakly exponential. Thus G is not exponential.

5. Exponential covering groups of splittable analytic groups

Now we generalize Theorem 3.5. Assume that G is a covering group of a splittable analytic subgroup of a $\mathrm{Gl}(V)$. Let π denote the covering map. Note that G has a splittable Lie algebra. If G is exponential, then $\pi(G)\subseteq\mathrm{Gl}(V)$ is exponential and if G is weakly exponential then $\pi(G)$ is weakly exponential.

Theorem 5.1. *Let G be a covering group of a splittable analytic group of a* $\mathrm{Gl}(V)$*. Then the following statements are equivalent:*

(i) *G is exponential.*

(ii) *For all nilpotent $x \in \mathfrak{g}$ and each $g_s \in Z_G(x)$ where $\pi(g_s)$ is semisimple, there is a semisimple $x_s \in \mathfrak{z}_{\mathfrak{g}}(x)$ such that $g_s = \exp x_s$.*

(iii) *For all nilpotent $x \in \mathfrak{g}$ the centralizer $Z_G(x)$ is weakly exponential.*

Proof. First, assume that G is exponential. Let $x \in \mathfrak{g}$ be nilpotent and $g_s \in Z_G(x)$ such that $\pi(g_s)$ is semisimple. We consider the element $g \exp x$. Since G is exponential there is a $y \in \mathfrak{g}$ with $g_s \exp x = \exp y$. Since $\mathfrak{g}$ is splittable there is a semisimple y_s and a nilpotent y_n, both in $\mathfrak{g}$, with $[y_s, y_n] = 0$ and $g_s \exp x = \exp(y_s + y_n) = \exp y_s \exp y_n$. Thus, we have $\pi(g_s)\pi(\exp x) = \pi(\exp y_s)\pi(\exp y_n)$. By the uniqueness of the Jordan decomposition we get $\pi(g_s) = \pi(\exp y_s)$ and $\pi(\exp x) = \pi(\exp y_n)$. Lemma 1.3 implies $x = y_n$. Then we get $\exp x = \exp y_n$ and hence $g_s = \exp y_s$. This implies $g_s \in \exp \mathfrak{z}_{\mathfrak{g}}(x)$.

Now suppose that (ii) is true. We prove that $Z_G(x)$ is weakly exponential for all nilpotent $x \in \mathfrak{g}$. By assumption and Theorem 3.5, the centralizer $Z_{\pi(G)}(x)$ is weakly exponential, and hence $\pi(G)$ is exponential. In particular, the Cartan subgroups of $\pi(G)$ are connected. Now let $g \in Z_G(x)$. Since by Proposition 3.2, $Z_{\pi(G)}(x)$ is splittable, there is a semisimple g_s and a unipotent g_u, both in $Z_{\pi(G)}(x)$, with $\pi(g) = g_s g_u = g_u g_s$. By Lemma 1.3, we find a nilpotent $x_n \in \mathfrak{g}$ with $g_u = \pi(\exp x_n)$. Furthermore, we know that g_s is in a (connected) Cartan subgroup of $Z_{\pi(G)}(x)$, hence there is a semisimple $x_s \in \mathfrak{z}_{\mathfrak{g}}(x)$, with $\pi(g_s) = \pi(\exp x_s)$. Thus $g = z \exp x_s \exp x_n$ with some $z \in \ker \pi$, and $\pi(z \exp x_s) = \pi(\exp x_s)$ is semisimple. By (ii) there is a $y \in \mathfrak{z}_{\mathfrak{g}}(x)$ with $z \exp x_s = \exp y$. Thus $g \in (Z_G(x))_0$, hence $Z_G(x)$ is connected.

Now suppose that, in addition, g is regular in $Z_G(x)$. Then $z \exp x_s$ is regular in $Z_G(x)$. Thus, y is exp-regular in $\mathfrak{z}_{\mathfrak{g}}(x)$. Because of $e^{\mathrm{ad}\, y} x_n = x_n$ and the exp-regularity of y, by Proposition 1.1 we conclude $[y, x_n] = 0$ and hence $g = \exp(y + x_n)$. Furthermore, we have $[y + x_n, x] = 0$. This implies the claim.

Next, we prove (iii)$\Rightarrow$(i). Let $g \in G$. We consider the Jordan decomposition $\pi(g)_s \pi(g)_u = \pi(g)_u \pi(g)_s = \pi(g)$. Since $\pi(g)_u$ is unipotent, there is a nilpotent $x \in \mathfrak{g}$ with $\pi(g)_u = \pi(\exp x)$. Moreover, we have $\pi(g)_s \in Z_{\pi(G)}(x)$. Now we consider the element $g \exp(-x)$. Since $\mathrm{Ad}_G = \mathrm{Ad}_{\pi(G)} \circ \pi$, we get $\mathrm{Ad}_G(g \exp(-x)) = \mathrm{Ad}_{\pi(G)}(\pi(g)\pi(\exp(-x))) = \mathrm{Ad}_{\pi(G)}(\pi(g)_s)$. So $g \exp(-x)$ is Ad-semisimple. Moreover, it is contained in $Z_G(x)$. Since $Z_G(x)$ is weakly exponential it is connected. So by Proposition 2.6, $g \exp(-x)$ is contained in a Cartan subgroup of $Z_G(x)$. Theorem 2.1 together with the weak exponentiality of $Z_G(x)$ implies that there is a $w \in \mathfrak{z}_{\mathfrak{g}}(x)$ with $g \exp(-x) = \exp w$. We conclude $g = \exp w \exp x = \exp(w + x)$. This implies statement (i). □

This theorem can be applied to real semisimple Lie groups. Recall that in this case an element $x \in \mathfrak{g} \subseteq \mathrm{gl}(V)$ is nilpotent if and only if it is ad-nilpotent. We note that $\mathrm{Ad}(G)$ is an analytic semisimple subgroup of $\mathrm{Gl}(\mathfrak{g})$ and according to [4, Proposition 4], it is closed. Because of the splittability of linear real semisimple Lie groups Theorem 5.1 still holds for the linear semisimple case. Thus we have:

Corollary 5.2. *Let G be a real semisimple Lie group and $\mathfrak{g} \subseteq \mathrm{gl}(V)$. Then G is exponential if and only if for each nilpotent $x \in \mathfrak{g}$ the centralizer $Z_G(x)$ is weakly exponential.*

Very recently, D. Ž. Djoković told me that he found the same characterization for real semisimple exponential Lie groups. Note that in particular a real semisimple Lie group cannot be exponential if there is any nilpotent $x \in \mathfrak{g} \subseteq \mathrm{gl}(V)$ such that the centralizer $Z_G(x)$ is not connected.

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