Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 39 (1998), No. 1, 31-36.

Surfaces in $\mathbb{P}^4$ with no Quadrisecant Lines *

Marina Bertolini **Cristina Turrini**

Dipartimento di Matematica, Universitá di Milano
e-mail: bertolin@vmimat.mat.unimi.it
turrini@vmimat.mat.unimi.it

Abstract. A classification of all smooth algebraic complex surfaces in $\mathbb{P}^4$, with no quadrisecant lines, is given.

1. Introduction

A classical problem in projective algebraic geometry has been that of describing those projective varieties for which the variety of multisecant lines has dimension different from the expected one (curves in $\mathbb{P}^3$ with no trisecant lines, or the Veronese surface in $\mathbb{P}^5$ are classical examples). Recently the problem has been reconsidered also in connection with the construction of surfaces in $\mathbb{P}^4$ (see e.g. [3]) or with the problem of $k-$spannedness of line bundles (see e.g. [1] e [4]). As to surfaces in $\mathbb{P}^4$, in [2] Aure gives a characterization of surfaces such that the union of trisecant lines does not fill $\mathbb{P}^4$. In this paper, making use of the enumerative formulas for $n-$secants due to Le Barz, we classify surfaces in $\mathbb{P}^4$ with no quadrisecant lines or, which is the same, for which the variety of quadrisecants has dimension less than 2.

More explicitly we prove the following results.

Theorem 1.1. *Let S be a smooth surface in $\mathbb{P}^4$ not containing infinitely many lines and with no quadrisecant lines. Then S is one of the following:*

i) *the projected Veronese surface;*

ii) *the smooth complete intersection of two quadrics;*

iii) *the Castelnuovo surface of degree 5, whose hyperplane sections correspond to plane quartics with a given double point and other seven base points;*

*Research done within the framework of the 40% project "Algebraic Geometry" of the Italian M.U.R.S.T.

iv) *the smooth complete intersection of a quadric and a cubic hypersurface;*

v) *the Bordiga surface of degree* 6*, whose hyperplane sections correspond to plane quartics with ten base points;*

vi) *the projection of the complete intersection of three quadric hypersurfaces in* $\mathbb{P}^5$ *from a point of it;*

vii) *the smooth complete intersection of two cubic hypersurfaces.*

In Theorem 1.1 we assume that S does not contain infinitely many lines in order to apply Le Barz's formulas for quadrisecants. Moreover, in the claim, we make the assumption of smoothness for S but indeed the proof is carried on, more generally, under the assumption that S has only ordinary singularities, in the classical sense (as in [9]). In this case, only one singular surface can appear (see Remark 3.1).

The case in which S contains infinitely many lines is easier and is settled by the following:

Theorem 1.2. *Let S be a smooth surface in $\mathbb{P}^4$ with infinitely many lines and with no quadrisecant lines. Then S is one of the following:*

i) *the rational normal scroll,*

ii) *the elliptic quintic scroll.*

We would like to thank E. Arrondo for useful discussions.

2. Preliminaries and Notation

In the sequel we will denote by S a complex, non degenerate, surface contained in $\mathbb{P}^4$ with ordinary singularities, i.e. S is either smooth or a generic projection of a smooth surface S' in $\mathbb{P}^5$. Hence, as it is classical, S has at most a finite number of ordinary double points. Moreover, following terminology of [9], if X is a subvariety of $\mathbb{P}^n$, a line l in $\mathbb{P}^n$ is a *k-secant* ($k \leq 1$) to X if $l \cap X$ is a scheme of length k.

We will denote by:

$n =$ degree of S

$g =$ sectional genus of S, i.e. the genus of a smooth hyperplane section of it

$t =$ number of trisecant lines of S passing through a generic point of $\mathbb{P}^4$, i. e. the number of triple points of a generic projection S^* of S

$d =$ degree of the double curve γ of S^*

$h =$ number of double points of a generic projection of γ

$\delta =$ number of double points of S.

3. Proof of Theorem 1.1

Lemma 3.1. *Let S be a surface in $\mathbb{P}^4$ with only ordinary singularities. If S has no quadrisecant lines then $n \leq 9$.*

Proof. If S has no quadrisecant lines, its generic hyperplane section C has no quadrisecants. Hence, from [10], we get

$$n(n-2)(n-3)(n-13) - 12d(d-4n+11) = 0 \tag{1}$$

where

$$2d = n^2 - 3n + 2 - 2g.$$

Now, from [5], there are the following two possibilities:

a) C lies on a quadric surface in $\mathbb{P}^3$, or

b) $3(2g-2) \leq n(n-3)$.

In case a), if C is contained in a smooth quadric Q, denote by (α, β) the bidegree of C as a divisor of Q. If either α or β is ≥ 4 then one of the rules of Q is at least a quadrisecant for C; so, if C has no quadrisecants deg $C = n \leq 6$. A similar argument applies if C is contained in a quadric cone.

In case b) we get

$$3d \geq n(n-3).$$

From (1) we have
$$n(n-2)(n-3)(n-13) \geq 4d(n^2 - 15n + 33).$$

As the coefficient of d is positive for $n \leq 2$ or $n \geq 13$ we can distinguish the two following cases:

i) $n \geq 13$ which implies $n^2 - 15n + 54 \leq 0$, contradiction;

ii) $n \leq 12$. In this second case, making use of (1) we get only the following integer solutions for d:

n	d
9	$7, 18$
7	$7, 10$
6	$6, 7$
5	$4, 5$
4	$2, 3$

(2)

Table 1

□

Lemma 3.2. *The only possible surfaces in $\mathbb{P}^4$ with ordinary singularities, not containing infinitely many lines, and with no quadrisecants are the ones listed in Table 2.*

In the following Table 2 we use standard notation: Bl for the blow-up, $V(-,-)$ for the complete intersection, H for the hyperplane section, l for a line. For the description of the surfaces we refer to the known classification of surfaces of small degree (e.g.[12]).

n	g	δ	description of S
9	10	0	V(3,3)
7	5	0	$Bl_p(K3)$, inner projection of $V(2,2,2)$
6	3	0	$S = Bl_{p_1,\dots,p_{10}}(\mathbb{P}^2)$, $H = 4l - p_1 - \cdots - p_{10}$
6	4	0	$V(2,3)$
5	1	1	projection of Del Pezzo surface S' of degree 5
5	2	0	$S = Bl_{2q,p_1,\dots,p_7}(\mathbb{P}^2)$, $H = 4l - 2q - p_1 - \cdots - p_7$
4	0	0	projected Veronese surface
4	1	0	$V(2,2)$

Table 2

Proof. From [11] we get the following relation

$$2\frac{n!}{4!(n-4)!} + t(n-3) + \frac{d(d-n+2) - 3t - \delta}{2} - d\frac{(n-2)!}{2!(n-4)!} = 0.$$

Substituting in this formula the values of n and d listed in Table 1, we see that the only possible integer numerical solutions are the following.

n	d	δ
9	18	$9t - 54$
9	7	$9t + 210$
7	10	$5t - 10$
7	7	$5t + 14$
6	7	$3t - 3$
6	6	$3t$
5	5	t
5	4	t
4	3	$-t + 1$
4	2	$-t$

Now applying the Hodge index theorem ([6]) to the surface S', we get the following inequality

$$2n^2d - 4nd - 8d^2 + 4n\delta + 6nt \leq 0.$$

By this, we see that the only possible numerical solutions are those appearing in Table 2 and the following ones:

1) $n = 5,\ g = 1,\ \delta = 0,$

2) $n = 4,\ g = 0,\ \delta = 1.$

As to these last cases, notice that the former should correspond to a smooth Del Pezzo surface of degree 5 in $\mathbb{P}^4$ which does not exist (e.g.[12]), and the latter is not a surface with ordinary singularities by [13]. □

To conclude the proof of Theorem 1.1 we have only to show that the smooth surfaces listed in Table 2 are indeed with no quadrisecant line. If not, the generic hyperplane through a quadrisecant line would cut S in a curve which is smooth (by Bertini) and which has no quadrisecant, since by [10] (1) applies not only to a generic, but to any smooth curve in $\mathbb{P}^3$.

Remark 3.1. *As to the singular quintic surface, we notice that the Del Pezzo surface S' in $\mathbb{P}^5$ of which S is a projection has a 4-secant plane as it is not three-spanned by [4]. So if S is obtained by S' by a projection from a point of the 4-secant plane, it has a quadrisecant line.*

4. Proof of Theorem 1.2

By [8, Theorem 1.4] it follows immediately:

Lemma 4.1. *Let S be a smooth surface in $\mathbb{P}^4$ with infinitely many lines. Then S is a smooth scroll.*

Now Theorem 1.2 follows from Lemma 4.1, observing that if S is a scroll in $\mathbb{P}^4$, by [7] it can only be the rational normal scroll or the quintic elliptic scroll. The first one indeed has no quadrisecants since its degree is four. The second one cannot have since otherwise it should be rational.

References

[1] Andreatta, M.: *Surfaces of genus ≤ 8 with no trisecant lines.* Arch. Math. **60** (1992), 85–95.

[2] Aure, A.: *The smooth surfaces in $\mathbf{P}^4$ without apparent triple points.* Duke Math. J. **57** 2 (1988), 423–430.

[3] Aure, A.; Ranestad, K.: *The smooth surfaces of degree 9 in $\mathbf{P}^4$.* London Math. Soc. Lecture Note Ser. **179** (1992), 513–540.

[4] Ballico, E.: *On k-spanned projective surfaces.* Algebraic Geometry Proc.(L'Aquila 1988), Lecture Notes in Math. **1417**, Springer Verlag.

[5] Gruson, L.; Peskine, C.: *Genre des courbes dans l'espace projectif.* Algebraic Geometry Proc. (Norway 1977), Lecture Notes in Math. **687**, Springer Verlag, 31–59.

[6] Hartshorne, R.: *Algebraic Geometry.* Grad. Texts in Math. **52**, Springer Verlag, New York - Heidelberg - Berlin, 1977.

[7] Lanteri, A.: *On the existence of scrolls in $\mathbb{P}^4$.* Rend. Accad. Naz. Lincei (VIII), **LXIX** (1980), 223–227.

[8] Lanteri, A.; Palleschi, M.: *Projective manifolds containing many rational curves.* Indiana Univ. Math. J. **36** (4) (1987).

[9] Le Barz, P.: *Formules pour les trisecantes des surfaces algebriques.* L'Enseignement Mathematique **33** (1987), 1–66.

[10] Le Barz, P.: *Validite de certaines formules de geometrie enumerative.* C. R. Acad. Sc. Paris **289**, Serie A, 755.

[11] Le Barz, P.: *Formules pour les multisecantes des surfaces.* C. R. Acad. Sc. Paris **292**, Serie I, 797.

[12] Livorni, E. L.: *On the existence of some surfaces.* Algebraic Geometry Proc. (L'Aquila 1988), Lecture Notes in Math. **1417**, Springer Verlag, 155–179.

[13] Swinnerton Dyer, H. P. F.: *An enumeration of all varieties of degree 4.* Amer. J. Math. **95** (1973), 403–418.

Received September 20, 1996