Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 39 (1998), No. 1, 7-18.

Generating Series for the Coxeter Groups and Applications

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Abstract. By forming generating series associated to convenient linear representations of Coxeter groups acting on spaces of constant curvature, we show that the mean values of some related quantities are given by linear recurrence sequences. The tools are the ones used to count words with a given length.

MSC 1991: 20F55, 51B10, 11J70

1. Introduction

Let W be a discrete group of Euclidean affine motions generated by the reflections in hyperplanes of a bounded fundamental domain A. It is clear that for $O \in A$ and $w \in W_a$:

$$\|\overrightarrow{Ow(O)}\| \leq l(w)\delta$$

where $\delta = \sup_{x,y\in A} \|x-y\|$ and $l(w)$ is the length of the word w. We shall see that we can say a little more about the $\|\overrightarrow{Ow(O)}\|$. In particular, we can get some mean values for a given length, Theorem 2 giving their asymptotic behavior.

Similarly, let W be a discrete group of hyperbolic motions generated by the inversions in hyperspheres of an unbounded fundamental domain C in a Poincaré half-space of $\mathbf{R}^m$. We suppose that the only point at infinity of C is ∞. We describe a multidimensional generalization of Hermite's algorithm [4] due to the author [7]. For z in the floor $\mathbf{R}^{m-1}$, let $w_n(C)$ be a sequence of adjacent domains through which the geodesic $(z\infty)$ passes and such that $l(w_n) = n$. We know that:

$$\left|z - p_n q_n^{-1}\right| \leq \frac{1}{\gamma\left|q_n\right|^2}.$$

0138-4821/93 \$ 2.50

where $p_n q_n^{-1} = w_n(\infty)$, and γ is a constant depending only on W. Here, p_n and q_n are in a Clifford algebra with (as vector space) a Euclidean norm $|\ |$ which restricts to $\|\ \|$ on $\mathbf{R}^{m-1}$. In the framework of the Diophantine approximation, it is interesting to have some results about the size of the denominators q_n. We show that it is possible to get the mean value of the $|q_n|^2$ produced by Hermite's algorithm at the step n. The Theorem 3 gives a general result.

In the two cases previously described, W is a Coxeter group. The results of Section 2 are all aimed at an explicit evaluation of the formal power series $\sum_w \rho(w) X^{l(w)}$ when ρ is a linear representation of W. The method is a generalization of the one which permits to get the number of words of a given length. For suitable choices of ρ (in symmetric algebras) we deduce the moments at any order of $\|\overrightarrow{Ow(O)}\|^2$ and $|q_n|^2$.

2. Generating series for Coxeter groups

For a general survey about the classical results for Coxeter groups, the reader is referred to [2] or [5]. But we give here the statements that are sufficient for the understanding of the sequel.

In this paper, W_S denotes a Coxeter group generated by a finite subset $S = \{s_1, \dots, s_p\}$ of distinct involutions with the defining relations

$$(s_i s_j)^{m_{i,j}} = Id \text{ with } 1 \leq m_{i,j} \leq \infty.$$

For w in W_S, the length $l_S(w)$ is the smallest integer q such that w is the product of q elements of S with the particular case $l_S(Id) = 0$. It is clear that we have $l_S(sw) = l_S(w) \pm 1$ for all s in S. It is well known that when W_S is finite, there exists a unique element w_S of maximal length. This element verifies $w_S^2 = Id$.

For $T \subset S$, the subgroup of W_S generated by all s in T is also a Coxeter group denoted by W_T. Such a subgroup is called a parabolic subgroup. A Coxeter group which is not a direct product of non-trivial parabolic subgroups is said to be irreducible. For all w in W_T, we have $l_S(w) = l_T(w)$. So we shall simply write l instead of l_S. We get a set of coset representatives for W_S/W_T by introducing the set W^T of w in W_S verifying $l(ws) > l(w)$ for all s in T. A main result for our purpose is that for a given w in W_S, there are unique $u \in W^T$ and $v \in W_T$ such that $w = uv$. Moreover, we have $l(w) = l(u) + l(v)$ and u is the unique element of smallest length in the coset wW_T. We have:

$$W_S = W^T \cdot W_T.$$

We denote the group algebra of W_S over a commutative ring R by $R[W_S]$.

Let K be a commutative field. We denote the quotient field of the polynomial ring $K[X]$ in one variable by $K(X)$, and the ring of the formal power series by $K[[X]]$. For any subset $H \subset W_S$, we denote by $H(X)$ the formal power series

$$\sum_{w \in H} w X^{l(w)} \text{ in } K[[X]][W_S].$$

In order to get an explicit expression for $W_S(X)$, we generalize a method which gives the number of elements of a given length [5]. The key formula is a consequence of the decomposition previously described :

$$W_S(X) = W^T(X) \cdot W_T(X) \text{ for all } T \subset S. \tag{1}$$

We note that

$$\sum_{T \subset S} (-1)^{|T|} W^T = \{w \in W_S, l(ws) = l(w) - 1 \text{ for all } s \text{ in } S\}.$$

This set will be denoted by B_S. If W_S is infinite then $B_S = \emptyset$, else $B_S = \{w_S\}$. It follows that:

$$B_S(X) = \sum_{T \subset S} (-1)^{|T|} W^T(X).$$

From (1), we can write

$$(B_S(X) - (-1)^{|S|})(W_S(X))^{-1} = \sum_{T \subsetneq S} (-1)^{|T|} (W_T(X))^{-1}. \tag{2}$$

We set $C_T(X) = (1 - (-1)^{|S|} B_S(X))^{-1}$. According to whether W_T is finite or not, we have:

$$C_T(X) = (1 - X^{2l(w_T)})^{-1} (1 + (-1)^{|T|} w_T X^{l(w_T)}) \quad \text{or} \quad C_T(X) = 1.$$

Hence (2) becomes:

$$(W_S(X))^{-1} = (-1)^{|S|+1} C_S(X) \sum_{T \subsetneq S} (-1)^{|T|} (W_T(X))^{-1}.$$

By induction on $|S|$, we obtain

$$(W_S(X))^{-1} = (-1)^{|S|+1} C_S(X) \sum (-1)^i C_{T_i}(X) \cdots C_{T_1}(X)$$

where the sum is taken over the $(T_1, \ldots, T_i)$ such that $\emptyset \neq T_1 \subsetneq \cdots \subsetneq T_i \subsetneq S$.

Let $P_0 \in K[X]$ be the generator of the ideal

$$\left\{P \in K[X], P(X)(W_S(X))^{-1} \in K[X][W_S]\right\}$$

such that $P_0(0) = 1$. There are unique polynomials P_i verifying the *main relation*

$$\boxed{P_0(X) = \left(\sum_{i \geq 1} P_i(X) w_i\right) W_S(X)} \tag{3}$$

where the w_i run over the finite set of the different products $w_{T_i} \cdots w_{T_1}$.

Example 1. Let $W_{\{s_1,s_2,s_3\}}$ be the Coxeter group defined by:

$$s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_1s_3)^2 = 1.$$

This group is isomorphic to $PGL_2(\mathbf{Z})$ acting on $\mathbf{R} \cup \{\infty\}$ by

$$\begin{cases} s_1(x) = 1/x \\ s_2(x) = 1 - x \\ s_3(x) = -x \end{cases}$$

The main relation (3) is

$$\begin{aligned}(1 - X^2)(1 - X^6) = \{&1 + 2X^2 + X^4 + X^6 - (X + X^3 + X^5 + X^7)s_1 \\ &-Xs_2 - (X + X^3 + X^5)s_3 + (X^2 + X^4 + X^6)s_1s_3 \\ &-X^4(s_1s_2 + s_2s_1) + (X^3 + X^5)s_1s_2s_1\} \cdot PGL_2(\mathbf{Z})(X).\end{aligned} \tag{4}$$

3. Linear recurrence sequences

Let A be a K-algebra. We say that a sequence $(u_n)_{n\geq 0}$ of elements in A is a linear recurrence sequence of length $k > 0$ over A (resp. over K) if there is an integer N and some $A_0 \neq 0, A_1, \ldots, A_k = 1$ in A (resp. in K) such that

$$\sum_{i=0}^{k} A_i u_{n+i} = 0 \text{ for } n \geq N. \tag{5}$$

To any sequence $(u_n)_{n\geq 0}$ in A, we associate the formal power series

$$u(X) = \sum_{n\geq 0} u_n X^n \text{ in } \mathsf{A}[[X]],$$

which is invertible if and only if u_0 is invertible. If $(u_n)_{n\geq 0}$ verifies (5) then

$$\left(\sum_{i=0}^{k} A_{k-i} X^i\right) u(X) \in \mathsf{A}[X]. \tag{6}$$

Reciprocally, let $u(X)$ be in $\mathsf{A}[[X]]$ and $v(X)$ be in $\mathsf{A}[X]$ such that $v(0)$ is invertible and $v(X)\, u(X) \in \mathsf{A}[X]$. It is easy to see that $(u_n)_{n\geq 0}$ is a linear recurrence sequence in A. Hence, from the main relation (3), the sequence defined by

$$u_n = \sum_{\substack{w \in W_S \\ l(w)=n}} w$$

is a linear recurrence sequence over $K[W_S]$.

We shall work with some linear representations of W_S. So we consider the case where A is the algebra $End_K(E)$ of the K-endomorphisms of a vector space E of dimension p over

K. One can see $End_K(E)$ as the algebra $Mat_p(K)$ of the square $p \times p$ matrices over K. For $R = K[X], K[[X]]$ or $K(X)$, we have also the isomorphisms:

$$R \otimes_K End_K(E) \approx Mat_p(R).$$

Let $(u_n)_{n\geq 0}$ be a recurrence sequence verifying (5) over $End_K(E)$. For all $n \geq N$, it is possible to get $(u_{n+1}, \ldots, u_{n+k})$ from $(u_n, \ldots, u_{n+k-1})$ using a linear operator D_A that we can write in the form

$$D_A = \begin{pmatrix} 0 & Id & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & Id \\ -A_0 & -A_1 & \cdots & \cdots & -A_{k-1} \end{pmatrix}, \ A_i \in End_K(E) \approx Mat_p(K).$$

Viewing D_A as an element of $End_K(E^k) \approx Mat_{p^k}(K)$, we define its characteristic polynomial in $K[X]$ by

$$P_A(X) = \det(X \cdot Id - D_A) = \det(\sum_{i=0}^{k} A_i X^i).$$

For $P(X) \neq 0$ in $K[X]$, we set $P^*(X) = X^{\deg(P)} P(X^{-1})$. From (6), the ideal

$$I(u) = \{P(X) \in K[X], P(X)u(X) \in K[X] \otimes_K End_K(E)\},$$

generated by a l.c.m. of the denominators of the entries of $u(X) \in Mat_p(K(X))$, contains P_A^*. We see that a linear recurrence sequence over $End_K(E)$ is a linear recurrence sequence over K :

Theorem 1. *Let E be a finite dimensional vector space over K and a representation $\rho : W_S \to Aut_K(E)$. The sequence defined by*

$$n \mapsto \sum_{\substack{w \in W_S \\ l(w)=n}} \rho(w)$$

is a linear recurrence sequence over K.

4. Some representations of Coxeter groups

Let $\chi : W_S \to \{1\}$ be the trivial character. Applying the results of Section 2, we can get the number N_l of elements with length l. Indeed N_l is the coefficient of X^l in $\chi(W_S)(X) = \sum_w \chi(w) X^{l(w)}$. There are general results for some Coxeter groups:

- If W_S is finite and irreducible, then $\chi(W_S)(X)$ is the Poincaré polynomial of W_S [3] and we have:

$$\chi(W_S)(X) = (1-X)^{-p} \prod_{i=1}^{p} (1 - X^{m_i+1})$$

  where the m_i are the exponents of W_S.

- If W_S is an affine Weyl group (see §5) then, using Morse theory [1], we have:

$$\chi(W_S)(X) = (1-X)^{1-p} \prod_{i=1}^{p-1} \frac{1-X^{m_i+1}}{1-X^{m_i}}$$

where the m_i are the exponents of the corresponding Weyl group. In this case, we obtain

$$N_l = \frac{l^{p-2}\prod_i (m_i+1)}{(p-1)!\prod_i m_i} + O(l^{p-3}) \tag{7}$$

Now, we introduce the geometric representation. We consider a bilinear symmetric form B on $E = \mathbf{R}^p$ verifying

$$B(e_i, e_j) = \begin{cases} -\cos(\pi/m_{i,j}) & \text{if} \quad m_{i,j} < \infty \\ r_{i,j} \leq -1 & \text{if} \quad m_{i,j} = \infty \end{cases}$$

where $(e_i)_{1\leq i \leq p}$ is the canonical basis of E. We obtain an injective representation of W_S as a discrete subgroup of the orthogonal group $O_B(E)$ by mapping each s_i on the reflection

$$x \mapsto x - 2B(e_i, x)e_i.$$

It is an important fact that the stabilizer $\{w \in W_S, w(x) = x\}$ of $x \in E$ is a Coxeter group generated by $S(x) = \{w \in S, s(x) = x\}$.

When B is positive, we can identify W_S with a discrete subgroup of an affine Euclidean group generated by orthogonal reflections. If B is hyperbolic, i.e. of signature $(*, 1)$, it is possible to identify W_S with a discrete group of Möbius transformations. These are the two cases that we will encounter in the sequel of this paper.

The geometric representation is sufficient to compute the mean value of the quantities which are of interest to us. But to get the moments of any orders, we shall use representations in symmetric algebras.

Let E^* be the dual space of a vector space E of dimension p over K. Let $\rho : W_S \to Aut_K(E)$ be a representation of W_S. First we get from ρ the contragredient representation $\rho^* : W_S \to Aut_K(E^*)$ following the rule

$$\langle \rho^*(w)x, y\rangle = \langle x, \rho(w^{-1})y\rangle$$

where $\langle\ ,\ \rangle$ is the duality bracket on $E^* \times E$. In practice, we have

$$\rho^*(w) = \rho(w^{-1})^t$$

where A^t denotes the transpose of a linear operator A. From the fact that $l(w^{-1}) = l(w)$, we get

$$\rho^*(W_S)(X) = \rho(W_S)^t(X).$$

We denote by $S(E^*)$ the symmetric algebra of E^*. This $\mathbf{Z}$ -graded K-algebra is the direct sum of the components $S^k(E^*)$ of degree k formed by the commutative products $x_1 \times \cdots \times x_k$

with x_i in E^*. Hence, the contragredient representation ρ^* induces a finite dimensional representation

$$\rho_k^* : W_S \to Aut_K(S^k(E^*)) \text{ for all } k \geq 0$$

by

$$\rho_k^*(w)(x_1 \times \cdots \times x_k) = \rho^*(w)(x_1) \times \cdots \times \rho^*(w)(x_k).$$

Let (ε_i) be the canonical basis of E and (ε_i^*) be its dual basis. The algebra $S(E^*)$ is isomorphic to the algebra $K\,[X_1, \cdots, X_p]$ of the polynomials in p variables over K and $S^k(E^*)$ corresponds to the vector space of the homogeneous polynomials of degree k. By identifying ε_i^* with X_i, the action of W_S on $K[X_1, \ldots, X_p]$ is simply given by

$$P(X_1, \ldots, X_p) \mapsto P(\sum_i a_{i,1} X_i, \cdots, \sum_i a_{i,n} X_i)$$

where $(a_{i,j})$ is the matrix of $\rho^*(w)$ with respect to the dual basis.

Remark 1. One can show that the geometric representation $\rho : W_S \to Aut_K(E)$ induces a faithful representation $K[W_S] \to End_K(S(E^*))$. This means that, working in $S(E^*)$, it is impossible to get a relation which is simpler than the main relation (3).

5. Affine Weyl groups

An irreducible Coxeter group W_S which admits a positive degenerate form B is isomorphic to a discrete group W_a of Euclidean transformations by mapping the s_i on orthogonal reflections of the affine space $\mathbf{R}^{p-1}$ where $p = |S|$. The group W_a is termed an affine Weyl group. We describe briefly the situation in terms of root systems.

The simplex A bounded by the hyperplanes of the s_i is a fundamental domain for the action of W_a on $\mathbf{R}^{p-1}$. There is a vertex O of A such that W_a is the semi-direct product of the stabilizer W_O of O with the subgroup T of the translations of W_a. The group W_O is a Weyl group (i.e. a finite crystallographic reflection group).

Let H_i, $i = 1, \ldots, p-1$, be the walls of A which contain O. We choose the vectors α_i of the H_i such that A is in the cone $C_O = \{O + x, (x, \alpha_i) \geq 0 \text{ for } i < p\}$ and that $(\alpha_i^\vee)$ is a basis of the lattice L of T where $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$. The set $R = W_O\,\{\alpha_i, i < p\}$ is the root system of W_a and $R^\vee = W_O\,\{\alpha_i^\vee, i < p\}$ is the inverse system of R. The roots are linear combinations of the α_i with integer coefficients. A root $\alpha = \sum_i a_i \alpha_i$ is said to be positive if $a_i \geq 0$ for all i. Denoting by R_+ the set of the positive roots, we have $R = R_+ \cup (-R_+)$. There is a unique positive root $\widetilde{\alpha} = \sum_i a_i \alpha_i$ such that $\sum_i a_i$ is maximum. This highest root $\widetilde{\alpha}$ is a vector of the wall H_p of A which does not contain O. We can suppose that the scalar product $(\ ,\ )$ is such that H_p is given by $(x, \widetilde{\alpha}) = 1$. This implies that there is a 1-1 correspondence between the hyperplanes $H_{\alpha,k}$ of equation $(x, \alpha) = k$, $\alpha \in R$ and $k \in \mathbf{Z}$, and the reflections of W_a. It is known that the length of w is the number of hyperplanes $H_{\alpha,k}$ which separate A and $w(A)$. In particular, the maximum length in W_O is $|R_+|$.

To establish the theorem 1, we shall work with real linear recurrence sequences and we need the following result

Lemma 1. *Let $a \in \mathbf{C}^m$, $a \neq (0)$, and $u_n = \Re(\sum_j a_j \exp(in\theta_j))$ with distinct θ_j in $]0,\pi]$. Then we have*

$$\liminf_n u_n < 0 < \limsup_n u_n.$$

Proof. We consider the mean values $m_n = \frac{1}{n}\sum_0^{n-1} u_k$ and $v_n = \frac{1}{n}\sum_0^{n-1} u_k^2$. From

$$\lim_n \frac{1}{n} \sum_{0\leq k<n} \exp(ik\theta) = \begin{cases} 0 \text{ if } \theta \neq 0 \\ 1 \text{ if } \theta = 0 \end{cases}$$

we deduce $\lim_n m_n = 0$ and $\lim_n v_n = \frac{1}{2}\sum_j |a_j|^2$. Let $M = \sum_j |a_j|$ and $v = \frac{1}{2}\sum_j |a_j|^2$.

Suppose that $0 \leq \liminf_n u_n$. For any $\varepsilon > 0$, there is an integer N such that $u_n \geq -\varepsilon$ for $n \geq N$. We have:

$$0 \leq \frac{1}{n} \sum_{N\leq k\leq n} (u_k+\varepsilon)^2 \leq \frac{1}{n}(M+\varepsilon) \sum_{N\leq k\leq n} (u_k+\varepsilon).$$

Making n tend to ∞, we deduce:

$$0 \leq v \leq \varepsilon M$$

which contradicts $a \neq (0)$.

Replacing a by $-a$, we obtain the desired inequality. □

Now, we are in position to prove :

Theorem 2. *For a given integer k, there are $a_0 > 0, a_j \in \mathbf{C}$, $\theta_j \in]0,\pi]$ such that the moment of order k of $\left\|\overrightarrow{Ow(O)}\right\|^2$ over $\{w \in W_a, l(w) = l\}$ is*

$$(a_0 + \Re(\sum_j a_j \exp(il\theta_j)))l^{2k} + O(l^{2k-1}).$$

Proof. In order to get (12) below, we will follow some arguments of [6]. For $w \in W_a$, there are unique $u, v \in W_O$ such that $w = uw'v$ where w' is the word of minimum length in the double coset $W_O w W_O$. The word $w'v$ is in $W^O = \{w \in W_a, l(s_i w) = l(w) - 1 \text{ for all } i \neq n\}$ which is also the set of the w such that $w(A) \subset C_O$. We denote the translation of vector $d \in L$ by t_d. For any d in $L \cap C_O$, the unicity of v implies that $\inf\{l(t_d w), w \in W_O\}$ is attained by a unique element $w_d \in W_O$. Moreover, for all $v \in W_O$, we have:

$$l(t_d w_d v) = l(t_d w_d) + l(v). \tag{8}$$

For a direction given by a positive root α, the number of hyperplanes $H_{\alpha,k}$ which separate A and $w(A)$ is 0 or $(d,\alpha) - 1$ according to whether $(d,\alpha) = 0$ or $(d,\alpha) \neq 0$. Hence, we get that

$$l(t_d w_d) = (d, \sum_{\alpha\in R_+} \alpha) - |\{\alpha \in R_+, (d,\alpha) > 0\}|. \tag{9}$$

For $d = \sum_i a_i \alpha_i^\vee$, we have (see the tables of [2]):

$$(d, \sum_{\alpha \in R_+} \alpha) = 2 \sum_i a_i. \tag{10}$$

Now, let $w \in W^O$ be of length l. We have $\left\| \overrightarrow{Ow(O)} \right\| = \|d\| \leq (\sum_i a_i) \sup_i \|\alpha_i^\vee\|$. From (8), (9) and (10):

$$\left\{ \begin{array}{c} 2\sum_i a_i - |R_+| \leq l(t_d w_d) \leq l \\ l \leq (d, \sum_{\alpha \in R_+} \alpha) + |R_+| \leq \|d\| \left\| \sum_{\alpha \in R_+} \alpha \right\| + |R_+|. \end{array} \right.$$

We deduce:

$$\frac{l - |R_+|}{\left\| \sum_{\alpha \in R_+} \alpha \right\|} \leq \left\| \overrightarrow{Ow(O)} \right\| \leq \frac{1}{2} \sup_i \|\alpha_i^\vee\| \, (l + |R_+|). \tag{11}$$

Let $w \in W$ be of length l. Then $\left\| \overrightarrow{Ow(O)} \right\| = \left\| \overrightarrow{Ow'v(O)} \right\|$ with $l - |R_+| \leq l(w'v) \leq l$ and $w'v \in W^O$. From (11):

$$\frac{l - |R|}{\left\| \sum_{\alpha \in R_+} \alpha \right\|} \leq \left\| \overrightarrow{Ow(O)} \right\| \leq \frac{1}{2} \sup_i \|\alpha_i^\vee\| \, (l + |R_+|). \tag{12}$$

For all x in the affine space $\mathbf{R}^{p-1}$ and w in W_a, we have $w(x) = w(O) + \widetilde{w}(\overrightarrow{Ox})$ where $\widetilde{w} \in W_O$ denotes the linear part of w. We consider a linear representation $\rho : W_a \to GL_p(\mathbf{Z})$ defined by

$$\rho(w) = \begin{pmatrix} M & d \\ 0 & 1 \end{pmatrix}$$

where $M \in GL_{p-1}(\mathbf{Z})$ is the matrix of $\widetilde{w}$ in the basis $(\alpha_i^\vee)$. Viewing $\| \quad \|^{2k}$ as a homogeneous polynomial of degree $2k$ in p variables over $\mathbf{R}$ and applying Theorem 1 with ρ_k^*, we see that $l \mapsto \sum_{l(w)=l} \left\| \overrightarrow{Ow(O)} \right\|^{2k}$ is a linear recurrence sequence. We fix k. From general theory of the linear recurrence sequences in $\mathbf{R}$, we have:

$$\sum_{l(w)=l} \left\| \overrightarrow{Ow(O)} \right\|^{2k} = \sum_{0 \leq p \leq m} (a_0^{(p)} + \Re(\sum_j a_j^{(p)} \exp(il\theta_j^{(p)})))l^p \text{ with } a^{(m)} \neq (0).$$

From Lemma 1, we get that:

$$\liminf_l \frac{\sum_{l(w)=l} \left\| \overrightarrow{Ow(O)} \right\|^{2k}}{l^m} < a_0^{(m)} < \limsup_l \frac{\sum_{l(w)=l} \left\| \overrightarrow{Ow(O)} \right\|^{2k}}{l^m},$$

and from (7) and (11):

$$0 < \liminf_l \frac{\sum_{l(w)=l} \left\| \overrightarrow{Ow(O)} \right\|^{2k}}{l^{p-2+2k}} \leq \limsup_l \frac{\sum_{l(w)=l} \left\| \overrightarrow{Ow(O)} \right\|^{2k}}{l^{p-2+2k}} < \infty.$$

This implies that $m = p - 2 + 2k$ and $a_0^{(m)} > 0$. $\square$

Example 2. We consider the affine Weyl group of type $\widetilde{B}_2$, i.e. with the defining relations:

$$s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^4 = (s_2 s_3)^4 = (s_1 s_3)^2 = 1.$$

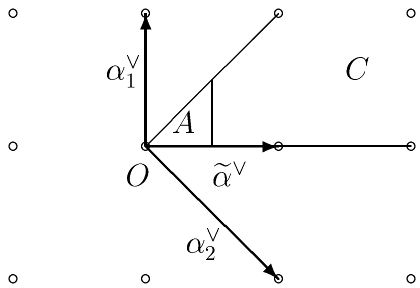

Figure 1: The affine Weyl group of type $\widetilde{B}_2$

The geometric elements are illustrated at figure 1. We have $L = \mathbf{Z}\alpha_1^\vee \oplus \mathbf{Z}\alpha_2^\vee$ and $\widetilde{\alpha}^\vee = \alpha_1^\vee + \alpha_2^\vee$. The $\rho(s_i) \in GL_3(\mathbf{Z})$ are given by

$$\rho(s_1) = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(s_2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \rho(s_3) = \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The last reflection corresponds to $\widetilde{\alpha}^\vee$. Using ρ_k^* for $k = 2$ and 4:

$$\begin{cases} \frac{1}{N_l} \sum_{l(w)=l} \left\| \overrightarrow{Ow(O)} \right\|^2 & = \frac{46}{108} l^2 + O(l) \\ \frac{1}{N_l} \sum_{l(w)=l} \left\| \overrightarrow{Ow(O)} \right\|^4 & = \frac{59}{5184} l^4 + O(l^3) \end{cases}$$

Remark 2. As the representation ρ can also be used to get $\sum_{l(w)=l} \overrightarrow{Ow(O)}$, it is not surprising that one can prove:

$$\frac{1}{N_l} \left\| \sum_{l(w)=l} \overrightarrow{Ow(O)} \right\| \in O(1).$$

6. Hyperbolic reflection groups

A Coxeter group which admits an hyperbolic form B is isomorphic to a discrete subgroup W_h of the full Möbius group Möb$(\widetilde{H}^m)$ acting on

$$\widetilde{H}^m = \{x \in \mathbf{R}^m, x_m \geq 0\} \cup \{\infty\}$$

where $m = \dim(E/E^{\perp}) - 1$. We say that W_h is a hyperbolic reflection group [9]. We get this representation by mapping the s_i on reflections in half-planes or half-spheres orthogonal to $\mathbf{R}^{m-1} = \{x \in \mathbf{R}^m, x_m = 0\}$. The mirrors of these reflections are the walls of a fundamental domain C for the action of W_S on the Poincaré half-space $H^m = \{x \in \mathbf{R}^m, x_m > 0\}$. From now on, we suppose that ∞ is the only point at infinity of C. In general there is no formula as (3) for the number N_l of elements with length l. But in this case, one can show that

$$0 < \liminf_l \frac{\ln(N_l)}{l} \leq \limsup_l \frac{\ln(N_l)}{l} < \infty.$$

Recall that, for a given $z \in \mathbf{R}^{m-1}$, the Hermite algorithm produces at the step n a word $w_n \in W_h$ of length n and we have:

$$\left|z - p_n q_n^{-1}\right| \leq \frac{1}{\gamma \left|q_n\right|^2}.$$

where $p_n q_n^{-1} = w_n(\infty)$, and γ is a constant depending only on W_h.

Theorem 3. *For any integer k, the sequence $l \mapsto \sum_{l(w)=l} |q(w)|^{2k}$ is a linear recurrence sequence.*

Proof. We use the geometric representation $\rho : W_S \to O_B(E)$ because (see [8]) $|q_n|^2 = -\frac{1}{2}B(w_n(v), v)$ for v (corresponding to ∞) in $F = E/E^{\perp}$.

For $u, v \in F$, we consider the values of the form $B^k(w(u), v)$. By identifying F and its dual F^*, we have

$$B^k(w(u), v) = \langle w(u), v\rangle^k = (\rho_k^*(w)(u^k))(v) = V^t \cdot M_w^{(k)} \cdot U$$

where the U_i are the coordinate of u^k in a basis (E_i) of $S_k(F^*)$, $V_i = E_i(v)$ and $M_w^{(k)}$ is the matrix of $\rho_k^*(w)$ respect to (E_i). Hence we have

$$\sum_{w \in W_S} |q(w)|^{2k} X^{l(w)} = (-\frac{1}{2})^k (E_i(e))^t \cdot \rho_k^*(W_S)(X) \cdot (e^k) \text{ in } \mathbf{R}[[X]] \cap \mathbf{R}(X). \qquad \square$$

Example 3. From (4), we get

$$\chi(PGL_2(\mathbf{Z}))(X) = \frac{(1 + X + X^2)(1 + X)^2}{1 - X^2 - X^3} \text{ in } \mathbf{Q}(X) \cap \mathbf{Z}[[X]] \tag{13}$$

hence

$$\lim_l \frac{\ln(N_l)}{l} \approx 0.28.$$

From $v = e_1^* = -2(e_2 + e_3)$, we get:

$$\sum_w |q(w)|^2 X^{l(w)} = \frac{X(1 - X)(1 + X)(1 + 2X + X^2 - 2X^3 + X^5 + X^6)}{1 - X - 2X^2 + 3X^4 - 2X^5 - X^6 + X^8}$$

and we deduce:

$$\lim_{l} \frac{1}{l} \ln \left(\frac{1}{N_l} \sum_{l(w)=l} |q(w)|^2 \right) \approx 0.317.$$

Similarly, we get

$$\lim_{l} \frac{1}{l} \ln \left(\frac{1}{N_l} \sum_{l(w)=l} |q(w)|^4 \right) \approx 0.642.$$

Remark 3. The same kind of results for the norm of the Möbius transformations of W_h are also valid.

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Received November 8, 1996