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# The Rank 2 Geometries of the Simple Suzuki Groups $Sz(q)$

**Dimitri Leemans**

*Université Libre de Bruxelles, Département de Mathématiques - C.P.216*
*Boulevard du Triomphe, B-1050 Bruxelles*
*e-mail: dleemans@cso.ulb.ac.be*

**Abstract.** We determine all firm and residually connected rank 2 geometries on which a Suzuki simple group $Sz(q)$ acts flag-transitively and residually weakly primitively.

## 1. Introduction

In 1986, Francis Buekenhout advocated in [1] a systematic search of the geometries to be obtained from a group and some of its subgroups.

In 1994, Michel Dehon presented in [10] a set of CAYLEY programs in order to classify all firm, residually connected and flag-transitive geometries for a given group $G$ with the additional restriction that each stabilizer of some element is a maximal subgroup of $G$. Several groups were investigated as for example, $U_4(2)$ [10], $M_{11}$ [6], etc. An atlas was also made, containing a collection of projective groups $PSL(2,q)$ [4].

This experimental work led to new rules for such classifications. In 1993, Francis Buekenhout and Michel Dehon changed the restriction on the subgroups forming the geometries, taking a residually weakly primitive condition. Again, experimental work was accomplished in that way. An atlas of residually weakly primitive geometries for small groups was achieved [5], and Harald Gottschalk determined all the geometries of the group $PSL(3,4)$ satisfying these conditions in his Diplomarbeit [13]. More recently, Dehon and Miller determined in [11] all geometries of $M_{11}$ satisfying these new conditions and Gottschalk and Leemans classified the geometries of $J_1$ [12].

From now on, it really seems that the residually weakly primitive restriction on the subgroups is a "good one".

The experience that we acquired leads us to be more ambitious. We now look at an infinite class of groups, and want to classify all geometries satisfying some conditions for all the groups of this class. We choose to study the family of the simple Suzuki groups. This choice is motivated by the well known fact that the structure of the Suzuki groups is particularly easy, compared to the other simple groups. Using the same conditions as in [5], except for the intersection property, we classify in this article all the firm, residually connected, flag-transitive and residually weakly primitive geometries of rank 2 that can be obtained from a Suzuki group $Sz(q)$ where $q = 2^{2e+1}$ and $e$ is at least 1.

For a history of the Suzuki groups, we refer to [3].

The paper is organized as follows.

In Section 2, we recall definitions and notation for incidence geometry and the Suzuki groups, and we explain some notation we use throughout this paper.
In Section 3, we give some properties on certain numbers that play an important role in our classification.
In Section 4, we recall the structure of maximal subgroups of a Suzuki group.
In Section 5, we give the structures of the maximal subgroups of each maximal subgroup of a Suzuki group.
In Section 6, we determine all rank 2 geometries that are obtained.
The results of the quite long discussion and case analysis of section 6 leads to a classification theorem that we state in Section 7.
We give in Section 8 the diagrams of the rank 2 geometries obtained for $Sz(8)$ which is the smallest case and which can still be completely determined using the help of MAGMA [8] or CAYLEY [7]. The whole classification of the geometries of $Sz(8)$ is available in [14]. The classification of RWPRI geometries of rank $> 2$ for $Sz(q)$, will be developed in a forthcoming paper.
Finaly, in Section 9, we give some observations and also mention some future work that will be accomplished on similar problems.

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## 2. Definitions and notation

In a first paragraph, we recall the basic concepts of incidence geometry. For a more detailed discussion about incidence geometry, we refer to [2].
In a second paragraph, we give the construction of the Suzuki groups $Sz(q)$ and some properties that will play an important role in this paper. For a more detailed introduction on Suzuki groups, see for example [16].
Finally, in a third paragraph, we fix some notation.

### 2.1. Groups and geometries

Most of the following ideas arise from [18] (see also [2], chapter 3).
Let $G$ be a group together with a finite family of subgroups $(G_i)_{i\in I}$. We define the *pre-geometry* $\Gamma = \Gamma(G,(G_i)_{i\in I})$ as follows. The set $X$ of *elements* of $\Gamma$ consists of all cosets $gG_i$, $g \in G$, $i \in I$. We define an *incidence relation* $*$ on $X$ by :

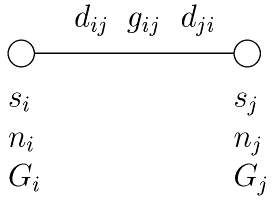
$$g_1G_i * g_2G_j \text{ iff } g_1G_i \cap g_2G_j \text{ is non-empty in } G.$$

The *type function* $t$ on $\Gamma$ is defined by $t(gG_i) = i$. The *type* of a subset $Y$ of $X$ is the set $t(Y)$; its *rank* is the cardinality of $t(Y)$ and we call $|X|$ the *rank* of $\Gamma$. The *Borel subgroup* of the pre-geometry is the subgroup $B = \cap_{i\in I}G_i$. A *flag* is a set of pairwise incident elements of $X$ and a *chamber* of $\Gamma$ is a flag of type $I$. An element of type $i$ is also called an *$i$-element*. The group $G$ acts on $\Gamma$ as an automorphism group, by left translation, preserving the type of each element.
As in [10], we call $\Gamma$ a *geometry* provided that every flag of $\Gamma$ is contained in some chamber and we call $\Gamma$ *flag-transitive* (FT) provided that $G$ acts transitively on all chambers of $\Gamma$, hence also on all flags of any type $J$, where $J$ is a subset of $I$.
Assuming that $\Gamma$ is a flag-transitive geometry and that $F$ is a flag of $\Gamma$, the *residue* of $F$ is the pre-geometry

$$\Gamma_F = \Gamma(\cap_{j\in t(F)}G_j, (G_i \cap (\cap_{j\in t(F)}G_j))_{i\in I\setminus t(F)})$$

and we readily see that $\Gamma_F$ is a flag-transitive geometry.
We call $\Gamma$ *firm* (F) (resp. *thick*, *thin*) provided that every flag of rank $|I|-1$ is contained in at least two (resp. three, exactly two) chambers. We call $\Gamma$ *residually connected* (RC) provided that the incidence graph of each residue of rank $\geq 2$ is a connected graph. We call $\Gamma$ *primitive* (PRI) provided that $G$ acts primitively on the set of $i$-elements of $\Gamma$, for each $i \in I$ and we call $\Gamma$ *residually primitive* (RPRI) if each residue $\Gamma_F$ of a flag $F$ is primitive for the group induced on $\Gamma_F$ by the stabilizer $G_F$ of $F$.
We call $\Gamma$ *weakly primitive* (WPRI) provided there exists some $i \in I$ such that $G$ acts primitively on the set of $i$-elements of $\Gamma$ and we call $\Gamma$ *residually weakly primitive* (RWPRI) provided that each residue $\Gamma_F$ of a flag $F$ is weakly primitive for the group induced on $\Gamma_F$ by the stabilizer $G_F$ of $F$.
A subgroup $H$ of a group $G$ is called *quasi-maximal* if there exists a unique chain of intermediate subgroups from $G$ to $H$.
We call $\Gamma$ *quasi primitive* (QPRI) provided that each subgroup $G_i$ forming the geometry is a quasi-maximal subgroup of $G$.
If $\Gamma$ is a geometry of rank 2 with $I = \{0,1\}$ such that each of its 0-elements is incident with each of its 1-elements, we call $\Gamma$ a *generalized digon*.
We say that $\Gamma$ satisfies the *rank two intersection property* $(IP)_2$ if in every rank 2 residue of $\Gamma$ other than a generalized digon, any two elements of the same type are incident with at most one element of the other type.
We call $\Gamma$ *locally 2-transitive* and we write $(2T)_1$ for this, provided that the stabilizer $G_F$ of any flag $F$ of rank $|I|-1$ acts 2-transitively on the residue $\Gamma_F$.
The *diagram* of a firm, RC, FT geometry $\Gamma$ is a graph together with additional structure,

whose vertices are the elements of $I$, which is further described as follows. To each vertex $i \in I$, we attach the *order* $s_i$ which is $|\Gamma_F| - 1$, where $F$ is any flag of type $I \backslash \{i\}$, the *number* $n_i$ *of varieties of type* $i$, which is the index of $G_i$ in $G$, and the subgroup $G_i$. Elements $i$, $j$ of $I$ are not joined by an edge of the diagram provided that a residue $\Gamma_F$ of type $\{i, j\}$ is a generalized digon. Otherwise, $i$ and $j$ are joined by an edge endowed with three positive integers $d_{ij}$, $g_{ij}$, $d_{ji}$ where $g_{ij}$ (the *gonality*) is equal to half the girth of the incidence graph of a residue $\Gamma_F$ of type $\{i, j\}$ and $d_{ij}$ (resp. $d_{ji}$), the *$i$-diameter* (resp. *$j$-diameter*) is the greatest distance from some fixed $i$-element (resp. $j$-element) to any other element in the incidence graph of $\Gamma_F$.
On a picture of the diagram, this structure will often be depicted as follows.




If $g_{ij} = d_{ij} = d_{ji} = n$, then $\Gamma_F$ is called a *generalized $n$-gon* and on a picture, we do not write $d_{ij}$ and $d_{ji}$.
As to notation for groups, we follow the conventions of the Atlas [9] up to slight variations. The symbol : stands for split extensions, the "hat" symbol ˆ stands for non split extensions and the symbol $\times$ stands for direct products.

In this article, we will consider only geometries of rank 2, i.e. geometries $\Gamma(G; G_0, G_1)$. We will call the quadruple $(G, G_0, G_1, G_{01})$ an *RWPRI configuration (of rank 2)* provided that $G_0$ is maximal in G and $G_{01}$ is maximal in both $G_0$ and $G_1$.

**Lemma 2.1.** *If* $\Gamma$ *is an RWPRI configuration, then* $\Gamma$ *is firm, residually connected and flag-transitive.*

*Proof.* This is a direct consequence of the results obtained in [10]. □

This lemma implies that all geometries of rank 2 which are RWPRI are also F, RC, and FT. Thus the search of the rank 2 geometries of the Suzuki group $Sz(q)$ that are F, RC, FT and RWPRI is reduced to the search of all RWPRI configurations.

The ordered pairs $(\Gamma, G)$ and $(\Gamma', G)$ are *isomorphic* (resp. *conjugate*) if there exists an automorphism (resp. *internal automorphism*) of $G$ mapping $\Gamma$ onto $\Gamma'$.

### 2.2. The Suzuki simple groups $Sz(q)$

We recall the definition of the Suzuki groups as given in [16]. Let $\mathcal{K}$ be a field of characteristic 2 with $|\mathcal{K}| > 2$. Let $\sigma$ be an automorphism of $\mathcal{K}$ such that $x^{\sigma^2} = x^2$ for each $x$ in $\mathcal{K}$. Let $\mathcal{B}$ be the 3-dimensional projective space over $\mathcal{K}$ and let $(x_0, x_1, x_2, x_3)$ be the coordinates of a point of $\mathcal{B}$. Let $E$ be the plane defined by the equation $x_0 = 0$ and put $U = (0, 1, 0, 0)\mathcal{K}$. We introduce coordinates in the affine space $\mathcal{B}_E$ by $x = \frac{x_2}{x_0}$, $y = \frac{x_3}{x_0}$, and $z = \frac{x_1}{x_0}$. Finally, let $\mathcal{D}$ be the set of points of $\mathcal{B}$ consisting of $U$ and all those points of $\mathcal{B}_E$ whose coordinates $(x, y, z)$ satisfy the equation

$$z = xy + x^{\sigma+2} + y^{\sigma}$$

We denote by $Sz(\mathcal{K}, \sigma)$ the group of all projective collineations of $\mathcal{B}$ which leave $\mathcal{D}$ invariant.

**Lemma 2.2.** [16]

- *If $\mathcal{K}$ is isomorphic to $GF(2^{2e+1})$ with $e \geq 1$, then $\mathcal{K}$ admits exactly one automorphism $\sigma$ with $x^{\sigma^2} = x^2$ for all $x$ in $\mathcal{K}$.*
- *If $\mathcal{K}$ is isomorphic to $GF(2^{2e})$, then $\mathcal{K}$ does not possess an automorphism $\sigma$ with $x^{\sigma^2} = x^2$ for all $x$ in $\mathcal{K}$.*

This theorem yields that we may write $Sz(q)$ instead of $Sz(\mathcal{K}, \sigma)$ if $\mathcal{K}$ is isomorphic to $GF(q)$ with $q = 2^{2e+1}$. The groups $Sz(q)$ are the *Suzuki groups* named after Michio Suzuki who found them in 1960. The generalizations $Sz(\mathcal{K}, \sigma)$ are due to Rimhak Ree and Jacques Tits (see for example [19]).

The set $\mathcal{D}$ is an *ovoid*, i.e. a non-empty point-set of a projective 3-space that satisfies the following three conditions.

1. No three points are collinear.
2. If $p \in \mathcal{D}$, there exists a plane $E$ of $\mathcal{B}$ with $\mathcal{D} \cap E = \{p\}$.
3. If $p \in \mathcal{D}$ and if $E$ is a plane of $\mathcal{B}$ with $\mathcal{D} \cap E = \{p\}$, then all lines $l$ through $p$ which are not contained in $E$ carry a point of $\mathcal{D}$ distinct from $p$.

For $a, b \in \mathcal{K}$, we denote by $\tau(a, b)$ the mapping defined by

$$(x, y, z)^{\tau(a,b)} = (x + a, y + b + a^\sigma x, z + ab + a^{\sigma+2} + b^\sigma + ay + a^{\sigma+1}x + bx)$$

We have $\tau(a, b)\tau(c, d) = \tau(a + c, ac^\sigma + b + d)$.

For $k \in \mathcal{K}^*$, we define the collineation $\eta(k)$ by

$$(x, y, z)^{\eta(k)} = (kx, k^{\sigma+1}y, k^{\sigma+2}z)$$

A straightforward computation shows that $\tau(a, b)\eta(k) = \eta(k)\tau(ka, k^{\sigma+1}b)$.

Let $\omega$ be the collineation of $\mathcal{B}$ defined by $(x_0, x_1, x_2, x_3)^\omega = (x_1, x_0, x_3, x_2)$.

The following lemma, due to Tits, will be used in a further section to study the structure of the stabilizer of a point of the ovoid.

**Lemma 2.3.** [19] *Let $\mathcal{K}$ be a commutative field of characteristic 2 with $|\mathcal{K}| > 2$ and assume that $\mathcal{K}$ admits an automorphism $\sigma$ such that $x^{\sigma^2} = x^2$ for all $x \in \mathcal{K}$. If $\mathcal{D}$ is the point set defined above in the projective space of dimension 3 over $\mathcal{K}$, then $Sz(\mathcal{K}, \sigma)$ acts doubly transitively on $\mathcal{D}$.*

*Moreover, if $\gamma \in Sz(\mathcal{K}, \sigma)_U$, then there exists exactly one triple $(k, a, b) \in \mathcal{K}^* \times \mathcal{K} \times \mathcal{K}$ with $\gamma = \eta(k)\tau(a, b)$, and if $\gamma \in Sz(\mathcal{K}, \sigma)\backslash Sz(\mathcal{K}, \sigma)_U$, then there exists exactly one 5-tuple $(k, a, b, c, d) \in \mathcal{K}^* \times \mathcal{K} \times \mathcal{K} \times \mathcal{K} \times \mathcal{K}$ with $\gamma = \eta(k)\tau(a, b)\omega\tau(c, d)$.*

### 2.3. Some notation

There are four numbers that play an important role in the structure of the subgroups of $Sz(q)$. They are respectively $q$, $q - 1$, $q + r + 1$ and $q - r + 1$ where $r = \sqrt{2q}$. We note $q + r + 1 = \alpha_q$ and $q - r + 1 = \beta_q$.

When $n$ is a proper divisor of $m$, we write $n \,|_P\, m$.

When $n$ is a maximal divisor of $m$, we write $n \,|_M\, m$.

## 3. Some useful properties

In this section, we give some properties concerning $q-1$, $\alpha_q$, and $\beta_q$.

**Lemma 3.1.** (Cataldi, 1603) *If $2^{2e+1}-1$ is a prime number, then $2e+1$ is a prime number.*

The following lemma implies that if $q-1$ is a prime number then the Suzuki group $Sz(q)$ does not have any subgroup isomorphic to a Suzuki group of lower order. Let us remark that the converse is not true.

**Lemma 3.2.** *If there exist $s = 2^{2f+1}$ and an integer $a$ such that $q = s^a$, then $q-1$ is not a prime number.*

*Proof.* $s = 2^{2f+1}$ and $q = 2^{2e+1} = 2^{a\cdot(2f+1)}$ implies that $2e+1$ is not a prime number. Suppose $q-1$ is a prime number. Then Lemma 3.1 says that $2e+1$ must be a prime number. Thus we have a contradiction. □

**Lemma 3.3.** *The numbers $q-1$, $\alpha_q$ and $\beta_q$ are pairwise coprime.*

*Proof.* $(q-1,\alpha_q) = p$ implies that $p$ divides $\alpha_q-(q-1)$ and $\alpha_q+(q-1)$. Thus $p$ divides $2^{e+1}+1$ and $2^{e+1}+2$. This implies $p=1$.
By a similar argument, we see that $(q-1,\beta_q) = p$ implies that $p$ divides both $2^{e+1}-2$ and $2^{e+1}-1$. Thus $p=1$.
At last, suppose $(\alpha_q,\beta_q) = p \neq 1$. Then $p$ divides $\alpha_q-\beta_q$ which is a power of 2. Thus $p$ is a even number dividing two odd numbers, a contradiction. □

**Lemma 3.4.** *Let $s = 2^{2e+1}$ and $q = s^{2f+1}$. Then $s^2+1$ divides $q^2+1$.*

*Proof.* $s^2+1 = 2^{4e+2}+1$ and $q^2+1 = 2^{(4e+2)(2f+1)}+1$. Thus we have

$$\frac{q^2+1}{s^2+1} = \sum_{i=1}^{2f+1} (-1)^{i+1}\cdot 2^{(4e+2)(2f+1-i)}$$

□

**Lemma 3.5.** *Let $s = 2^{2e+1}$ and $q = s^{2f+1}$. Then, $s^2+1$ does not divide $\alpha_q$ nor $\beta_q$.*

*Proof.* The following argument is due to J.A. Thas.
Assume first that $s^2+1 \mid \alpha_q$. Then

$$2^{2(2e+1)}+1 \mid 2^{(2e+1)(2f+1)}+2^{2ef+e+f+1}+1$$

Suppose first that $f$ is odd. Then

$$\begin{aligned} 2^{2(2e+1)}+1 &\mid (2^{2(2e+1)f}+1-1)2^{2e+1}+2^{2ef+e+f+1}+1 \\ &\mid -2^{2e+1}+2^{2ef+e+f+1}+1 \end{aligned}$$

Because $f$ is odd, we can write $f=2r+1$ with $r$ an integer. Then

$$\begin{aligned} 2^{2(2e+1)}+1 &\mid -2^{2e+1}+2^{4er+2e+2r+1+e+1}+1 \\ &\mid -2^{2e+1}+2^{2(2e+1)r+3e+2}+1 \end{aligned}$$

Suppose that $r$ is odd. Then

$$\begin{aligned} 2^{2(2e+1)}+1 &\mid -2^{2e+1}+(2^{2(2e+1)r}+1-1)2^{3e+2}+1 \\ &\mid -2^{2e+1}-2^{3e+2}+1 \\ &\mid 2^{2e+1}+2^{3e+2}-1 \end{aligned}$$

But $2^{4e+2}+1 > 2^{2e+1}+2^{3e+2}-1$, thus we have a contradiction.
Suppose that $r$ is even. Then $r = t+1$, with $t$ odd and we have the following.

$$\begin{array}{ll} 2^{2(2e+1)}+1 & \mid \; -2^{2e+1}+(2^{2(2e+1)t}+1-1)2^{7e+4}+1 \\ & \mid \; -2^{2e+1}-2^{7e+4}+1 \\ & \mid \; (2^{2(2e+1}+1-1)2^{3e+2}+2^{2e+1}+1 \\ & \mid \; 2^{3e+2}-2^{2e+1}-1 \end{array}$$

But $2^{4e+2}+1 > 2^{3e+2}-2^{2e+1}-1$, thus we have a contradiction.
Hence we have proved that $f$ cannot be odd. The same reasoning can be made for $f$ even. And the same arguments can be used to show that $2^{2(2e+1)}+1$ cannot divide $\beta_q$. □

**Lemma 3.6.** *Let $s = 2^{2e+1}$ and $q = s^{2f+1}$, with $e$ and $f$ some positive integers. Then $\alpha_q$ and $\beta_q$ are not prime.*

*Proof.* Easy to see thanks to Lemmas 3.4, 3.5. □

**Lemma 3.7.** (Fermat, 1640) *If $p$ is a prime number then $p$ divides $2^p - 2$.*

## 4. The maximal subgroups of $Sz(q)$

Table 1 gives a description of the conjugacy classes of maximal subgroups of $Sz(q)$. Those results have been obtained by Suzuki [17]. It is from these subgroups that we start to classify all RWPRI configurations in the subgroup lattice of $Sz(q)$.
For a Suzuki simple group $Sz(q)$, with $q = 2^{2e+1}$, if $2e+1$ is a prime number, there are no proper subgroups isomorphic to a Suzuki simple group. This implies that if $2e+1$ is a prime number, the group $Sz(q)$ has only four conjugacy classes of maximal subgroups. Some Suzuki groups may have subgroups of Suzuki type which are not isomorphic. For example, $Sz(32768)$ has maximal subgroups that are respectively isomorphic to $Sz(8)$ and $Sz(32)$. This means we don't have five classes of maximal subgroups there but six. In fact, the number of non-isomorphic classes of $Sz(s)$ is equal to the number of prime divisors of $2e+1$. Dividing $2e+1$ by each of them gives us a number $2f+1$ for which there exists a $Sz(2^{2f+1})$ which is a maximal subgroup of $Sz(q)$.

## 5. The maximal subgroups of the maximal subgroups of $Sz(q)$

**Lemma 5.1.** *The maximal subgroups of $(q \cdot q) : (q-1)$ have one of the following structures.*

- $(q \cdot q) : n$ *where $n$ is a maximal divisor of $q-1$.*
- $q : (q-1)$

*Proof.* The subgroup $G_0$ is the stabilizer of a point. Let us take $U$ as the fixed point. Then, a complete description of this group is given in [16]. Lüneburg also shows that the stabilizer of a circle is of order $q(q-1)$. A circle is obtained by intersecting the ovoid with a plane. If we take the plane $\Pi$ defined by a cartesian equation $x = 0$, we see that the $\tau(0,b)$ and the $\eta(k)$ stabilize this plane for $b \in GF(q)$ and $k \in GF(q)^* = GF(q) - \{0\}$. Looking at the subgroup generated by these collineations, we have a subgroup of order $q(q-1)$ which is the stabilizer of a circle. This group has a normal subgroup formed by the $\tau(0,b)$, of order $q$.

| **Structure** | **Order** | **Index** | **Description** |
|---|---|---|---|
| $(q \cdot\!\!\cdot q) : (q-1)$ | $q^2 \cdot (q-1)$ | $q^2+1$ | Normalizer of a 2-Sylow, stabilizer of a point of the ovoid |
| $D_{2(q-1)}$ | $2 \cdot$ (q-1) | $\frac{(q^2+1)\cdot q^2}{2}$ | Stabilizer of a pair of points of the ovoid |
| $\alpha_q : 4$ | $\alpha_q \cdot 4$ | $\frac{q^2(q-1)}{4\beta_q}$ | Normalizer of a $\alpha_q$-Sylow |
| $\beta_q : 4$ | $\beta_q \cdot 4$ | $\frac{q^2(q-1)}{4\alpha_q}$ | Normalizer of a $\beta_q$-Sylow |
| $Sz(2^{2f+1})$ with $2f+1 \mid_M 2e+1$ | $(s^2+1) \cdot s^2 \cdot (s-1)$ | | |

Table 1: The maximal subgroups of $Sz(q)$

And its structure is $q : (q-1)$. We denote it by $G_C$. We still have to show that it is a maximal subgroup of $G_0$. Thanks to Lemma 2.3, every element of $G_0$ can be written in a unique form as $\eta(k)\tau(a,b)$ with $a, b \in GF(q)$ and $k \in GF(q)^*$. Thus we see that the only elements we could add are of the form $\tau(a,0)$ with $a \in GF(q)$, $a \neq 0$. If we add any of these elements to $G_C$, the $\eta(k)$ act such that every $\tau(a,0)$ is generated, and thus $G_0$ is generated. We can then conclude that $G_C$ is maximal in $G_0$.

Taking all $\tau(a,b)$, we obtain a subgroup of structure $q \cdot\!\!\cdot q$ which is normal in $G_0$. The subgroup formed by the $\eta(k)$, with $k \in GF(q)^*$ is a cyclic group of order $q-1$. It then has maximal subgroups which are the subgroups of order $n$ with $n$ a maximal divisor of $q-1$. If we take one generator of such a subgroup, namely $\eta(k)$ and we look at the subgroup generated by the $\tau(a,b)$ and this $\eta(k)$, we obtain a subgroup of order $q^2 \cdot n$. Again, $q \cdot\!\!\cdot q$ is normal in it and thus this subgroup has structure $(q \cdot\!\!\cdot q) : n$. The only elements that are missing use $\eta(l)$ with $l$ not appearing in the multiplicative group generated by $k$. This multiplicative group is a maximal subgroup of $GF(q)^*$. We can conclude that $(q \cdot\!\!\cdot q) : n$ is maximal in $G_0$.

The group generated by all the $\tau(a,0)$ and the $\eta(k)$, with $a \in GF(q)$ and $k \in GF(q)^*$ is $G_0$. Thus no other maximal subgroups can be found. □

**Lemma 5.2.** *The maximal subgroups of $n : 4$ have the following structure.*

- $m : 4$ *where $m$ is a maximal divisor of $n$*
- $n : 2$

*Proof.* Obviously, $n : 4$ has a maximal subgroup whose structure is $n : 2$. In $n : 4$, there exists a normal subgroup isomorphic to $m$ for each divisor $m$ of $n$. Extending this cyclic group by 4, we obtain a group of order $4m$ which has structure $m : 4$ because $(m, 4) = 1$. If $m$ is a maximal divisor of $n$, then $[n{:}4 : m{:}4]$ is a prime number, hence $m{:}4$ is a maximal subgroup of $n{:}4$. □

**Lemma 5.3.** *The maximal subgroups of $D_{2(q-1)}$ have the following structure.*

- *the cyclic group of order $q-1$*
- *$D_{2n}$ where $n$ is a maximal divisor of $q-1$. If $q-1$ is a prime number then $D_2$ is the cyclic group of order 2.*

*Proof.* Straightforward □

As to the maximal subgroups that are isomorphic to some Suzuki simple group, we refer to Table 1.

## 6. Rank 2 RWPRI geometries

Let $G$ be a group. Let $G_0$ and $G_1$ be subgroups of $G$ and let $G_{01} = G_0 \cap G_1$. The RWPRI condition in rank 2 asks that either $G_0$ or $G_1$ must be a maximal subgroup of $G$ and that $G_{01}$ must be a maximal subgroup of $G_0$ and $G_1$. We use the following method to determine all the RWPRI geometries of the Suzuki group $Sz(q)$. We choose a subgroup $G_0$ and we impose that it must be a maximal subgroup of $Sz(q)$. Then we determine all the $G_{01}$ possible groups. They are the maximal subgroups of $G_0$. Finally, we look at all the $G_1$ candidates which must contain $G_{01}$ as a maximal subgroup and which must be different from $G_0$.

**Lemma 6.1.** *Let $G \cong Sz(q)$. There are no RWPRI geometries of rank $> 1$ with $G_0 \cong (q \cdot\!\!\!\cdot q) : (q-1)$.*

*Proof.* We start from the maximal subgroups of $G_0$. They have been determined in Lemma 5.1. We see that the only subgroups of $Sz(q)$ that contain one of the $G_{01}$ candidates are thus isomorphic to $G_0$. Because the normalizer of each $G_{01}$ candidate must be a subgroup of $G_0$, we obtain that $G_1$ must be equal to $G_0$ which is a contradiction to the fact that $G_{01}$ is their intersection. □

**Lemma 6.2.** *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let $G \cong Sz(q)$*
*If $2e+1$ and $q-1$ are prime numbers, the RWPRI rank 2 geometries of $G$ with $G_0 \cong D_{2(q-1)}$ are isomorphic to one of the geometries appearing in Table 2.*
*If $2e+1$ is prime and $q-1$ is not prime, the only RWPRI rank 2 geometry of $G$ with $G_0 \cong D_{2(q-1)}$ is obtained with $G_1 \cong q : (q\text{-}1)$.*
*If $2e+1$ is not prime, the RWPRI rank 2 geometries of $G$ with $G_0 \cong D_{2(q-1)}$ are isomorphic to one of the geometries appearing in Table 3.*

*Proof.* By Lemma 5.3, $G_{01}$ can be isomorphic to either $q-1$ of $D_{2n}$ where $n$ is a maximal divisor of $q-1$.

Suppose first that $G_{01} = q-1$.
The $G_1$ candidates are neither subgroups of $\alpha_q : 4$, nor of $\beta_q : 4$ because of Lemma 3.3. The normalizer of a $q-1$ is $G_0$. Thus two dihedral groups $\mathrm{D}_{2(q-1)}$ never intersect on a cyclic group of order $q-1$ in $Sz(q)$, and $G_1$ cannot be a $D_{2(q-1)}$. So the only candidates must be subgroups of $(q \cdot\!\!\!\cdot q) : (q-1)$ or $Sz(s)$ with $q = s^a$ for some integer $a$.

Suppose $G_1 < (q \cdot\!\!\!\cdot q) : (q-1)$. Then, in order to have $G_{01}$ maximal in $G_1$ the only candidate $G_1$ is $q : (q-1)$. This gives rise to an RWPRI configuration. If we fix $G_1$ and

| $G_1$ | $G_{01}$ | $\#\Gamma$ |
|---|---|---|
| $q:(q-1)$ | $q-1$ | 1 |
| $D_{2(q-1)}$ | 2 | $\frac{(q-2)(q+4)}{4(2e+1)}+1$ |
| $D_{2n}$ *where $n$ is a prime divisor of $\alpha_q$ or $\beta_q$* | 2 | $\frac{(q-2)(q+2)}{4(2e+1)}+1$ |
| $2^2$ | 2 | $\frac{q-2}{2(2e+1)}$ |
| 4 | 2 | $\frac{q-2}{2(2e+1)}+1$ |

Table 2: $G_0 = D_{2(q-1)}$, and $2e+1$ and $q-1$ are prime numbers

| $G_1$ | $G_{01}$ | $\#\Gamma$ |
|---|---|---|
| $q:(q-1)$ | $q-1$ | 1 |
| $Sz(s)$ *iff $s-1$ is a maximal divisor of $q-1$* | $D_{2(s-1)}$ | 1 |

Table 3: $G_0 = D_{2(q-1)}$ and $2e+1$ is not prime

$G_{01}$ (this can be done because all subgroups isomorphic to $G_{01}$ are conjugate in $G_1$), there is only one $D_{2(q-1)}$ intersecting $G_1$ in $G_{01}$. Thus we have only one such configuration up to isomorphism.

Suppose $G_1 < Sz(s)$. As $Sz(s)$ exists, it implies that $q-1$ is not a prime number. No maximal subgroup of $Sz(s)$ contains a cyclic group of order $q-1$. Thus $G_1$ must be $Sz(s)$. But $q-1$ is not a maximal subgroup of $Sz(s)$.

Now suppose that $G_{01} = D_{2n}$ where $n$ is a maximal divisor of $q-1$.
Let us first examine the case where $q-1$ is a prime number. Then $G_{01}$ is a cyclic group of order 2. This gives rise to the following candidates for $G_1$.

- $D_{2(q-1)}$
- $D_{2m}$ where $m$ is a prime divisor of $\alpha_q$ or $\beta_q$.
- $2^2$
- 4

All of these subgroups can be taken as $G_1$ to form an RWPRI configuration.

Suppose $G_1 = D_{2(q-1)}$. We can fix $G_0$ and $G_{01}$ because all subgroups isomorphic to $G_{01}$ are conjugate in $G_0$. We know that there are $\frac{q^2\cdot(q^2+1)}{2}$ subgroups isomorphic to $G_0$ in $Sz(q)$. We also know that $G_0$ contains $q-1$ subgroups of order 2. And the number of involutions in $Sz(q)$ is equal to $(q^2+1)(q-1)$. Thus we can conclude that there are $\frac{q^2}{2}$ subgroups isomorphic to $G_0$ intersecting in the same $G_{01}$. If we want to fix the types of our geometries, $G_{01}$ cannot transform one of these geometries into another. If we do not fix the types, then we have to count how many of our geometries are self-dual by conjugacy. Suppose we have a geometry $\Gamma(Sz(q); D_{2(q-1)} = G_0, D_{2(q-1)} = G_1)$. We know that the normalizer of $G_{01}$ is a group of order $q^2$ isomorphic to $q \dot{\cdot} q$. By looking at the action of $H = q \dot{\cdot} q/G_{01}$ on the $\frac{q^2}{2}$ subgroups isomorphic to $D_{2(q-1)}$ containing $G_{01}$, we see that it is transitive and hence strictly transitive because its order is equal to its degree. Because $H$ contains $q-1$ involutions, there are $q-1$ geometries that are self-dual, and the $\frac{q^2-2q}{2}$ elements of order 4 spread the $\frac{q^2}{2}$ subgroups into squares. Every square appears two times, that is one time in each way. And no two disjoint squares can have a common vertex. This means that through every of the $\frac{q^2}{2}$ points, there are $\frac{q^2-2q}{4}$ disjoint squares that fuse $\frac{q^2-2q}{2}$ geometries inside sets of 2 geometries. So finally, we have $q-1+\frac{q^2-2q}{4}=1+\frac{(q-2)(q+4)}{4}$ non-conjugate geometries in this case. In $Aut(Sz(q))$, the stabilizer of $G_0$ and $G_{01}$ is a cyclic group of order $2(2e+1)$. Its normalizer is two times bigger. The normalizer of $G_1$ is $G_1:(2e+1)$. And each subgroup isomorphic to $G_1:(2e+1)$ contains $q-1$ subgroups isomorphic to $Z_{2(2e+1)}$. This implies that there are two subgroups isomorphic to $G_1:(2e+1)$ containing the same $Z_{2(2e+1)}$. Thus one geometry is fixed under the action of $Aut(Sz(q))$ and all the others are fused inside sets of $2e+1$. That is, we have $1+\frac{(q-2)(q+4)}{4(2e+1)}$ non-isomorphic RWPRI configurations and we observe that this number is indeed an integer thanks to Lemma 3.7.

Suppose $G_1=D_{2m}$ where $m$ is a prime divisor of $\alpha_q$ or $\beta_q$. We can decide to fix it without loss of generality. All its subgroups isomorphic to $G_{01}$ are conjugate in it. This implies that we can also fix $G_{01}$. The normalizer of $G_1$ in $G$ is $m{:}4$. So the stabilizer of $G_1$ and $G_{01}$ is a cyclic group of order 4. And there are $\frac{q^2}{2}$ subgroups isomorphic to $G_0$ that intersect $G_1$ in $G_{01}$. The cyclic group of order 4 fuses them inside sets of two. So there are $\frac{q^2}{4}$ non-conjugate geometries. The subgroup which stabilizes $G_0$ and $G_{01}$ in $Aut(G)$ is isomorphic to $Z_{2(2e+1)}$. The subgroup stabilizing $G_1$ in $Aut(G)$ is isomorphic to $G_1:(2e+1)$. Again, a simple counting argument permits us to conclude that there is only one subgroup isomorphic to $G_1$ which is fixed under the action of $Z_{2(2e+1)}$. Thus, from the $\frac{q^2}{4}$ non-conjugate geometries, one is fixed under the action of $Aut(G)$ and all other are fused inside sets of $2e+1$ geometries. This means we have $1+\frac{(q-2)(q+2)}{4(2e+1)}$ non-isomorphic RWPRI configurations of that kind. And, again, thanks to Lemma 3.7, we observe that this number is indeed an integer.

Suppose $G_1=2^2$. Then $G_1$ contains three subgroups of order 2. Also, $G_1$ is contained in the stabilizer of some point. In each point-stabilizer, there are $q-1$ $Z_2$. We must combine two of them to obtain subgroups isomorphic to $2^2$. We first choose one $Z_2$. Then, the second one can be chosen in $\frac{q-2}{2}$ ways. But there we have taken each possibility 3 times. So the number of $2^2$ is $\frac{(q^2+1)(q-1)(q-2)}{6}$. From this, we can count the number of $2^2$ containing the same $Z_2$. There are $\frac{q-2}{2}$ such subgroups. Thus, by fixing $G_0$ and $G_{01}$, we have $\frac{q-2}{2}$ possible $G_1$. These give $\frac{q-2}{2}$ non-conjugate geometries. The normalizer of $G_1$ in $Aut(G)$ is $G_1:(2e+1)$ and this subgroup does not contain the stabilizer of $G_0$ and $G_{01}$ in $Aut(G)$. Thus no configuration of that kind is fixed under the action of $Aut(G)$. This means that all of them are fused in sets of $2e+1$ geometries. Hence we have $\frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations like that. Let us observe that in the smallest case (i.e. for $q=8$), there is only one geometry of that kind up to isomorphism.

Suppose $G_1=4$. Let us count the number of cyclic groups of order 4 contained in a $Sz(q)$. Each of them is contained in the stabilizer of a point (there are $q^2+1$ such stabilizers). In each stabilizer, the $\tau(a,b)$ with $a\neq 0$ are elements of order 4. There are $(q-1)q$ such elements. In a cyclic group of order 4, there are 2 elements of order 4. Thus the number of cyclic groups of order 4 in $Sz(q)$ equals $(q^2+1)(q-1)\frac{q}{2}$. We already know that the number of $Z_2$ in $Sz(q)$ is $(q^2+1)(q-1)$. And it is well known that every $Z_4$ contains only one $Z_2$. Hence the number of $Z_4$ containing the same $Z_2$ is $\frac{q}{2}$. Now we fix $G_0$ and we can also fix $G_{01}$ because all the subgroups that are isomorphic to $G_{01}$ are conjugate in $G_0$. The stabilizer of

| $G_1$ | $G_{01}$ | $\#\Gamma$ |
|---|---|---|
| $\alpha_q : 4$ | 4 | $\frac{q-2)}{2(2e+1)}$ |
| $n : 4$ *where* $n$ *is a prime divisor of* $\beta_q$ | 4 | $\frac{q-2}{2(2e+1)}+1$ |
| $4 \times 2$ | 4 | $\frac{q-2}{2(2e+1)}$ |

Table 4: The case $G_0 = \alpha_q : 4$ and $\alpha_q$ prime.

| $G_1$ | $G_{01}$ | $\#\Gamma$ |
|---|---|---|
| $Sz(s)$ *iff* $\alpha_s$ *or* $\beta_s$ *is a maximal divisor of* $\alpha_q$ | $\alpha_s : 4$ *or* $\beta_s : 4$ | 1 |

Table 5: The case $G_0 = \alpha_q : 4$ and $\alpha_q$ not prime.

$G_0$ and $G_{01}$ is $G_{01}$. Thus it fixes $G_1$ also. This means we have $\frac{q}{2}$ non-conjugate geometries. The normalizer of $G_1$ in $Aut(G)$ being a subgroup isomorphic to $4 : (2e + 1)$, it contains the stabilizer of $G_0$ and $G_{01}$ in $Aut(G)$. This means that there is one configuration which is fixed under the action of $Aut(G)$ and all the others are fused under $Aut(G)$ inside sets of $2e + 1$ geometries. Thus we have $1 + \frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations of that kind.

Suppose now that $q - 1$ is not a prime number. Then $G_{01}$ is a dihedral group $D_{2n}$. This means that the $G_1$ candidates will neither be subgroups of $\alpha_q : 4$, nor of $\beta_q : 4$. Here we have two candidates for $G_1$. They are respectively $D_{2(q-1)}$ and $Sz(s)$ if $s - 1 = n$. The case $G_1 = D_{2(q-1)}$ is not possible. This is due to the fact that $N_{Sz(q)}(D_{2n}) = D_{2n}$. While fixing $G_0$, we can also fix $G_{01}$ because all the subgroups isomorphic to $G_{01}$ in $G_0$ are conjugate. It is easy to see that there is only one subgroup isomorphic to $Sz(s)$ which contains $G_{01}$. So the $G_1$ is uniquely determined. This gives us only one geometry up to conjugacy and thus up to isomorphism. □

An example of a configuration with $G_1 = Sz(s)$ can be obtained in $Sz(512)$ by taking as $G_0$ (resp. $G_1$, $G_{01}$) a subgroup isomorphic to $D_{2(511)}$ (resp. $Sz(8)$, $D_{14}$).

**Lemma 6.3.** *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let $G \cong Sz(q)$*
*If $\alpha_q$ is a prime number, the RWPRI rank 2 geometries of $G$ with $G_0 \cong \alpha_q : 4$ are isomorphic to one of the geometries appearing in Table 4.*
*If $\alpha_q$ is not a prime number, the RWPRI rank 2 geometries of $G$ with $G_0 \cong \alpha_q : 4$ are isomorphic to one of the geometries appearing in Table 5.*

*Proof.* By Lemma 5.2, $G_{01}$ can be isomorphic to either $\alpha_q : 2$ or $n : 4$ where $n$ is a maximal divisor of $\alpha_q$.

Suppose first that $G_{01} = n : 4$.
Let us first consider the case where $n = 1$ (i.e. $\alpha_q$ is a prime number). In this case, several candidates appear for $G_1$.

- $4 \times 2$
- $m : 4$ where $m \mid \beta_q$, and $m$ is a prime number.
- $\alpha_q : 4$

All these cases give rise to RWPRI candidates.

Suppose $G_1 = 4\times 2$. Let us count the number of subgroups isomorphic to $4\times 2$ in $Sz(q)$. Each of them is contained in the stabilizer of a point. In each point-stabilizer, there are $(q-1)\frac{q}{2}$ cyclic groups of order 4, as we saw before. The $4\times 2$ are constructed by adding to these $Z_4$ an element of order 2. This element must be different from the one in the $Z_4$. Thus we have $q-2$ such possible elements. But a $4\times 2$ contains 3 subgroups of order 2. This implies that there are $\frac{q-2}{2}$ subgroups isomorphic to $4\times 2$ which contain the same $Z_4$. And every $4\times 2$ contains 2 subgroups of order 4. Thus the number of $4\times 2$ is $(q^2+1)(q-1)\frac{q}{2}\cdot\frac{q-2}{2}\cdot\frac{1}{2} = \frac{(q^2+1)q(q-1)(q-2)}{8}$. Now let us count the number of non-conjugate and non-isomorphic geometries. We can fix $G_0$. All the subgroups isomorphic to $Z_4$ are conjugate in $G_0$. This implies that we can also fix $G_{01}$. The stabilizer of $G_0$ and $G_{01}$ being $G_{01}$, we have $\frac{q-2}{2}$ non-conjugate geometries. The $(2e+1)$ classes of subgroups isomorphic to $4\times 2$ in $G$ are fused under the action of $Aut(G)$. This means that the normalizer of a subgroup isomorphic to $4\times 2$ will not contain elements of order $(2e+1)$. The stabilizer of $G_0$ and $G_{01}$ in $Aut(G)$ is a subgroup isomorphic to $4:(2e+1)$. Thus, there is no configuration which is fixed under the action of $Aut(G)$. Then, all the non-conjugate configurations are fused under the action of $Aut(G)$ inside sets of $2e+1$ geometries. Hence we have $\frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations like that. Let us observe that in the smallest case (i.e. for $q=8$), there is only one such geometry up to isomorphism.

Suppose $G_1 = m{:}4$. The subgroup $G_1$ has its normalizer in $\beta_q : 4$. Thus it is self-normalizing. This gives us the number of subgroups isomorphic to $m{:}4$ in $Sz(q)$. There are $\frac{(q^2+1)q^2(q-1)}{4m}$ such subgroups. Each of them contains $m$ cyclic subgroups of order 4. And we already know that there are $\frac{(q^2+1)q(q-1)}{2}$ cyclic groups of order 4 in $Sz(q)$. Thus every $Z_4$ is contained in $\frac{q}{2}$ subgroups $m{:}4$. We can fix $G_0$. All the subgroups isomorphic to $Z_4$ are conjugate in $G_0$. Thus we can also fix $G_{01}$. The stabilizer of $G_0$ and $G_{01}$ is $G_{01}$. This implies that there are $\frac{q}{2}$ non-conjugate geometries here. The stabilizer of $G_1$ and $G_{01}$ in $Aut(G)$ being isomorphic to $4:(2e+1)$, there is only one configuration which is fixed under the action of $Aut(G)$ and all the others are fused inside sets of $2e+1$ geometries. Thus we have $1+\frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations in that case.

Suppose $G_1 = \alpha_q : 4$. Then there are $\frac{q}{2}$ subgroups isomorphic to $G_1$ that contain the same $Z_4$. One of them is $G_0$ so we have here $\frac{q}{2}-1$ non-conjugate geometries if we decide to fix the types. It is easy to see that $N_G(Z_4)$ is a group which is of order $2q$ and which is isomorphic to $q{\cdot}2$. By looking at the action of $(q{\cdot}2)/4$ on the set of $\frac{q}{2}$ subgroups isomorphic to $\alpha_q : 4$, we see that this group is transitive and hence strictly transitive. It is also easy to see that all its elements (except the identity) are involutions. This means that all the $\frac{q}{2}-1$ geometries we found are self-dual. Thus there are $\frac{q-2}{2}$ non-conjugate geometries. The stabilizer of $G_0$ and $G_{01}$ in $Aut(G)$ is isomorphic to $4:(2e+1)$. Thus there is no geometry fixed under the action of $Aut(G)$ and all of them are fused inside sets of $2e+1$ elements. This means we have $\frac{q-2}{2(2e+1)}$ non-isomorphic RWPRI configurations of that kind. Let us observe that in the smallest case (i.e. for $q=8$), there is only one geometry up to isomorphism.

Now suppose $n \neq 1$, which means that $\alpha_q$ is not a prime number. Then $G_1$ is not a subgroup of $\beta_q : 4$ (resp. $D_{2(q-1)}$) because $(\alpha_q, \beta_q)$ (resp. $(\alpha_q, q-1)) = 1$. Two candidates arise for $G_1$. They are $\alpha_q : 4$ and $Sz(s)$ if and only if $\alpha_s$ or $\beta_s$ is a maximal divisor of $\alpha_q$.

The normalizer $N_{Sz(q)}(n:4)$ is $n:4$. Thus the case $G_1 = \alpha_q : 4$ leads to a contradiction.

| $G_1$ | $G_{01}$ | $\#\Gamma$ |
|---|---|---|
| $\beta_q : 4$ | 4 | $\frac{q-2)}{2(2e+1)}$ |
| $n : 4$ *where* $n$ *is a prime divisor of* $\beta_q$ | 4 | $\frac{q-2}{2(2e+1)}+1$ |
| $4 \times 2$ | 4 | $\frac{q-2}{2(2e+1)}$ |

Table 6: The case $G_0 = \beta_q : 4$ and $\beta_q$ prime.

| $G_1$ | $G_{01}$ | $\#\Gamma$ |
|---|---|---|
| $Sz(s)$ *iff* $\alpha_s$ *or* $\beta_s$ *is a maximal divisor of* $\beta_q$ | $\alpha_s : 4$ *or* $\beta_s : 4$ | 1 |

Table 7: The case $G_0 = \beta_q : 4$ and $\beta_q$ not prime.

We then have just one candidate left, that is the one with $G_1 = Sz(s)$. We can fix $G_0$ and $G_{01}$ because $G_0$ conjugates all its subgroups isomorphic to $G_{01}$, the stabilizer being $G_{01}$. And there is only one $Sz(s)$ containing $G_{01}$. Thus there is only one such geometry up to conjugacy, and hence up to isomorphism.

Suppose that $G_{01} = \alpha_q : 2$.

This implies that $G_1$ must be a subgroup of $Sz(s)$. So $q-1$ cannot be a prime number. The order of $G_{01}$ must divide the order of $G_1$.

Suppose $G_1$ is a subgroup of $(s{\cdot}s) : (s-1)$ or $D_{2(s-1)}$. Then $\alpha_q$ divides $s-1$ which is a contradiction to the fact that $(q-1, \alpha_q) = 1$ and $s-1 \mid q-1$.

Suppose $G_1$ is a subgroup of $\alpha_s$:4 (resp. $\beta_s : 4$). Then the order of $G_1$ is smaller than the order of $G_{01}$ which is also a contradiction. Finally, suppose $G_1 = Sz(s)$. Then $G_{01}$ must be a maximal subgroup of $Sz(s)$. This yields $s-1 = \alpha_q$ and $s-1 \mid q-1$, a contradiction. □

**Lemma 6.4.** *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let $G \cong z(q)$.*
*If $\beta_q$ is a prime number, the RWPRI rank 2 geometries of $G$ with $G_0 \cong \beta_q : 4$ are isomorphic to one of the geometries appearing in Table 6.*
*If $\beta_q$ is not a prime number, the only RWPRI rank 2 geometries of $G$ with $G_0 \cong \beta_q : 4$ are isomorphic to one of the geometries appearing in Table 7.*

*Proof.* Same as proof of lemma 6.3 □

At this stage, we have determined the RWPRI configurations for the rank 2 geometries of the Suzuki groups $Sz(q)$ where $2e+1$ is a prime number. If $2e+1$ is not a prime number, $Sz(q)$ may have other maximal subgroups which are isomorphic to $Sz(s)$. Before starting the discussion about this case, we demonstrate a few lemmas that are used in the lemma giving the RWPRI rank 2 geometries of $Sz(q)$ with $G_0 \cong Sz(s)$.

**Lemma 6.5.** *For every subgroup isomorphic to $(s{\cdot}s) : (s-1)$, there is only one subgroup isomorphic to $Sz(s)$ containing it.*

*Proof.* Let $P$ be a subgroup isomorphic to $(s{\cdot}s) : (s-1)$. Every subgroups isomorphic to $Sz(s)$ are conjugate in $Sz(q)$. We only have to show that $N_{Sz(q)}(P) \leq Sz(s)$ in order to prove

that there exists only one $Sz(s)$ containing $P$. $Sz(s)$ has an orbit of length $s^2+1$. In this orbit, $P$ fixes a point $x$ and exchanges all the $s^2$ other points. We know that $N_{Sz(q)}(P)$ fixes $x$. It is thus a subgroup of the stabilizer $G_x$ of $x$ which has structure $(q{\cdot}q):(q-1)$. In $G_x$, there is a cyclic subgroup $Z_{q-1}$ which has exactly two fixed points. And so does $Z_{s-1} \leq Z_{q-1}$. Thus the second fixed point is in the orbit of $s^2$ points. We also know that $P$ has one orbit of length 1, one of length $s^2$ and all the others of length $\geq s^2$. It can be easily seen that the other orbits have length $s^2(s-1)$.
Because $s \neq 2$, the normalizer $N_{Sz(q)}(P)$ must stabilize the orbit of 1 point and the orbit of $s^2$ points. Thus it must be a subgroup of $Sz(s)$. □

**Lemma 6.6.** *For every subgroup isomorphic to $D_{2(s-1)}$, there is only one subgroup isomorphic to $Sz(s)$ containing it.*

*Proof.* The normalizer $N_{Sz(q)}(D_{2(s-1)})$ must be a subgroup of $D_{2(q-1)}$. Because $q-1$ and $s-1$ are odd numbers, $D_{2(s-1)}$ is self-normalizing in $D_{2(q-1)}$. □

**Lemma 6.7.** [17] *Let $\Pi$ be the partition consisting of four classes of conjugate subgroups of $Sz(q)$:*

- *the Sylow 2-subgroups of $Sz(q)$,*
- *one class of cyclic groups of order $q-1$,*
- *one class of cyclic groups of order $q+r+1$,*
- *one class of cyclic groups of order $q-r+1$.*

*If $W$ is a subgroup of $Sz(q)$, then there exists $X \in \Pi$ with $W \subseteq N_{Sz(q)}(X)$ or $W$ is isomorphic to $Sz(s)$ with $s \leq q$.*

**Lemma 6.8.** *For every subgroup isomorphic to $\alpha_s:4$, there is only one subgroup isomorphic to $Sz(s)$ that contains it.*

*Proof.* The number of $Sz(s)$ in $Sz(q)$ is equal to $[Sz(q):Sz(s)]$ because $N_{Sz(q)}(Sz(s)) = Sz(s)$. By lemma 6.7, the normalizer $N_{Sz(q)}(\alpha_s{:}4)$ is either $Sz(s)$ or a subgroup of $\alpha_q:4$ or $\beta_q:4$. Suppose it is $Sz(s)$. Then $Sz(s)$ is not simple, a contradiction. So it must be a subgroup of either $\alpha_q:4$ or $\beta_q:4$. Anyway, $\alpha_s:4$ is self-normalizing in the one that contains it. Thus $N_{Sz(q)}(\alpha_s:4) = \alpha_s:4$ and the number of $\alpha_s:4$ in $Sz(q)$ is equal to $[Sz(q):\alpha_s:4]$. Every $Sz(s)$ contains exactly $[Sz(s):\alpha_s:4]$ subgroups isomorphic to $\alpha_s{:}4$. Thus the number of $Sz(s)$ containing the same $\alpha_s:4$ is equal to 1. □

**Lemma 6.9.** *Let $e>0$ be a natural number. Let $q=2^{2e+1}$ and let $G \cong Sz(q)$. Let $G_0 \cong Sz(s)$ be a maximal subgroup of $Sz(q)$ and suppose $G_1$ is not one of $D_{2(q-1)}$, $\alpha_q:4$, or $\beta_q:4$ (these cases have been analysed in Lemmas 6.2, 6.3 and 6.4. Then the RWPRI rank 2 geometries of $G$ with $G_0 \cong Sz(s)$ are isomorphic to one of the geometries appearing in Table 8.*

| $G_1$ | $G_{01}$ | $\#\Gamma$ |
|---|---|---|
| $(q \colon s) : (s-1)$ | $(s \colon s) : (s-1)$ | 1 |
| $Sz(s')$ *when it is a maximal subgroup and* $s' \neq s$ | $Sz(t)$ | 1 |
| $D_{2n}$ *where* $s-1 \mid_M n \mid_P$ *q-1* | $D_{2(s-1)}$ | 1 |
| $n:4$ *where* $\alpha_s \mid_M n \mid_P \alpha_q$ *or* $\beta_q$ | $\alpha_s : 4$ | 1 |
| $n:4$ *where* $\beta_s \mid_M n \mid_P \alpha_q$ *or* $\beta_q$ | $\beta_s : 4$ | 1 |

Table 8: The case $G_0 = Sz(s)$.

*Proof.* Recall that the maximal subgroups of $Sz(s)$ have one of the following structures.

- $(s \colon s) : (s-1)$
- $D_{2(s-1)}$
- $\alpha_s$:4
- $\beta_s$:4
- $Sz(t)$ if and only if there exists a natural number $b$ such that $s = t^b$.

Suppose $G_{01} = (s \colon s) : (s-1)$.
Because $G_{01}$ is the stabilizer of a point of the subovoid, $G_1$ must be either a subgroup of $Sz(s)$ or $(q \colon q) : (q-1)$. Lemma 6.5 implies that $G_1$ cannot be a subgroup isomorphic to $Sz(s)$. Thus we have to look at $(q \colon q) : (q-1)$ and its subgroups to find some $G_1$ candidate. All the elements of $G_{01}$ are of the form $\eta(k)\tau(a,b)$ where $a$, $b \in GF(s)$ and $k \in GF(s)^*$. The subgroup $\langle G_{01}, \tau(0,c)\rangle$ with $c \in GF(q) - GF(s)$ has order $q \cdot s \cdot (s-1)$ because there are elements of the form $\eta(z)\tau(x,y)$ with $x \in GF(s)$, $y \in GF(q)$ and $z \in GF(s)^*$. It contains $G_{01}$ as a maximal subgroup because $GF(s)$ is a maximal subfield of $GF(q)$. It is easy to see that it has structure $(q \colon s) : (s-1)$.
The subgroup $\langle G_{01}, \tau(c,0)\rangle$ with $c \in GF(q) - GF(s)$ has order $q^2(s-1)$ because it generates all elements of the form $\eta(z)\tau(x,y)$ with $x$, $y \in GF(q)$, $z \in GF(s)^*$. It contains $(q \colon s) : (s-1)$ and thus $G_{01}$ is not maximal in it.
The subgroup $\langle G_{01}, \eta(l)\rangle$ with $l \in GF(q) - GF(s)$ is the group $(q \colon q) : (q-1)$ which does not contain $G_{01}$ as a maximal subgroup.
We can fix $G_1$ and $G_{01}$ because $G_1$ conjugates all its subgroups isomorphic to $G_{01}$. Because $G_{01}$ is self-normalizing as it was demonstrated in the previous lemma, the stabilizer of $G_1$ and $G_{01}$ is $G_{01}$. And there is only one subgroup isomorphic to $Sz(s)$ which contains $G_{01}$. This means we have here only one geometry up to conjugacy, and hence up to isomorphism.

Suppose $G_{01} = D_{2(s-1)}$.
Because $s-1$ divides $q-1$ the only $G_1$ candidates are $Sz(s)$ and $D_{2n}$ where $n$ is a divisor of $q-1$, and $s-1$ is a maximal divisor of $n$. In fact, we can suppose that $n$ is a proper divisor of $q-1$ because the case $n = q-1$ has been treated while determining the configurations with $G_0 = D_{2(q-1)}$. Lemma 6.6 shows that two subgroups isomorphic to $Sz(s)$ never intersect in a $D_{2(s-1)}$. Thus the only RWPRI candidates here are obtained by taking $Sz(s)$ as $G_0$, and $D_{2n}$ as $G_1$ where $n$ is a proper divisor of $q-1$, and $s-1$ is a maximal divisor of $n$. It is easy to see that there is only one such geometry up to conjugacy and hence up to isomorphism, for each such $n$.

Suppose $G_{01} = \alpha_s : 4$.
This implies that $G_1$ will certainly not be a subgroup of $D_{2(q-1)}$ or $(q{:}q) : (q-1)$. The candidates are then another $Sz(s)$ or $n : 4$ where $n$ is a proper divisor either of $\alpha_q$ or $\beta_q$ and $\alpha_s$ is a maximal divisor of $n$. Lemma 6.8 implies that $G_1$ must be an $n : 4$. We can fix $G_1$ because all subgroups isomorphic to $G_1$ in $Sz(q)$ are conjugate. We can also fix $G_{01}$ because all the subgroups isomorphic to $G_{01}$ are conjugate in $G_1$. The stabilizer of $G_1$ and $G_{01}$ is then $G_{01}$. And there is only one $Sz(s)$ containing $G_{01}$. Thus there is only one such geometry up to conjugacy and hence up to isomorphism.

Suppose $G_{01} = \beta_s : 4$.
Almost the same discussion as in the case $G_{01} = \alpha_s : 4$ leads us to the conclusion that $G_1$ must be $n : 4$ where $n$ is a proper divisor either of $\alpha_q$ or of $\beta_q$, and $\beta_s$ is a maximal divisor of $n$, and that there is only one such geometry up to conjugacy and hence up to isomorphism.

Finally, suppose $G_{01} = Sz(t)$.
The only candidates are subgroups isomorphic to $Sz(s')$ because of the order of $Sz(t)$. The case $s = s'$ leads to a contradiction because, as $Sz(q)$ is a simple group, and as the only subgroup between $Sz(q)$ and $Sz(t)$ is $Sz(s)$ which is also simple, the normalizer of $Sz(t)$ in $Sz(q)$ is $Sz(t)$. Thus we have here $G_1 = Sz(s')$ with $s \neq s'$ that gives a geometry. We can fix $G_0$ and $G_{01}$ here and there is only one subgroup isomorphic to $G_1$ that contains $G_{01}$. Thus there is only one such geometry up to conjugacy and hence up to isomorphism. □

## 7. Classification Theorem

The results of the preceding section permit us to give a complete classification theorem giving all the firm, residually connected , flag-transitive and residually weakly primitive geometries of rank 2 for a Suzuki group $Sz(q)$ up to isomorphism. The symbol $\#\Gamma$ in the tables below is the number of non-isomorphic geometries that arise from an RWPRI configuration. The last column of the tables tells whether or not the corresponding geometry satisfies $(2T)_1$. This condition is so easy to test in rank 2 that we do not give details about the results obtained.

**Theorem 7.1.** *Let $e > 0$ be a natural number. Let $q = 2^{2e+1}$ and let $Sz(q)$ be the Suzuki simple group over GF(q).*
*If the numbers $2e+1$ and $q-1$ are prime, the F, RC, FT, RWPRI rank 2 geometries of $Sz(q)$ are isomorphic to one of the geometries appearing in Table 9.*
*If $2e+1$ is prime and $q-1$ is not prime, the F, RC, FT, RWPRI rank 2 geometries of $Sz(q)$ are isomorphic to one of the geometries appearing in Table 10.*
*If $2e+1$ is not a prime number, the F, RC, FT, RWPRI rank 2 geometries of $Sz(q)$ are isomorphic to one of the geometries appearing in Table 11.*

*Proof.* This theorem is a compilation of the results obtained in Lemmas 6.1, 6.2, 6.3, 6.4, and 6.9. Thanks to Lemma 3.6, in the case $2e+1$ is not a prime number, we are sure that $\alpha_q$ and $\beta_q$ are not prime numbers. □

| $G_0$ | $G_1$ | $G_{01}$ | $\#\Gamma$ | $(2T)_1$ |
|---|---|---|---|---|
| $D_{2(q-1)}$ | $D_{2(q-1)}$ | 2 | $\frac{(q-2)(q+4)}{4(2e+1)}+1$ | *NO* |
| | $D_{2n}$ *where* $n$ *is a prime divisor of* $\alpha_q$ *or* $\beta_q$ | 2 | $\frac{(q-2)(q+2)}{4(2e+1)}+1$ | *NO* |
| | $2^2$ | 2 | $\frac{q-2}{2(2e+1)}$ | *NO* |
| | 4 | 2 | $\frac{q-2}{2(2e+1)}+1$ | *NO* |
| | $q:(q-1)$ | $q-1$ | 1 | *YES* |
| $\alpha_q$ *prime* | | | | |
| $\alpha_q:4$ | $\alpha_q:4$ | 4 | $\frac{q-2}{2(2e+1)}$ | *NO* |
| | $n:4$ *where* $n$ *is a prime divisor of* $\beta_q$ | 4 | $\frac{q-2}{2(2e+1)}+1$ | *NO* |
| | $4\times 2$ | 4 | $\frac{q-2}{2(2e+1)}$ | *NO* |
| $\beta_q$ *prime* | | | | |
| $\beta_q:4$ | $\beta_q:4$ | 4 | $\frac{q-2}{2(2e+1)}$ | *YES iff q=8* |
| | $n:4$ *where* $n$ *is a prime divisor of* $\alpha_q$ | 4 | $\frac{q-2}{2(2e+1)}+1$ | *NO* |
| | $4\times 2$ | 4 | $\frac{q-2}{2(2e+1)}$ | *YES iff q=8* |

Table 9: The case $2e+1$ prime and $q-1$ prime

| $G_0$ | $G_1$ | $G_{01}$ | $\#\Gamma$ | $(2T)_1$ |
|---|---|---|---|---|
| $D_{2(q-1)}$ | $q:(q-1)$ | *q-1* | *1* | *YES* |
| $\alpha_q$ *prime* | | | | |
| $\alpha_q:4$ | $\alpha_q:4$ | 4 | $\frac{q-2}{2(2e+1)}$ | *NO* |
| | $n:4$ *where* $n$ *is a prime divisor of* $\beta_q$ | 4 | $\frac{q-2}{2(2e+1)}+1$ | *NO* |
| | $4\times 2$ | 4 | $\frac{q-2}{2(2e+1)}$ | *NO* |
| $\beta_q$ *prime* | | | | *NO* |
| $\beta_q:4$ | $\beta_q:4$ | 4 | $\frac{q-2}{2(2e+1)}$ | *NO* |
| | $n:4$ *where* $n$ *is a prime divisor of* $\alpha_q$ | 4 | $\frac{q-2}{2(2e+1)}+1$ | *NO* |
| | $4\times 2$ | 4 | $\frac{q-2}{2(2e+1)}$ | *NO* |

Table 10: The case $2e+1$ prime and $q-1$ not prime

| $G_0$ | $G_1$ | $G_{01}$ | $\#\Gamma$ | $(2T)_1$ |
|---|---|---|---|---|
| $D_{2(q-1)}$ | $q:(q-1)$ | $q-1$ | 1 | *YES* |
| | $Sz(s)$ *if* $s-1$ *is a maximal divisor of* $q-1$ | $D_{2(s-1)}$ | 1 | *NO* |
| $\alpha_q:4$ | $Sz(s)$ *if* $\alpha_s$ *or* $\beta_s$ *is a maximal divisor of* $\alpha_q$ | $\alpha_s:4$ *or* $\beta_s:4$ | 1 | *NO* |
| $\beta_q:4$ | $Sz(s)$ *if* $\alpha_s$ *or* $\beta_s$ *is a maximal divisor of* $\beta_q$ | $\alpha_s:4$ *or* $\beta_s:4$ | 1 | *NO* |
| $Sz(s)$ | $(q\,\hat{\cdot}\,s):(s-1)$ | $(s\,\hat{\cdot}\,s):(s-1)$ | 1 | *NO* |
| | $Sz(s')$ *when it is a maximal subgroup and* $s'\neq s$ | $Sz(t)$ | 1 | *NO* |
| | $D_{2n}$ *where* $s-1\mid_M n\mid_P q-1$ | $D_{2(s-1)}$ | 1 | *NO* |
| | $n:4$ *where* $\alpha_s\mid_M n\mid_P \alpha_q$ *or* $\beta_q$ | $\alpha_s:4$ | 1 | *NO* |
| | $n:4$ *where* $\beta_s\mid_M n\mid_P \alpha_q$ *or* $\beta_q$ | $\beta_s:4$ | 1 | *NO* |

Table 11: The case $2e+1$ not prime (and thus $q-1$ not prime)

| $e$ | $\#(q-1)$ | $\#\alpha_q$ | $\#\beta_q$ | $\#\Gamma$ | $e$ | $\#(q-1)$ | $\#\alpha_q$ | $\#\beta_q$ | $\#\Gamma$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 29 | 2 | 1 | 1 | 1 | 177 |
| 3 | 1 | 2 | 1 | 2411 | 4 | 2 | 2 | 2 | 5 |
| 5 | 2 | 1 | 2 | 375 | 6 | 1 | 2 | 2 | $\geq 6.10^6$ |
| 7 | 3 | 2 | 3 | 14 | 8 | 1 | 2 | 2 | $\geq 1.10^9$ |
| 9 | 1 | 1 | 3 | $\geq 1.10^{10}$ | 10 | 3 | 3 | 3 | 15 |
| 11 | 2 | 3 | 2 | 1 | 12 | 3 | 2 | 3 | 8 |
| 13 | 3 | 3 | 3 | 5 | 14 | 3 | 1 | 2 | $\geq 3.10^7$ |
| 15 | 1 | 3 | 2 | $\geq 2.10^{17}$ | 16 | 4 | 3 | 3 | 15 |
| 17 | 4 | 3 | 3 | 17 | 18 | 2 | 2 | 3 | 1 |
| 19 | 4 | 4 | 5 | 22 | 20 | 2 | 3 | 2 | 1 |
| 21 | 3 | 3 | 2 | 1 | 22 | 6 | 5 | 5 | 23 |
| 23 | 3 | 3 | 1 | $\geq 7.10^{12}$ | 24 | 2 | 4 | 2 | 6 |
| 25 | 5 | 4 | 5 | 23 | 26 | 3 | 2 | 2 | 1 |
| 27 | 6 | 3 | 3 | 20 | 28 | 4 | 4 | 4 | 19 |
| 29 | 2 | 2 | 5 | 1 | 30 | 1 | 2 | 4 | $\geq 1.10^{35}$ |
| 31 | 6 | 5 | 5 | 22 | 32 | 3 | 5 | 4 | 20 |
| 33 | 2 | 2 | 4 | 1 | 34 | 4 | 4 | 4 | 17 |
| 35 | 3 | 4 | 2 | 1 | 36 | 3 | 4 | 1 | $\geq 3.10^{20}$ |
| 37 | 7 | 7 | 5 | 27 | 38 | 4 | 4 | 5 | 20 |
| 39 | 3 | 3 | 1 | $\geq 1.10^{23}$ | 40 | 6 | 5 | 5 | 9 |
| 41 | 2 | 3 | 3 | 1 | 42 | 3 | 4 | 6 | 22 |
| 43 | 6 | 5 | 2 | 22 | 44 | 1 | 2 | 2 | $\geq 5.10^{51}$ |
| 45 | 5 | 5 | 5 | 24 | 46 | 3 | 5 | 4 | 18 |
| 47 | 5 | 5 | 4 | 24 | 48 | 2 | 5 | 4 | 1 |
| 49 | 8 | 6 | 6 | 28 | 50 | 2 | 3 | 3 | 1 |
| 51 | 2 | 3 | 3 | 1 | 52 | 10 | 7 | 7 | 54 |
| 53 | 1 | 3 | 3 | $\geq 4.10^{62}$ | 54 | 2 | 3 | 3 | 1 |
| 55 | 6 | 6 | 4 | 25 | 56 | 5 | 4 | 1 | $\geq 2.10^{32}$ |
| 57 | 6 | 6 | 6 | 30 | 58 | 9 | 9 | 7 | 36 |
| 59 | 6 | 5 | 7 | 30 | 30 | 4 | 3 | 3 | 8 |

Table 12: The number of prime divisors of $q-1$, $\alpha_q$ and $\beta_q$, and the number of non-isomorphic geometries of rank 2.

**Corollary 7.1.** *If $2e+1$ is a prime number and $q-1$, $\alpha_q$ and $\beta_q$ are not prime, then the group $Sz(q)$ has only one geometry of rank 2.*

*Proof.* This is a direct consequence of theorem 7.1. □

Table 12 contains the number of prime divisors of $q-1$ ($\#(q-1)$), $\alpha_q$ ($\#\alpha_q$), $\beta_q$ ($\#\beta_q$) and the number $\#\Gamma$ of non-isomorphic geometries obtained for every Suzuki group up to $e=60$. Thanks to Corollary 7.1, we know that there is only one geometry for $e = 11$, 18, 20, 21, 26, 29, 33, 35, 41, 48, 50, 51 and 54.

## 8. A particular case: the Suzuki group $Sz(8)$

In this section we mention all the F, RC, FT, RWPRI geometries of rank 2 that can be constructed from $Sz(8)$. These geometries have been determined using the theorem that has been demonstrated in this article. The diagrams have been studied further using MAGMA. Besides the numbering of each geometry, there is a number in parenthesis which indicates the number of non-conjugate geometries that are fused under the action of $Aut(G)$. The classification of all the F, RC, FT, RWPRI geometries of $Sz(8)$ (i.e. for all ranks) is available in [14].

**Corollary 8.1.** *Up to isomorphism, there are 29 firm, residually connected geometries of rank 2 on which $Sz(8)$ acts flag-transitively and residually weakly primitively. Their diagrams are given below.*

*Proof.* Straightforward by theorem 7.1. □

| | | | | |
|---|---|---|---|---|
| **1. (3)** | | 7 2 7 | | |
| | | | | |
| | 6 | | 6 | RPRI |
| | 2080 | | 2080 | |
| | $D_{14}$ | | $D_{14}$ | B = 2 |
| **2. (3)** | | 7 3 7 | | |
| **3. (3)** | | | | |
| **4. (3)** | 6 | | 6 | RPRI |
| **5. (3)** | 2080 | | 2080 | |
| **6. (1)** | $D_{14}$ | | $D_{14}$ | B = 2 |
| **7. (3)** | | 7 4 7 | | |
| | | | | |
| | 6 | | 6 | RPRI |
| | 2080 | | 2080 | |
| | $D_{14}$ | | $D_{14}$ | B = 2 |
| **8. (3)** | | 6 2 6 | | |
| **9. (3)** | | | | |
| **10. (3)** | 12 | | 6 | QPRI |
| | 2080 | | 1120 | |
| | $D_{14}$ | | $D_{26}$ | B = 2 |

**11. (3)**
**12. (3)**
**13. (1)**

6 3 6

| | | |
|---|---|---|
| 12 | 6 | QPRI |
| 2080 | 1120 | |
| $D_{14}$ | $D_{26}$ | B = 2 |

**14. (3)**

8 2 8

| | | |
|---|---|---|
| 4 | 6 | QPRI |
| 2080 | 2912 | |
| $D_{14}$ | $D_{10}$ | B = 2 |

**15. (3)**

8 3 8

| | | |
|---|---|---|
| 4 | 6 | QPRI |
| 2080 | 2912 | |
| $D_{14}$ | $D_{10}$ | B = 2 |

**16. (3)**
**17. (3)**
**18. (3)**

8 4 8

| | | |
|---|---|---|
| 4 | 6 | QPRI |
| 2080 | 2912 | |
| $D_{14}$ | $D_{10}$ | B = 2 |

**19. (1)**

9 3 10

| | | |
|---|---|---|
| 4 | 6 | QPRI |
| 2080 | 2912 | |
| $D_{14}$ | $D_{10}$ | B = 2 |

**20. (3)**

13 5 14

| | | |
|---|---|---|
| 1 | 6 | |
| 2080 | 7280 | |
| $D_{14}$ | $2^2$ | B = 2 |

**21. (3)**

13 5 14

| | | |
|---|---|---|
| 1 | 6 | |
| 2080 | 7280 | |
| $D_{14}$ | 4 | B = 2 |

**22. (1)**

12 6 12

| | | |
|---|---|---|
| 1 | 6 | |
| 2080 | 7280 | |
| $D_{14}$ | 4 | B = 2 |

**23. (1)**

10 5 10

| | | |
|---|---|---|
| 7 | 1 | QPRI |
| 2080 | 520 | $(2T)_1$ |
| $D_{14}$ | $2^3{:}7$ | B = 7 |

**24. (3)** $\bigcirc \overset{5\quad 3\quad 5}{\rule{5em}{0.4pt}} \bigcirc$

| | | |
|---|---|---|
| 12 | 12 | RPRI |
| 560 | 560 | |
| 13:4 | 13:4 | B = 4 |

**25. (3)** $\bigcirc \overset{6\quad 4\quad 6}{\rule{5em}{0.4pt}} \bigcirc$

| | | |
|---|---|---|
| 12 | 4 | RPRI |
| 560 | 1456 | |
| 13:4 | 5:4 | B = 4 |

**26. (1)** $\bigcirc \overset{5\quad 3\quad 6}{\rule{5em}{0.4pt}} \bigcirc$

| | | |
|---|---|---|
| 12 | 4 | RPRI |
| 560 | 1456 | |
| 13:4 | 5:4 | B = 4 |

**27. (3)** $\bigcirc \overset{8\quad 5\quad 8}{\rule{5em}{0.4pt}} \bigcirc$

| | | |
|---|---|---|
| 1 | 12 | QPRI |
| 560 | 3640 | |
| 13:4 | $4 \times 2$ | B = 4 |

**28. (3)** $\bigcirc \overset{8\quad 5\quad 8}{\rule{5em}{0.4pt}} \bigcirc$

| | | |
|---|---|---|
| 4 | 4 | RPRI |
| 1456 | 1456 | $(2T)_1$ |
| 5:4 | 5:4 | B = 4 |

**29. (3)** $\bigcirc \overset{14\quad 7\quad 14}{\rule{5em}{0.4pt}} \bigcirc$

| | | |
|---|---|---|
| 1 | 4 | QPRI |
| 1456 | 3640 | $(2T)_1$ |
| 5:4 | $4 \times 2$ | B = 4 |

## 9. Observations and future work

While computing the exact number of non-isomorphic geometries for every $Sz(2^{2e+1})$ with $e \leq 60$, we observed the following facts.

- If $Sz(q)$ does not have proper subgroups isomorphic to a Suzuki group, then Table 12 shows that its number of non-isomorphic geometries is either 1 or a number that grows very fast like 2411 for $e = 3$, more than 6 millions for $e = 6$, more than 10 billions for $e = 9$, etc.
- If $Sz(q)$ has proper subgroups isomorphic to some Suzuki simple group, then its number of non-isomorphic geometries is more than reasonable.

This seems a little bit intriguing. We could have thought that when a Suzuki ovoid has subovoids, its collineation group should have more geometries. This is not the case.

Maybe we should add some condition in order to restrict the number of geometries that appear. If we impose $(2T)_1$, we see that when $e > 2$, there is only one geometry that survives this test (the case $e = 1$ gives 3 such geometries). Thus the huge number of geometries we obtained in these cases disappear and the number of geometries obtained is quite small.

Anyway, a geometry which seems to play a central role is the one given by the RWPRI configuration ($Sz(q)$, $D_{2(q-1)}$, $q : (q-1)$, $q-1$). It is the geometry that appears in all cases. It satisfies $(2T)_1$. This geometry will be studied further in another paper.
Also, we intend to classify the rank $> 2$ RWPRI geometries of $Sz(q)$ and we want to consider the $PSL(2, q)$ because of their similarity with the Suzuki.

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