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Multiplicities of Ideals in Noetherian Rings

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Abstract. Given any ideal I in a Noetherian ring R, we get a multiplicity function $\mathrm{mult}_I\colon \mathrm{Spec}(R) \to \mathbf{N}$. This paper characterizes all such functions.

All rings are assumed to be Noetherian.

Given a function $\psi\colon \mathrm{Spec}(R) \to \mathbf{N}$, how can we tell whether there exists an ideal $I \subseteq R$ such that $\psi(\mathfrak{p}) = \mathrm{mult}_I(\mathfrak{p})$ is the multiplicity of $\mathfrak{p}$ at R/I for all $\mathfrak{p} \in \mathrm{Spec}(R)$? It turns out that there is a simple criterion which characterizes such functions ψ, which will be stated and proved below.

First we recall the definitions and simple preliminary facts. The following set of equivalent definitions of mult_I is taken from [3]. Given any prime ideal $\mathfrak{p}$ in R, let $J = \bigcup_{j>0}(I\colon \mathfrak{p}^j)$, which is the intersection of the primary components of I whose associated primes do not contain $\mathfrak{p}$ (see [1, 3.13]). Note that J is independent of the chosen primary decomposition for I. We define $\mathrm{mult}_I(\mathfrak{p})$ to be the length r of a maximal strictly increasing chain of ideals

$$I \subseteq J_r \subset J_{r-1} \subset \cdots \subset J_2 \subset J_1 \subset J, \tag{1}$$

where each J_i equals $P \cap J$ for some $\mathfrak{p}$-primary ideal P. Equivalently,

(2) $\mathrm{mult}_I(\mathfrak{p})$ is the length of the module $JR_\mathfrak{p}/IR_\mathfrak{p}$;

(3) $\mathrm{mult}_I(\mathfrak{p})$ is the length of the largest ideal of finite length in $R_\mathfrak{p}/IR_\mathfrak{p}$.

Note that $\mathrm{mult}_I(\mathfrak{p}) > 0$ if and only if $\mathfrak{p}$ is associated to I, and that because of (3), we have $\mathrm{mult}_I(\mathfrak{q}) = \mathrm{mult}_{IR_\mathfrak{p}}(\mathfrak{q}R_\mathfrak{p})$ for all primes $\mathfrak{q} \subseteq \mathfrak{p}$; i.e. multiplicity localizes. In particular, $IR_\mathfrak{p} = JR_\mathfrak{p}$ implies $\mathrm{mult}_I(\mathfrak{p}) = \mathrm{mult}_J(\mathfrak{p})$. We are now ready to state the main result.

Theorem 1. *Let $\psi\colon \mathrm{Spec}(R) \to \mathbf{N}$. Then $\psi = \mathrm{mult}_I$ for some ideal I of R if and only if the support $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid \psi(\mathfrak{p}) > 0\}$ of ψ is finite and for every $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\psi(\mathfrak{p}) > \mathrm{mult}_0(\mathfrak{p})$ there exists $\mathfrak{q} \subset \mathfrak{p}$ with $\psi(\mathfrak{q}) < \mathrm{mult}_0(\mathfrak{q})$.*

The idea behind this investigation originally stemmed from the question of which finite sets $X \subseteq \mathrm{Spec}(R)$ satisfy $X = \mathrm{Ass}(R/I)$ for some ideal I of R. We call such a set X an *asset.* The answer to this question turns out to be a little awkward to state, but it follows directly from Theorem 1, using the fact that $\mathfrak{p} \in \mathrm{Ass}(R/I) \Leftrightarrow \mathrm{mult}_I(\mathfrak{p}) \geq 1$.

Corollary 2. *Let $X \subseteq \mathrm{Spec}(R)$ be a finite set. Then X is an asset if and only if every element $\mathfrak{p} \in X$ satisfies at least one of the following conditions:*

(i) $\mathfrak{p} \in \mathrm{Ass}(R)$.

(ii) $\mathfrak{p}$ *contains a prime* $\mathfrak{q}$ *with* $\mathrm{mult}_0(\mathfrak{q}) > 1$.

(iii) $\mathfrak{p}$ *contains a prime in* $\mathrm{Ass}(R) \setminus X$.

In particular, if R is a domain then every finite set of nonzero primes is an asset.

Proof. One only needs to construct a multiplicity function that is 1 on X and 0 elsewhere.

This theorem and corollary are best understood geometrically, when, for instance, R is a finitely generated algebra (not necessarily reduced) over a field. Finding an ideal I such that $\psi = \mathrm{mult}_I$ essentially amounts to finding functions that vanish on various components to specified degrees. Of course, the existence of such functions first depends on the set from which we are looking, i.e. the ring R. The minimal primes of R correspond to the maximal irreducible components of the scheme for which R is the coordinate ring, while the multiplicity of a minimal prime at 0 is the maximum degree to which a function in R can vanish on the corresponding irreducible component. The other associated primes of R correspond to components embedded in larger components on which a function may vanish to a higher degree than on any larger component as a whole. The multiplicity of an embedded prime at 0 represents how much more it is possible to vanish on the corresponding embedded component than on the components in which it is contained. The first point, which will be demonstrated by Proposition 3, is that once a function is required to vanish to the highest possible degree on *all* of the components of R (including the embedded components), it is impossible to vanish to a higher degree anywhere—there is no room left in which to thicken any subset. However, given any leeway whatsoever, which corresponds to letting $\psi(\mathfrak{q}) < \mathrm{mult}_0(\mathfrak{q})$ for some prime $\mathfrak{q}$ contained in $\mathfrak{p}$, it is possible to thicken the subset corresponding to $\mathfrak{p}$ to any desired degree. Note that the only primes $\mathfrak{q}$ which can possibly satisfy $\mathrm{mult}_I(\mathfrak{q}) < \mathrm{mult}_0(\mathfrak{q})$ are the associated primes of R.

The proof of the theorem will follow after a number of smaller results, the first of which will be instrumental in proving the necessity of the condition on ψ. Lemma 4 enables us to add a component (with not too much control over how thick it ends up), while Lemma 5 allows us to reduce over-thickened components to the correct thickness.

Proposition 3. *Let R be a Noetherian ring and $I \subseteq R$ an ideal. The following are equivalent:*

(i) $I = 0$

(ii) $\mathrm{mult}_I(\mathfrak{q}) = \mathrm{mult}_0(\mathfrak{q})$ *for all* $\mathfrak{q} \in \mathrm{Spec}(R)$

(iii) $\mathrm{mult}_I(\mathfrak{q}) \geq \mathrm{mult}_0(\mathfrak{q})$ *for all* $\mathfrak{q} \in \mathrm{Spec}(R)$

(iv) $\mathrm{mult}_I(\mathfrak{q}) \geq \mathrm{mult}_0(\mathfrak{q})$ *for all* $\mathfrak{q} \in \mathrm{Ass}(R)$

Proof. (i) $\Rightarrow$ (ii): Definition.
(ii) $\Rightarrow$ (iii): Obvious.
(iii) $\Rightarrow$ (iv): Obvious.
(iv) $\Rightarrow$ (i): Since $I = 0 \Leftrightarrow IR_\mathfrak{p} = 0$ for all primes $\mathfrak{p} \in \mathrm{Ass}(R)$ and multiplicity localizes, it suffices to show that (iv) $\Rightarrow$ (i) for local rings $(R, \mathfrak{p})$. This will be accomplished by induction on $n := \dim(R, \mathfrak{p})$. If $n = 0$, then $(R, \mathfrak{p})$ is a local Artinian ring with unique prime $\mathfrak{p} \supseteq I$. Hence $R_\mathfrak{p}/IR_\mathfrak{p} = R/I$ has finite length $\mathrm{mult}_I(\mathfrak{p}) = \mathrm{length}(R) - \mathrm{length}(I) = \mathrm{mult}_0(\mathfrak{p}) - \mathrm{length}(I)$. Since $\mathrm{mult}_I(\mathfrak{p}) \geq \mathrm{mult}_0(\mathfrak{p})$, we must have $\mathrm{length}(I) = 0$, so $I = 0$ as required.

Now suppose $n > 0$ and that (iv) $\Rightarrow$ (i) for all local rings of dimension less than n. If I is an ideal of $(R, \mathfrak{p})$ satisfying (iv), then the inductive hypothesis implies that $IR_\mathfrak{q} = 0$ for all $\mathfrak{q} \in \mathrm{Ass}(R) \setminus \{\mathfrak{p}\}$. If $\mathfrak{p} \notin \mathrm{Ass}(R)$ then this already implies $I = 0$, so we may assume that $\mathfrak{p} \in \mathrm{Ass}(R)$. Let $f \in \mathfrak{p}$ and $S = R[f^{-1}]$. Then $\mathrm{Ass}(S) = \{\mathfrak{q}S \mid \mathfrak{q} \in \mathrm{Ass}(R),\ f \notin \mathfrak{q}\}$, so the fact that $IS \cdot S_{\mathfrak{q}S} = IR_\mathfrak{q} = 0$ for all $\mathfrak{q}S \in \mathrm{Ass}(S)$ implies that $IS = 0$. Hence $f^k I = 0$ for some $k \in \mathbf{N}$, whence $\mathfrak{p}^l I = 0$ for some $l \in \mathbf{N}$, so that I has finite length. Thus I is contained in the largest ideal $J \subseteq R$ of finite length. It follows that the largest ideal in R/I of finite length is J/I, and its length is $\mathrm{mult}_I(\mathfrak{p}) = \mathrm{length}(J) - \mathrm{length}(I) = \mathrm{mult}_0(\mathfrak{p}) - \mathrm{length}(I)$. Since we have assumed that $\mathrm{mult}_I(\mathfrak{p}) \geq \mathrm{mult}_0(\mathfrak{p})$, we must have $\mathrm{length}(I) = 0$, i.e. $I = 0$.

Lemma 4. *Let $X \subseteq \mathrm{Spec}(R)$ be a finite subset and $\mathfrak{p} \in X$ a maximal element. Suppose $J \subseteq R$ with $\mathrm{Ass}(R/J) = X \setminus \{\mathfrak{p}\}$ and that P is a $\mathfrak{p}$-primary ideal. Then, writing I for $P \cap J$, we have $\mathrm{mult}_I(\mathfrak{q}) = \mathrm{mult}_J(\mathfrak{q})$ for all $\mathfrak{q} \neq \mathfrak{p}$ and $\mathrm{mult}_I(\mathfrak{p}) = \mathrm{mult}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p})$.*

Proof. The first statement follows because both multiplicities are 0 if $\mathfrak{q} \notin X$ and $IR_\mathfrak{q} = PR_\mathfrak{q} \cap JR_\mathfrak{q} = JR_\mathfrak{q}$ if $\mathfrak{q} \in X \setminus \{\mathfrak{p}\}$. The second statement follows because multiplicity localizes.

Lemma 5. *Let $I' \subseteq R$ and $\mathfrak{p} \in \mathrm{Ass}(R/I')$ be a maximal element. If $i \leq \mathrm{mult}_{I'}(\mathfrak{p})$ then there is an ideal $I \subseteq R$ such that $\mathrm{mult}_I(\mathfrak{q}) = \mathrm{mult}_{I'}(\mathfrak{q})$ for all $\mathfrak{q} \neq \mathfrak{p}$ and $\mathrm{mult}_I(\mathfrak{p}) = i$.*

Proof. Let $J = \bigcup_{j>0}(I' : \mathfrak{p}^j)$ and let P be a $\mathfrak{p}$-primary ideal with $P \cap J = J_i$ as in (1). Then $\mathrm{mult}_{J_i}(\mathfrak{p}) = i$ by construction and $\mathrm{mult}_{J_i}(\mathfrak{q}) = \mathrm{mult}_{I'}(\mathfrak{q})$ for all $\mathfrak{q} \neq \mathfrak{p}$ by the previous lemma. Take $I = J_i$.

Proof of Theorem 1. If $\mathrm{mult}_I(\mathfrak{q}) \geq \mathrm{mult}_0(\mathfrak{q})$ for every $\mathfrak{q}$ strictly contained in some given $\mathfrak{p} \in \mathrm{Spec}(R)$, then

$$\mathrm{mult}_I(\mathfrak{p}) = \mathrm{mult}_{IR_\mathfrak{p}}(\mathfrak{p}R_\mathfrak{p}) \leq \mathrm{mult}_0(\mathfrak{p}R_\mathfrak{p}) = \mathrm{mult}_0(\mathfrak{p}),$$

where the middle inequality is because of Proposition 3 applied to the ring $R_\mathfrak{p}$. Since the finiteness follows from our Noetherian assumptions, the condition on ψ is proved.

Assume henceforth that the function ψ satisfies the conditions of Theorem 1, and let X denote the support of ψ, i.e.

$$X = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \psi(\mathfrak{p}) > 0\}.$$

The sufficiency of the condition will be proven by induction on the number n of primes $\mathfrak{p} \in X$ such that $\psi(\mathfrak{q}) < \mathrm{mult}_0(\mathfrak{q})$ for some prime $\mathfrak{q}$ strictly contained in $\mathfrak{p}$.

In the case $n = 0$ an ideal I with $\psi = \mathrm{mult}_I$ is constructed as follows. The hypothesis that $n = 0$ means that if $\mathfrak{p} \in X$ is maximal, we have $\psi(\mathfrak{q}) \geq \mathrm{mult}_0(\mathfrak{q})$ for every $\mathfrak{q}$ strictly contained in $\mathfrak{p}$. Applying our hypothesis about ψ to every prime contained in $\mathfrak{p}$ (including $\mathfrak{p}$ itself), we have $\psi(\mathfrak{q}) = \mathrm{mult}_0(\mathfrak{q})$ for all primes $\mathfrak{q}$ strictly contained in $\mathfrak{p}$ and $\psi(\mathfrak{p}) \leq \mathrm{mult}_0(\mathfrak{p})$. Fix a primary decomposition $0 = \bigcap_{\mathfrak{q}\in \mathrm{Ass}(R)} P^{\mathfrak{q}}$ with each $P^{\mathfrak{q}}$ a $\mathfrak{q}$-primary ideal, and define

$$I' = \bigcap_{\mathfrak{q}\in X} P^{\mathfrak{q}}.$$

Now $I'R_{\mathfrak{p}} = 0 \cdot R_{\mathfrak{p}}$ for every $\mathfrak{p}$ maximal in X, whence $\mathrm{mult}_{I'}(\mathfrak{q}) = \mathrm{mult}_0(\mathfrak{q})$ for all $\mathfrak{q} \in X$ because multiplicity localizes. Applying Lemma 5 to I' successively for each maximal $\mathfrak{p} \in X$ (in any order) we obtain the desired ideal I.

Now assume that $n > 0$, so that we can fix a maximal element $\mathfrak{p} \in X$ and a prime $\mathfrak{q}$ strictly contained in $\mathfrak{p}$ such that $\psi(\mathfrak{q}) < \mathrm{mult}_0(\mathfrak{q})$. Let $\psi'\colon \mathrm{Spec}(R) \to \mathbf{N}$ be the function which takes the same values as ψ except at $\mathfrak{p}$, where $\psi'(\mathfrak{p}) = 0$. By induction, we fix an ideal $J \subseteq R$ with $\psi' = \mathrm{mult}_J$. It remains to show that we can choose I with $\mathrm{mult}_I(\mathfrak{q}) = \mathrm{mult}_J(\mathfrak{q})$ for all $\mathfrak{q} \neq \mathfrak{p}$ and $\mathrm{mult}_I(\mathfrak{p})$ arbitrary. Using Lemma 4, Lemma 5, and the characterization of multiplicity as in (1), all that is needed is a chain of strict containments

$$J \supset P_1 \cap J \supset P_2 \cap J \supset \cdots \tag{4}$$

with each P_r a $\mathfrak{p}$-primary ideal. By Lemma 4 we may assume $(R, \mathfrak{p})$ is a local ring. The filtration $R \supset \mathfrak{p} \supset \mathfrak{p}^2 \supset \cdots$ is $\mathfrak{p}$-stable by definition, so the Artin-Rees lemma implies that (4) is $\mathfrak{p}$-stable as well if we take $P_r = \mathfrak{p}^r$. But $\mathrm{mult}_{\mathfrak{p}^r\cap J}(\mathfrak{q}) = \mathrm{mult}_J(\mathfrak{q}) < \mathrm{mult}_0(\mathfrak{q})$ for all r by Lemma 4, so none of the ideals $\mathfrak{p}^r \cap J$ is zero, by Proposition 3. Nakayama's lemma now prevents the chain from stabilizing.

It is interesting to note that if the local ring in the last paragraph of the proof is assumed to be $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$, then taking the preimage in R, we see that the symbolic powers $\mathfrak{p}^{(r)}$ suffice to complete the induction.

There may be more to say about the functions ψ described in Theorem 1. For instance, is it possible to describe the "realization space" of a given ψ – that is, what does the set of ideals $\{I \subseteq R \mid \mathrm{mult}_I = \psi\}$ look like? As a (meager) first step, Proposition 3 gives the answer when $\psi = \mathrm{mult}_0$. There is a possibility that this question has something to do with Hilbert schemes, but if so, the connection would seem to be fairly indirect, because Hilbert schemes deal with Hilbert polynomials, which can easily be equal for ideals with very different primary decompositions. In particular, the notion of degree appears only implicitly in the present case (where we choose a multiplicity function) at the primes where the multiplicity function is nonzero. It does seem natural, however, to ask what kinds of Hilbert polynomials (or Hilbert functions) can arise from ideals with the same multiplicity functions.

References

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