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Clones Closed with Respect to Permutation Groups or Transformation Semigroups

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Abstract. This paper presents a new approach to the classification of finite algebras with respect to actions of permutation groups or transformation semigroups on the clone of term operations. In particular it is a contribution to the structural investigation of the (uncountable) lattice $\mathcal{L}_A$ of clones on a finite set A, which can be divided into (sometimes finitely many) intervals represented by so-called G-clones. A Galois theory for G-clones is developed. Concrete results are presented for clones — and, more generally, for closed classes — on base sets A with two or three elements.

Introduction

This paper contributes to the classification of finite algebras with the same underlying set A. A finite algebra $\mathcal{A} = \langle A; F\rangle$ consists of a finite set A together with a (possibly infinite) set of (fundamental) operations (e.g. finite groups, rings, lattices, ...). For fixed A with $|A| \geq 3$, the set of all algebras on A is uncountable. The set $T(\mathcal{A})$ of all term operations is a natural closure of F consisting of the projections and all compositions of the fundamental operations, i.e. $T(\mathcal{A})$ is the so-called clone of operations $\langle F\rangle_{O_A}$ generated by F. Thus we can compare algebras $\mathcal{A} = \langle A; F\rangle$ and $\mathcal{A}' = \langle A; F'\rangle$ via their clone of operations:

$$\mathcal{A} \leq \mathcal{A}' \iff T(\mathcal{A}) = \langle F\rangle_{O_A} \subseteq \langle F'\rangle_{O_A} = T(\mathcal{A}').$$

With respect to this preorder two algebras are "the same" if they are *term equivalent* ([23, 4.139]), i.e. if they have the same underlying set and the same term operations. Every algebra $\mathcal{A} = \langle A; F\rangle$ is term equivalent to the algebra $\mathcal{A} = \langle A; T(\mathcal{A})\rangle$ and the algebras of the form $\langle A; \langle F\rangle_{O_A}\rangle$ form a transversal for term equivalence.

Thus, up to term equivalence, the classification of finite algebras with underlying set A leads to the classification of clones, i.e. to the investigation of the lattice[1] $\mathcal{L}_A$ of all clones on A.

More generally, the same approach can be done with *closed classes* instead of clones. The closed class generated by $F \subseteq O_A$ is the set of all compositions of functions from F (without using the projections as in the clone case).

The lattice $\mathcal{L}_A$ is completely known (by a result of E.L. Post obtained in 1921, cf. [27]) for Boolean functions ($|A| = 2$), it is countable; however $\mathcal{L}_A$ is uncountable for $|A| \geq 3$ ([16]). There is not much hope to get an effective description of the clone lattice on an at least three-element set. Many efforts were made to understand the structure of $\mathcal{L}_A$, e.g. it is atomic and dually atomic, all maximal clones (= dual atoms = lower neighbours of the largest clone O_A) are described (there are finitely many [17], [28]) while a complete description of all minimal clones (= atoms) still is an open problem.

There are two general strategies: one can restrict to special sublattices (e.g. intervals) of $\mathcal{L}_A$ or one introduces equivalence relations between clones in order to partition the lattice $\mathcal{L}_A$ into smaller pieces.
From the universal algebraic point of view isomorphic algebras are the same. However there is a strong restriction: the algebras under consideration must be of the same type. In order to compare algebras of various types one has to use the more general notion of *weak isomorphism* ([22], [8], [11]) which, in fact, induces an isomorphism of the corresponding clones, too (for an approach via crypto-isomorphisms with respect to a general closure operator we refer to [26]). Two algebras $\mathcal{A} = \langle A; F\rangle$ and $\mathcal{A}' = \langle A; F'\rangle$ are weakly isomorphic if $\mathcal{A}$ is term equivalent to $\mathcal{A}^\sigma = \langle A; F^\sigma\rangle$ for some permutation $\sigma : A \mapsto A$ (cf. 1.2 below). But also these weak isomorphisms cannot reduce considerably the amount of algebras or corresponding clones (there remain uncountably many). Consequently, one should look for stronger equivalences between clones.

In this paper we shall develop the following approach. If a function $f : A^n \to A$ is given (in a clone F) we shall assume that every function f^s with the same behaviour as f but "up to renaming (via s) the elements of A" is also given (where $s \in G$ is a permutation on A in a subgroup $G \leq S_A$). This action of S_A on clones gives a closure operator $F \mapsto {}^G F$ on $\mathcal{L}_A$ (cf. 2.2). Then two clones (or algebras) will be "($\vee_G$-)equivalent" if their G-closures (called G-clones) coincide; the G-clone ${}^G F$ generated by F is the largest element in each equivalence class. The special case of self-dual clones (and closed classes) or homogeneous algebras (where $f^s = f$ for each element f of the clone) is considered e.g. in [4], [5], [6], [20] and [3]. Further references can be found also in [21].

Every clone determines a least interval (in $\mathcal{L}_A$) between G-clones to which it belongs.

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[1] $\mathcal{L}_A$ is an algebraic lattice with respect to inclusion. The lattice operations are given in 1.1.

It turns out (by results of V. Gorlov and Nguen van Hoa [14]) that in several cases, e.g. for $G = S_A$, the lattice of G-clones even becomes finite. Thus the general problem to describe the structure of $\mathcal{L}_A$ now divides into two parts:

- describe the lattice of G-clones (or closed classes) and
- describe the equivalence classes (w.r.t. $\sim_{\vee_G}$ or $\sim_{\wedge_G}$, see 2.4 for definition).

In the present paper we are mainly concerned with the first problem. We shall develop this with respect to permutation groups G (sections 1–4). We also develop a Galois theory for G-clones and corresponding relational G-clones (section 4).
Instead of permutations one can also use arbitrary unary mappings $h : A \mapsto A$. This leads to H-clones with respect to transformation semigroups H on A (sections 5–6). There are large differences between the group and the semigroup actions on clones, but both might be helpful for the structural analysis of the lattice $\mathcal{L}_A$.
For two- or three-element sets A we present (for special G and H) a list of all G- and H-clones as well as of closed classes (sections 3 and 6).

1. Preliminaries: Notions and Notations

1.1. Let $O_A^{(n)}$ be the set of all n-ary operations $f : A^n \longrightarrow A$ (of *arity* $ar\,f := n$) on a set A ($n \in \mathbf{N}^+ = \{1, 2, \ldots, \}$ and $O_A := \bigcup_{n=1}^{\infty} O_A^{(n)}$. The projections $e_i^n \in O_A^{(n)}$ and constants $c_a^n \in O_A^{(n)}$ are defined by $e_i^n(x_1, \ldots, x_n) = x_i$ and $c_a^n(x_1, \ldots, x_n) = a$ (for $a \in A$). Let J_A denote the set of all projections. For simplicity, in case of finite A, often we shall use $k := \{0, 1, \ldots, k-1\}$ or $\underline{k} := \{1, 2, \ldots, k\}$ instead of A. The (full) *iterative (preiterative, resp.) algebra* on A is the algebra

$$\mathcal{P}_A = \langle O_A; \tau, \zeta, \Delta, \nabla, * \rangle \quad (\mathcal{P}_A^* = \langle O_A;\ \tau, \zeta, \Delta, * \rangle, \text{resp.})$$

of type (1,1,1,1,2) ((1,1,1,2), resp.), where the operations are defined as follows:
Let $f \in O_A^{(n)}, g \in O_A^{(m)}$, then

$$\begin{aligned}
(\tau f)(x_1, \ldots, x_n) &:= f(x_2, x_1, x_3, \ldots, x_n), \\
(\zeta f)(x_1, \ldots, x_n) &:= f(x_2, x_3, \ldots, x_n, x_1), \\
(\Delta f)(x_1, \ldots, x_{n-1}) &:= f(x_1, x_1, \ldots, x_{n-1}), \\
(\nabla f)(x_1, \ldots, x_{n+1}) &:= f(x_2, x_3, \ldots, x_{n+1}), \\
(f * g)(x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n-1}) &:= f(g(x_1, \ldots x_m), x_{m+1}, \ldots q, x_{m+n-1}).
\end{aligned}$$

For $n = 1$ we define $\tau f = \zeta f = \Delta f = f$.
Let $F \subseteq O_A$. Then $\langle F \rangle_{\mathcal{P}_A}$ denotes the subalgebra of $\mathcal{P}_A$ generated by F. F is called a *closed class* if $F = \langle F \rangle_{\mathcal{P}_A}$. If, in addition, F contains all projections e_i^n then F is a *clone*. Clones can be characterized as subalgebras of the *full function algebra*

$$\underline{O_A} := \langle O_A;\ e, \tau, \zeta, \Delta, \nabla, * \rangle$$

(where $e = e_1^1$ is the identity operation). The subalgebra of $\underline{O_A}$ generated by F is denoted by $\langle F \rangle_{O_A}$, for short $\langle F \rangle$.

Let $\mathcal{L}_A$ denote the lattice (w.r.t. inclusion) of all clones on A. The lattice operations meet $\wedge$ and join $\vee$ are given by $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \vee F_2 = \langle F_1 \cup F_2 \rangle_{O_A}$.

1.2. Let S_A be the full symmetric group on A. For a permutation $s \in S_A$ and $x \in A$ let x^s denote the image of x under s. For $s \in S_A$ and $f \in O_A^{(n)}$ define f^s by

$$f^s(x_1, \dots, x_n) := (f(x_1^{s^{-1}}, \dots, x_n^{s^{-1}}))^s.$$

The mapping $(\ )^s : O_A \longrightarrow O_A : f \mapsto f^s$ is an *inner* [19] automorphism of the iterative algebra $\mathcal{P}_A$.
Note that

$$F^s := \{f^s \mid f \in F\}$$

is a closed class or clone, resp., whenever F is. If $F^s = F$ for the clone $F = T(\mathcal{A}) := \langle\{f_i \mid i \in I\}\rangle_{O_A}$ of an algebra $\mathcal{A} = \langle A; (f_i)_{i \in I}\rangle$, then s is called *weak automorphism* of $\mathcal{A}$ ([10]). A function with $f^s = f$ (or equivalently, if s is an automorphism of $\langle A; f\rangle$) is called *s-selfdual.*

1.3. A theorem of A. I. Mal'cev (cf. e.g. [19, 6.2:I]) states (here slightly reformulated):

1. *All automorphisms of $\mathcal{P}_A$ are inner; consequently $Aut\,\mathcal{P}_A \cong S_A$. (The same holds for $\underline{O_A}$.)*
2. *All endomorphisms $h : \mathcal{P}_A \longrightarrow \mathcal{P}_A$ which are not automorphisms are of the form*

$$s_a : f \mapsto c_a^{(n)} \quad or \quad \tilde{s}_a : f \mapsto c_a^{(1)}$$

for $f \in O_A^{(n)}$, $n \in \mathbf{N}$ and fixed $a \in A$.

Thus the endomorphism monoid $End\,\mathcal{P}_A$ is the group S_A together with constants s_a and $\tilde{s}_a$ $(a \in A)$.

1.4. Remarks. In case $|A| = k = 2$ we have $Aut\,F \leq S_2$ for all closed classes $F \leq \mathcal{P}_2$, however for $k \geq 3$ there exist closed classes with non-inner automorphisms ([12]). Obviously, every automorphism of $\mathcal{P}_A$ induces an automorphism of the lattice $\mathcal{L}_A$. Conversely, there are no other automorphisms of $\mathcal{L}_A$ (by a result of the first author (unpublished)), i.e.

$$Aut\ \mathcal{P}_A \cong Aut\ \mathcal{L}_A \cong S_A \quad \text{for} \quad |A| \geq 2.$$

2. *G*-closed classes and *G*-clones

The following definition covers several special cases considered e.g. in [4], [20], [5], [6].

2.1. Definition. *Let G be a subgroup of S_A. For $F \subseteq O_A$ we define $F^G := \{f^s \mid f \in F, s \in G\}$. A clone $F \leq O_A$ or a closed class $F \leq \mathcal{P}_A$, resp., is called G-clone (or G-closed, resp.), if $f^s \in F$ for all $s \in G$ and $f \in F$ (cf. [14]). Let $\mathcal{L}_{A,G}$ ($\mathcal{L}'_{A,G}$, resp.) denote the set of all G-clones (G-closed classes, resp.).*
A function which is s-selfdual (cf. 1.2) for all $s \in G$ is called G-homogeneous. A clone or a closed class F is G-homogeneous if every $f \in F$ is G-homogeneous (i.e. $G \subseteq Aut\langle A; F\rangle$). Obviously, every G-homogeneous clone is an G-clone.

Note that F is G-closed iff $F^G = F$ iff $F^s = F$ for all $s \in G$. Moreover, $\{e\}$-clones are just the ordinary clones.
For simplicity, in the following we mainly consider clones. However, the results can be used for arbitrary closed classes, too.

2.2. Theorem. *Let $G \leq S_A$, and $F \leq O_A$.*

(a) ${}^G F := \langle F^G \rangle = \langle \bigcup_{s\in G} F^s \rangle_{O_A}$ *is the least G-clone containing F.*

(b) ${}_G F := \bigcap_{s\in G} F^s$ *is the greatest G-clone contained in F.*

(c) *The set $\mathcal{L}_{A,G}$ of all G-clones is a sublattice of the lattice $\mathcal{L}_A$ with least element J_A and greatest element O_A: For G-clones F_1, F_2 the clones $\langle F_1 \cup F_2 \rangle_{O_A}$ and $F_1 \cap F_2$ are again G-clones.*

Proof. (a) Obviously, every G-clone containing F must contain F^s for every $s \in G$, too. It remains to prove that the clone $\langle F^G \rangle$ is G-closed.
Let $s \in G$, $f \in F^{(n)}$, $g \in F^{(m)}$. Then $e^s = e$, $(\sigma f)^s = \sigma(f^s)$ for every operator $\sigma \in \{\tau, \zeta, \Delta, \nabla\}$ and $(f * g)^s = f^s * g^s$ (what immediately follows from the definition). Consequently $\langle F^G \rangle^s \subseteq \langle F^G \rangle$ for $s \in G$ and therefore $\langle F^G \rangle^G = \langle F^G \rangle$.
(b) $\bigcap_{s\in G} F^s$ is a clone (as intersection of clones) which is contained in F (put $s = e$), and it is G-closed because of

$$(\bigcap_{s'\in G} F^{s'})^{s''} = \bigcap_{s'\in G} F^{s's''} = \bigcap_{s\in G} F^s.$$

Moreover, if F' is a G-clone contained in F then $F'^s = F'$ is contained in every F^s, i.e. $F' \subseteq {}_G F$.
(c) directly follows from (a) and (b) because $F = F^G = {}^G F = {}_G F$ for any G-closed F. □

2.3. Remark. The clones ${}^G F$ and ${}_G F$ in 2.2 coincide with the join $\bigvee_{s\in G} F^s$ and meet $\bigwedge_{s\in G} F^s$ of the clones $F^s (s \in G)$ in the lattice $\mathcal{L}_A$. For $G \leq G' \leq S_A$ and a clone $F \leq O_A$ we always have

$$\bigwedge_{s\in G'} F^s \subseteq \bigwedge_{s\in G} F^s \subseteq F \subseteq \bigvee_{s\in G} F^s \subseteq \bigvee_{s\in G'} F^s.$$

This motivates the following definitions.

2.4. Definitions. *Let $G \leq S_A$. For a clone $F \leq O_A$ let*

$$G\text{-}Int(F) := [{}_G F, {}^G F]_{\mathcal{L}_A} = [\bigwedge_{s\in G} F^s, \bigvee_{s\in G} F^s]_{\mathcal{L}_A}$$

be its corresponding G-interval in the lattice $\mathcal{L}_A$. Two clones $F_1, F_2 \leq O_A$ are called $\wedge_G$-equivalent ($F_1 \sim_{\wedge_G} F_2$) or $\vee_G$-equivalent ($F_1 \sim_{\vee_G} F_2$), resp., if

$$\bigwedge_{s\in G} F_1^s = \bigwedge_{s\in G} F_2^s \quad \textit{or} \quad \bigvee_{s\in G} F_1^s = \bigvee_{s\in G} F_2^s, \quad \textit{respectively.}$$

2.5. Remarks. a) Obviously F is a G-clone iff $G\text{-}Int(F) = \{F\}$ (i.e. $F = {}_GF = {}^GF$). Both, $\sim_{\wedge_G}$ and $\sim_{\vee_G}$, are equivalence relations on $\mathcal{L}_A$. Two clones F_1, F_2 are both, $\wedge_G$- and $\vee_G$-equivalent, iff the corresponding intervals coincide: $G\text{-}Int(F_1) = G\text{-}Int(F_2)$.
The set of G-clones is a transversal for the set of all $\wedge_G$-equivalence classes as well as for all $\vee_G$-equivalence classes.

b) $\sim_{\wedge_G}$ is a congruence in the semilattice $\langle \mathcal{L}_A; \wedge\rangle$ and $\sim_{\vee_G}$ is a congruence in $\langle \mathcal{L}_A; \vee\rangle$, i.e. $F_1 \sim_{\wedge_G} F_2$, $F_1' \sim_{\wedge_G} F_2'$ implies $F_1 \wedge F_1' \sim_{\wedge_G} F_2 \wedge F_2'$ (analogously for $\sim_{\vee_G}$).

c) Obviously $F_1 \subseteq F_2$ implies ${}_GF_1 \subseteq {}_GF_2$ and ${}^GF_1 \subseteq {}^GF_2$.

d) For clones $F_1, F_2 \leq O_A$ we have ${}_GF_1 \wedge {}_GF_2 = {}_G(F_1 \wedge F_2)$ and ${}^GF_1 \vee {}^GF_2 = {}^G(F_1 \vee F_2)$.

e) The set of intervals $[F_1, F_2]_{\mathcal{L}_A}$ with G-closed clones F_1, F_2 forms a lattice w.r.t. inclusion. Meet and join in this lattice are given by

$$[F_1, F_2] \wedge [F_1', F_2'] = [F_1 \vee F_1', F_2 \wedge F_2']$$

$$[F_1, F_2] \vee [F_1', F_2'] = [F_1 \wedge F_1', F_2 \vee F_2'].$$

All G-intervals belong to this lattice, however in general it is an open problem under which conditions meet or join of G-intervals are again G-intervals.

f) The G-closure is a tool for the investigation of the lattice $\mathcal{L}_A$. As mentioned in the introduction there is no hope to find a satisfactory full description of the uncountable lattice $\mathcal{L}_A$ ($|A| \geq 3$) as in the countable case of Post's lattice $\mathcal{L}_2$ of clones of Boolean functions. The G-closure may naturally reduce this lattice even to a finite sublattice of $\mathcal{L}_A$. Note that under this reduction process for every G-interval $[F_1, F_2]$ the (possibly infinite) set $\{F \mid G\text{-}Int(F) = [F_1, F_2]\}$ reduces to two clones (namely the least and greatest elements F_1 and F_2 in this interval). For $|A| = 2$ and $|A| = 3$ concrete results are given in the next sections. An external characterization of the G-intervals is given in 4.5.

2.6. From the group theoretical point of view subgroups of S_A are considered usually up to conjugation, i.e. the groups $G \leq S_A$ and $G' := s^{-1}Gs$ are "the same". Note that then the corresponding lattices $\mathcal{L}_{A,G}$ and $\mathcal{L}_{A,G'}$ are isomorphic ($F \mapsto F^s$ is an isomorphism).
In fact, if F is G-closed then we have $(F^s)^{s^{-1}Gs} = F^{Gs} = F^s$, i.e. F^s is G'-closed. Moreover, $(G\text{-}Int(F))^s = G'\text{-}Int(F^s)$, i.e. the intervals $G\text{-}Int(F)$ and $G'\text{-}Int(F^s)$ are isomorphic for arbitrary $F \leq O_A$. In case $k = 2$ conjugate groups ($\leq S_2$) coincide. For $k \geq 3$ the G-intervals of F for conjugate groups G may differ.

Obviously, for $H \leq G \leq S_A$, every G-clone is also an H-clone (the converse does not hold in general), i.e. $\mathcal{L}_{A,G} \subseteq \mathcal{L}_{A,H}$.

We mention here the conjecture that $\mathcal{L}_{A,G} \neq \mathcal{L}_{A,H}$ for different groups $G \neq H$. If this is true, the subgroup lattice of S_A would be dual to the lattice (w.r.t. inclusion) of all lattices $\mathcal{L}_{A,G}$ ($G \leq S_A$).

2.7. From the universal algebraic point of view G-clones come into play as follows: The clone of operations $F := \langle\{f_i \mid i \in I\}\rangle$ of an algebra $\mathcal{A} = \langle A; (f_i)_{i\in I}\rangle$ is a G-clone ($G \leq S_A$) iff $G \subseteq wAut\,\mathcal{A}$, where $wAut\,\mathcal{A}$ denotes the group of weak automorphisms of $\mathcal{A}$ (c.f. 1.2 and [10]). Analogously, F is an homogeneous G-clone iff $G \subseteq Aut\,\mathcal{A}$.
S_A-clones are of special interest. If, moreover, every $s \in S_A$ fixes the fundamental operations

f_i of $\mathcal{A}$, then $f^s = f$ for every $f \in F$ and $\mathcal{A}$ is a homogeneous algebra (for investigation of homogeneous algebras we refer to [3]).

3. G-clones on $A = \{0,1\}$ and $A = \{0,1,2\}$

The lattice $\mathcal{L}_2$ of all clones on $A = \{0,1\}$ is well-known. Here we use the notations for the clones given in [15][2]. Since S_2 contains a single non-trivial function (negation $\neg : x \mapsto \neg x$), S_2-clones F are those which are self-dual as sets (i.e. $F^s = F$ for negation $s = \neg$). The following result also can be derived from Theorem 1 and Proposition 1 in [10, p. 101,102].

3.1. Theorem. *There are exactly* 14 S_2*-clones on* $A = \{0,1\}$ *:*

$$\mathsf{O}_1, \mathsf{O}_4, \mathsf{O}_8, \mathsf{O}_9, \mathsf{L}_1, \mathsf{L}_4, \mathsf{L}_5, \mathsf{D}_2, \mathsf{D}_1, \mathsf{D}_3, \mathsf{A}_4, \mathsf{A}_1, \mathsf{C}_4, \mathsf{C}_1.$$

There is one more S_2*-closed class which is not a clone, namely* O_7*.*

3.2. Here we describe the clones in 3.1 by a generating set:

$$\begin{aligned}
\mathsf{O}_1 &= J_A \text{ (projections)},\\
\mathsf{O}_4 &= \langle \neg \rangle \text{ (projections and their negations)},\\
\mathsf{O}_8 &= \langle c_0, c_1 \rangle \text{ (constants)},\\
\mathsf{O}_9 &= \langle c_0, c_1, \neg \rangle \text{ (essentially unary functions)},\\
\mathsf{L}_4 &= \langle g \rangle \text{ (linear idempotent functions, } g(x,y,z) := x+y+z),\\
\mathsf{L}_5 &= \langle g, \neg \rangle \text{ (linear self-dual functions)},\\
\mathsf{L}_1 &= \langle c_1, + \rangle \text{ (all linear functions, } + : (x,y) \mapsto x+y),\\
\mathsf{D}_2 &= \langle h \rangle \text{ (self-dual monotone functions,}\\
&\ h(x,y,z) := (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)),\\
\mathsf{D}_1 &= \langle g, h \rangle \text{ (self-dual idempotent functions)},\\
\mathsf{D}_3 &= \langle h, \neg \rangle \text{ (self-dual functions)},\\
\mathsf{A}_4 &= \langle \wedge, \vee \rangle \text{ (monotone idempotent functions)},\\
\mathsf{A}_1 &= \langle c_0, c_1, \wedge, \vee \rangle \text{ (monotone functions)},\\
\mathsf{C}_4 &= \langle \wedge, \vee, g \rangle \text{ (idempotent functions)},\\
\mathsf{C}_1 &= O_A \text{ (all functions)}.
\end{aligned}$$

O_7 is the closed class generated by the constants c_0, c_1.
The lattice $\mathcal{L}_{2,S_2}$ of S_2-clones is given in Fig. 1.

3.3. Remark. As pointed out in [10, Prop.1] there are 7 clones F among the above 14 clones for which S_2 is not only the group of weak automorphisms but also the automorphism group (of the algebra $\langle A; F \rangle$). These clones are just the S_2-homogeneous clones in the sense of Definition 2.1. They are $\mathsf{D}_3, \mathsf{D}_1, \mathsf{D}_2, \mathsf{L}_5, \mathsf{L}_4, \mathsf{O}_4, \mathsf{O}_1$ (i.e. the interval $[\mathsf{O}_1, \mathsf{D}_3]_{\mathcal{L}_{2,S_2}}$, see Fig. 1) and there are no more S_2-homogeneous closed classes.

[2] In order to avoid confusion we used different fonts for the notation of closed classes defined in [15] or [14], namely e.g. $\mathsf{S}_2, \mathsf{D}, \mathsf{O}$. One should notice the difference to e.g. S_2, O_3.

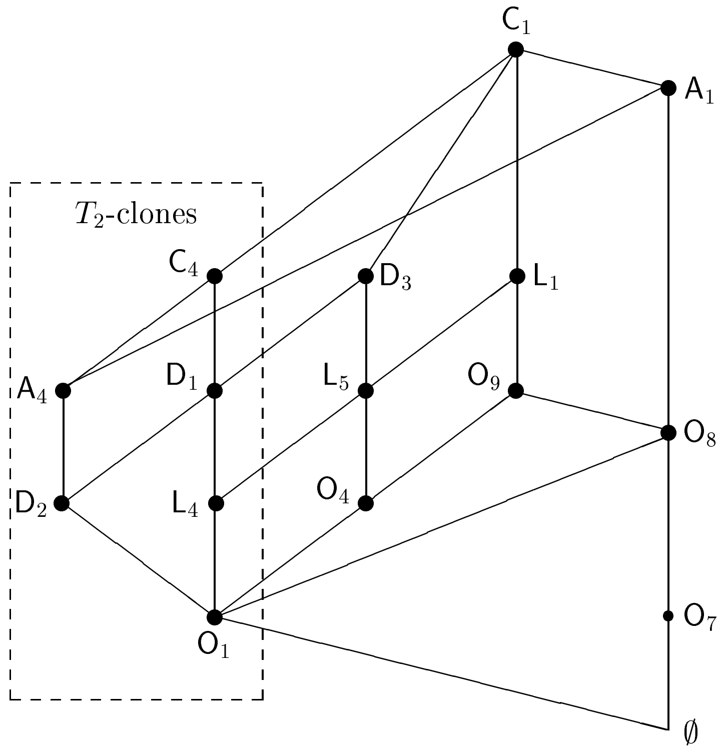

Figure 1: The lattice $\mathcal{L}_{2,S_2}$ ($\mathcal{L}_{2,T_2}$) of S_2-clones (T_2-clones, resp.) and closed classes

3.4. We are going to describe all intervals S_2-*Int*(F) for Boolean clones $F \leq O_2$. Since $\mathcal{L}_2$ is known this can be done without difficulties. Note that because of 3.1 the number of possible intervals is limited. Take some clone F and its dual F' ($= F^s$ for $s = \neg$) and form the interval $[F \wedge F', F \vee F']_{\mathcal{L}_2}$. Proceed as far no new intervals can be obtained.
Many covering pairs, say F_1, F_2 in the lattice of Fig. 1 are also covering pairs in the lattice $\mathcal{L}_2$. In that case $[F_1, F_2]_{\mathcal{L}_2}$ cannot be an interval of the form S_2-*Int*(F), since there is no F between F_1 and F_2. We list here all possible intervals $[F_1, F_2]$ of the form

$$Int(F) := S_2\text{-}Int(F)$$

which are non-degenerate (i.e. $F_1 \neq F_2$), give one example for F (using Post's notation[3]) and add in parenthesis the cardinality $|\{F \mid Int(F) = [F_1, F_2]\}|$ of clones F to which this interval corresponds.

$$\begin{array}{llll}
[\mathsf{O}_1, \mathsf{O}_8] = Int(\mathsf{O}_6) & (2), & [\mathsf{O}_8, \mathsf{A}_1] = Int(\mathsf{P}_6) & (2), \\
[\mathsf{A}_4, \mathsf{A}_1] = Int(\mathsf{A}_3) & (2), & [\mathsf{C}_4, \mathsf{C}_1] = Int(\mathsf{C}_3) & (2), \\
[\mathsf{L}_4, \mathsf{L}_1] = Int(\mathsf{L}_3) & (2), & & \\
[\mathsf{D}_2, \mathsf{A}_4] = Int(\mathsf{F}_6^2) & (2), & [\mathsf{D}_2, \mathsf{C}_4] = Int(\mathsf{F}_1^2) & (2), \\
[\mathsf{D}_2, \mathsf{A}_1] = Int(\mathsf{F}_3^2) & (2), & [\mathsf{D}_2, \mathsf{C}_1] = Int(\mathsf{F}_4^2) & (2), \\
[\mathsf{O}_1, \mathsf{A}_4] = Int(\mathsf{F}_2^\infty) & (\aleph_0), & [\mathsf{O}_1, \mathsf{C}_4] = Int(\mathsf{F}_1^\infty) & (\aleph_0), \\
[\mathsf{O}_1, \mathsf{A}_1] = Int(\mathsf{F}_3^\infty) & (\aleph_0), & [\mathsf{O}_1, \mathsf{C}_1] = Int(\mathsf{F}_4^\infty) & (\aleph_0).
\end{array}$$

[3]with notations from 3.2 we have e.g. $\mathsf{O}_6 = \langle c_0 \rangle$, $\mathsf{A}_3 = \langle \wedge, \vee, c_0 \rangle$, $\mathsf{L}_3 = \langle c_0, g \rangle$, $\mathsf{F}_6^2 = \langle \wedge, h \rangle$, $\mathsf{P}_6 = \langle \wedge, c_0, c_1 \rangle$, $\mathsf{C}_3 = \langle \wedge, c_0, g \rangle$.

The problem mentioned in 2.5, e) now can be answered: The join of G-intervals is again a G-interval and the meet of G-intervals is either empty or also again a G-interval (possibly of degenerate form $[F, F]$ with G-clone F).

The lattice $\mathcal{L}_3$ of clones on the base set $A = \{0, 1, 2\}$ is uncountable. However, as mentioned in the introduction, there are only finitely many S_3-clones. On the basis of [5] and [20] the following result was found by N.v. Hoa and V.V. Gorlov and proved in [14].

3.5. Theorem. *There are* 48 *S_3-clones on $A = \{0, 1, 2\}$. In addition there are* 4 *non-empty S_3-closed classes which are not clones.*

3.6. Remark. Among the 52 S_3-closed classes there are 8 non-empty S_3-homogeneous closed classes all of which are clones, namely

$$\mathsf{S}_2, \mathsf{L}_1\mathsf{S}, \mathsf{S}_2\mathsf{D}_1, \mathsf{S}_2\mathsf{L}_4, \mathsf{S}_2\mathsf{D}_2, \mathsf{S}_2\mathsf{O}_1, \mathsf{S}_2^*\mathsf{D}_2, \mathsf{O}(= J_3).$$

This is the interval $[J_3, \mathsf{S}_2]_{\mathcal{L}_{3,S_3}}$ (see Fig. 2).

3.7. The lattice $\mathcal{L}_{3,S_3}$ of S_3-clones together with the 4 S_3-closed classes is given in Fig. 2 [4]. It suffices to define explicitly the $\wedge$-irreducible elements in the lattice $\mathcal{L}_{3,S_3}$ (17 clones and one class). All other clones (classes, resp.) are intersections of $\wedge$-irreducibles as can be seen from Fig. 2 (where the $\wedge$-irreducibles are marked by rectangular points). Below we list all the $\wedge$-irreducibles since the description of all classes in [14] is given in a relatively complicated form. We introduce the following notations.
Let $f : A^n \to A$ and $a, b \in A$ (here $A = \{0, 1, 2\}$). Then $f_{a,b} : \{a, b\}^n \longrightarrow A$ denotes the restriction of f to the set $\{a, b\}$. If f preserves $\{a, b\}$, then $f_{a,b} : \{a, b\}^n \longrightarrow \{a, b\}$ will be considered as Boolean function (on the set $\{a, b\}$ instead of $\{0, 1\}$). For $K \subseteq O_2$ and $a, b \in A$, let $K_{a,b} \subseteq O_{\{a,b\}}$ be the corresponding copy of K given by $K_{a,b} = \{f^s \mid f \in K\}$ where $s : \{0, 1\} \to \{a, b\} : 0 \mapsto a, 1 \mapsto b$ and $f^s(x_1, \ldots, x_n) := s(f(s^{-1}(x_1), \ldots, s^{-1}(x_n)))$ (cf. 1.2). We write $f_{a,b} \sim f_{0,1}$ if $f_{a,b} = f^s_{0,1}$ for $s : 0 \mapsto a, 1 \mapsto b$. $Im\, f$ denotes the image of f.

List of $\wedge$-irreducible S_3-clones and S_3-closed classes

(1) $\mathsf{P}_3 := \mathsf{O}_3$;

(2) $\mathsf{CL}_3 := J_3 \cup O_3^{(1)} \cup \{f \mid |Im\, f| \leq 2\}$ (Słupecki class);

(3) $\mathsf{BQ}_4 := \langle s_1 \rangle \cup \{f \mid |Im\, f| \leq 2\}$ where $s_1(x) := x + 1\, (mod\, 3)$ (=permutation $s_1 = (012) \in S_3$);

(4) $\mathsf{BQ}_5 := J_3 \cup \{f \mid |Im\, f| \leq 2\}$;

(5) $\mathsf{LQ} := \langle O_3^{(1)} \rangle \cup F_0$, where F_0 is the set of all $f \in O_3^{(n)}$ ($n \in \mathbf{N}$) such that there are $h_1, \ldots, h_n : \{0, 1, 2\} \to \{0, 1\}$ and $h_0 : \{0, 1\} \to \{0, 1, 2\}$ with $f(x_1, \ldots, x_n) = h_0(h_1(x_1) \oplus \ldots \oplus h_n(x_n))$ ($\oplus =$ addition $mod\, 2$);

(6) $\mathsf{P}_3^1 := \langle O_3^{(1)} \rangle$ (essentially unary functions);

(7) $\mathsf{Q}_2 := \langle S_3 \rangle$ (essentially unary permutations);

[4] Our thanks are due to B.Ganter for his help in designing this diagram. Using "formal concept analysis" some errors in the diagram of [14] could be detected.

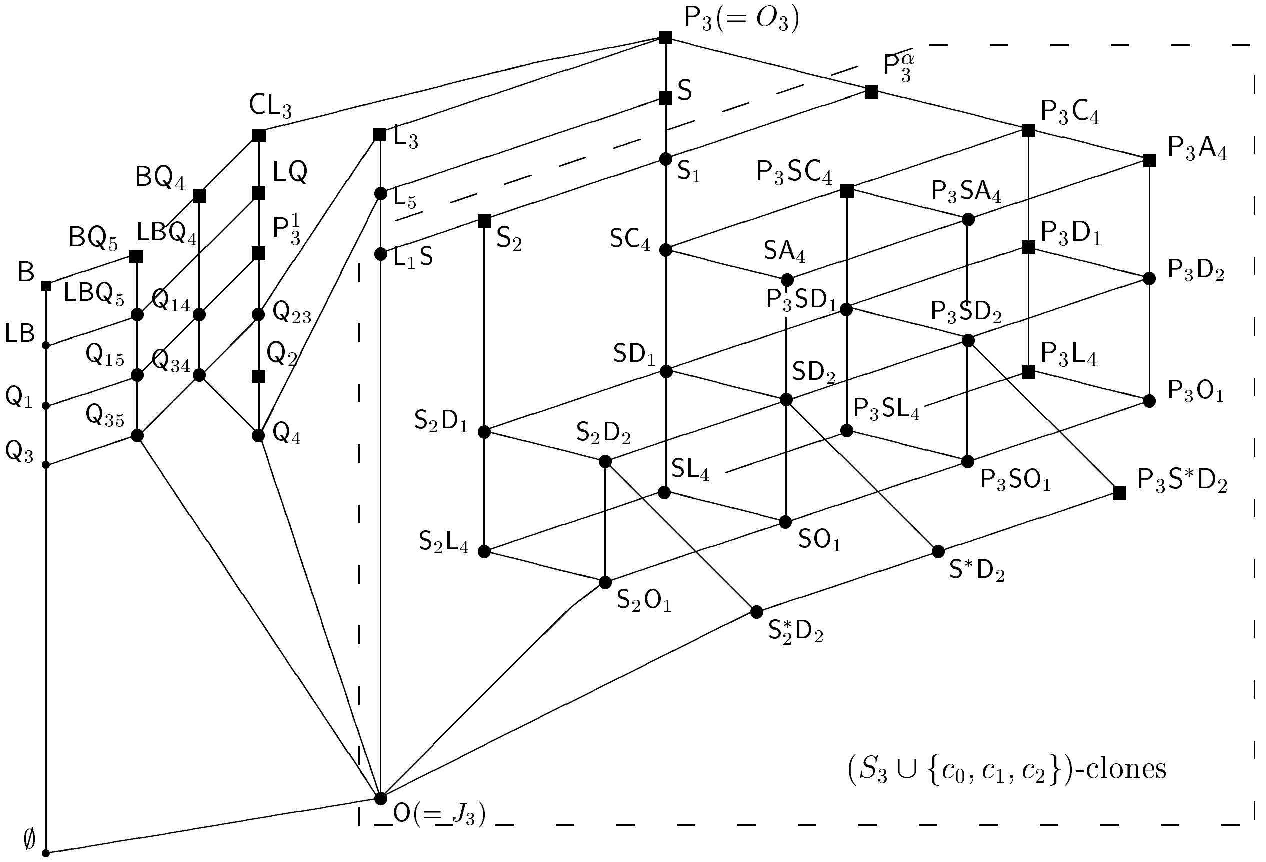

Figure 2: The lattice $\mathcal{L}_{3,S_3}$ of S_3-clones

(8) $\mathsf{L}_3 := \langle +, \cdot \rangle$ (clone of linear functions, $+, \cdot$ are addition and multiplication $mod\, 3$);

(9) $\mathsf{S} := \{f \mid f^{s_1} = f\}$ (s_1-self dual functions, cf.(3));

(10) $\mathsf{S}_2 := \{f \mid f^{s_0} = f \text{ and } f^{s_1} = f\}$, s-self dual functions for all $s \in S_3$, $s_0(x) := 2 \cdot x$ (=permutation (12)) ;

(11) $\mathsf{P}_3^\alpha := \{f \mid f(x, \dots, x) = x\}$ (idempotent functions);

(12) $\mathsf{P}_3\mathsf{C}_4 := \{f \mid f_{a,b} \in \mathsf{C}_{4,a,b} \text{ for } (a, b) \in \{(0, 1), (1, 2), (2, 0)\}\}$;

(13) $\mathsf{P}_3\mathsf{SC}_4 := \{f \mid f \in \mathsf{P}_3\mathsf{C}_4 \text{ and } f_{a,b} \sim f_{0,1} \text{ for } (a, b) \in \{(1, 2), (2, 0)\}\}$;

(14) $\mathsf{P}_3\mathsf{D}_1 := \{f \mid f_{a,b} \in \mathsf{D}_{1,a,b} \text{ for } a, b \in \{0, 1, 2\}\}$;

(15) $\mathsf{P}_3\mathsf{A}_4 := \{f \mid f_{a,b} \in \mathsf{A}_{4,a,b} \text{ for } a, b \in \{0, 1, 2\}\}$;

(16) $\mathsf{P}_3\mathsf{L}_4 := \{f \mid f_{a,b} \in \mathsf{L}_{4,a,b} \text{ for } (a, b) \in \{(0, 1), (1, 2), (2, 0)\}\}$;

(17) $\mathsf{P}_3\mathsf{S}^*\mathsf{D}_2 := \{f \mid f \in \mathsf{P}_3\mathsf{SD}_2 \text{ and } (*)\ \}$,
where $\mathsf{P}_3\mathsf{SD}_2 := \mathsf{P}_3\mathsf{SC}_4 \cap \mathsf{P}_3\mathsf{D}_1 \cap \mathsf{P}_3\mathsf{A}_4$ and (*) is the following property:
For every g obtained from f by identification of variables (including $g = f$) we have: $g_{0,1} \in J_2$ implies $g \in J_3$.

The S_3-closed classes which are not clones can be obtained from the following $\wedge$-irreducible class B by intersection with $\mathsf{P}_3, \mathsf{LQ}, \mathsf{P}_3^1$ and L^3.

(18) $\mathsf{B} := \mathsf{BQ}_5 \setminus J_3 = \{f \mid |Im\, f| \leq 2\}$.

3.8. Remarks. a) To determine the interval $S_3\text{-}Int(F)$ for any concretely given clone F one can proceed as follows. Let $\mathcal{I}$ be the set of all $\wedge$-irreducible clones from the list (1)–(17) in 3.7. Then

$$S_3\text{-}Int(F) = [\bigvee_{C\in\mathcal{T}_F} C, \bigwedge_{C\in\mathcal{M}_F} C]_{\mathcal{L}_3}$$

where

$$\mathcal{M}_F := \{C \in \mathcal{I} \mid F \subseteq C\} \text{ and } \mathcal{T}_F := \{\bigwedge_{D\in\mathcal{M}'} D \mid \mathcal{M}' \subseteq \mathcal{I}\setminus\mathcal{M}_F, \bigwedge_{D\in\mathcal{M}'} D \subseteq F\}.$$

Instead of $\mathcal{T}_F$ one could use also the set of clones, which are $\vee$-irreducible in the lattice $\mathcal{L}_{3,S_3}$ and are contained in F.
Analogously (using (1)–(18) in 3.7) one can determine $S_3\text{-}Int(F)$ for closed classes. We shall not go into further details. As an example we mention the celebrated Janov-Muchnik-class JM_3 generated by all functions

$$f_n(x_1,\ldots,x_n) = \begin{cases} 1 & \text{if } \exists j : x_j = 1 \text{ and } \forall i \neq j : x_i = 2 \\ 0 & \text{otherwise.}\end{cases}$$

JM_3 is a closed class with countable basis and its corresponding interval is $S_3\text{-}Int(\mathsf{JM}_3) = [\emptyset, \mathsf{B}]$, i.e. $\bigwedge_{s\in S_3} \mathsf{JM}_3^s = \emptyset$, $\langle \bigvee_{s\in S_3} \mathsf{JM}_3^s\rangle_{\mathcal{P}_3} = \mathsf{B}$.

b) Because of 2.3, for $G \leq G' \leq S_A$, every G'-clone is a G-clone, too. The set $\mathcal{L}_{3,S_3}$ of S_3-clones is finite (3.5), however there are infinitely many A_3-clones (A_3 being the alternating group) as shown in [6].

4. General Galois theory for *G*-clones

In this section we outline a characterization of G-clones by invariant relations based on a Galois connection between operations and relations. Restricting to G-clones, all results more or less straightforward can be derived from corresponding results for arbitrary clones (see e.g. [25], [24]). Therefore we omit the proofs[5] in this section.

4.1. For a base set A let

$$R_A^{(m)} := \{\varrho \mid \varrho \subseteq A^m\}$$

be the set of all m-ary relations and $R_A := \bigcup_{m=1}^{\infty} R_A^{(m)}$. Elements ($m$-tuples) $r \in A^m$ we often write as $r = (r(1),\ldots,r(m))$, considering them as functions $r : \underline{m} \to A : i \mapsto r(i)$. A function $f \in O_A^{(n)}$ *preserves* the relation $\varrho \in R_A$ (f is a *polymorphism* of ϱ, ϱ is *invariant* for f) if, for all $r_1,\ldots,r_n \in A^m$, $r_1,\ldots,r_n \in \varrho$ implies $f[r_1,\ldots,r_n] \in \varrho$ where $f[r_1,\ldots,r_n] \in A^m$ is defined by $f[r_1,\ldots,r_n](i) := f(r_1(i),\ldots,r_n(i))$ for $i \in \underline{m}$.
For $F \subseteq O_A$ and $Q \subseteq R_A$,

$$Inv_A F := \{\varrho \in R_A \mid \forall f \in F : f \text{ preserves } \varrho\}$$

[5] see [13]. Some proofs in detail were also given by J.-U. Grabowski and T. Rönsch (TU Dresden).

is the set of invariant relations of F and

$$Pol_A Q := \{f \in O_A \mid \forall \varrho \in Q : f \text{ preserves } \varrho\}$$

is the set of polymorphisms of G.
The operators *Pol-Inv* give a Galois connection between functions and relations on A, and for finite A, the Galois-closed elements $Pol\,Inv\,F$ and $Inv\,Pol\,Q$ are just the clones and so-called relational clones: $\langle F\rangle = Pol\,Inv\,F$, $[Q] = Inv\,Pol\,Q$ (for more details, see [25], [24]). Here Q is called a *relational clone* if $Q = Inv\,Pol\,Q$.

4.2. Definition. *For $s \in S_A$ and $\varrho \in R_A^{(m)}$ let*

$$\varrho^s := \{(x_1^s, \ldots, x_m^s) \mid (x_1, \ldots, x_m) \in \varrho\}.$$

(Remark: ϱ^s consists of all mappings $\underline{m} \to A$ of the composed form $\underline{m} \xrightarrow{r} A \xrightarrow{s} A$ with $r \in \varrho$). *A relational clone Q is called a relational G-clone for $G \leq S_A$, if $Q^G = Q$ (where $Q^G := \{\varrho^s \mid \varrho \in Q, s \in G\}$). An operation $f \in O_A$ G-preserves ϱ if f preserves ϱ^s for all $s \in G$. For $F \subseteq O_A$, $Q \subseteq R_A$ let*

$$G\text{-}Pol_A\, Q := Pol_A\, Q^G,\ G\text{-}Inv_A\, F := Inv_A\, F^G.$$

In the following Lemma, (i)-(iii) describes the interplay between dualization w.r.t. s and preservation while (iv)-(v) show that the operators *G-Pol* and *G-Inv* form a Galois connection (induced by the preservation property).

4.3. Lemma. *Let $\varrho \in Q \subseteq R_A$, $f \in F \subseteq O_A$, $s \in S_A$, $G \leq S_A$ and $Q \subseteq R_A$. Then we have:*
- (i) *f preserves ϱ iff f^s preserves ϱ^s,*
- (ii) $(Pol_A Q)^s = Pol_A(Q^s)$,
- (iii) $(Inv_A F)^s = Inv_A(F^s)$.
- (iv) $G\text{-}Pol\,Q = \{f \in O_A \mid \forall \varrho \in Q : f$ *G-preserves* $\varrho\}$ *and*
  $G\text{-}Pol\,Q$ *is a G-clone.*
- (v) $G\text{-}Inv\,F = \{\varrho \in R_A \mid \forall f \in F : f$ *G-preserves* $\varrho\}$ *and*
  $G\text{-}Inv\,F$ *is a relational G-clone.*

Using these results one can characterize the Galois-closed sets of functions and relations (w.r.t. the above Galois connection) as follows:

4.4. Theorem. *For $F \subseteq O_A$, $Q \subseteq R_A$ and $G \leq S_A$ we have*
- (1) *$G\text{-}Pol\,G\text{-}Inv\,F$ is the G-clone generated by F:*
  $^G F = \langle F^G\rangle = G\text{-}Pol\,G\text{-}Inv\,F = Pol\,G\text{-}Inv\,F$,
- (2) *$G\text{-}Inv\,G\text{-}Pol\,Q = Inv\,G\text{-}Pol\,Q$ is the relational G-clone generated by Q (the least relational G-clone containing Q).*

Moreover, for a clone $F \leq O_A$ we have

$$_G F = \bigcap_{s\in G} F^s = \bigcap_{s\in G} Pol\,Inv\,F^s = \bigcap_{s\in G} Pol(Inv\,F)^s$$
$$= Pol\Big(\bigcup_{s\in G}(Inv\,F)^s\Big) = Pol(Inv\,F)^G = G\text{-}Pol\,Inv\,F.$$

Combining this with 4.4(1) we get the following characterization of the lower and upper bound $_G F$ and $^G F$, resp., of $G\text{-}Int\,F$:

4.5. Proposition. *Let $F \leq O_A$ be a clone and $G \leq S_A$. Then*

$$G\text{-}Int\,F = [G\text{-}Pol\,Inv\,F,\ Pol\,G\text{-}Inv\,F]_{\mathcal{L}_A}.$$

□

Remark. Analogously one can define and characterize G-intervals corresponding to relational clones.

5. H-clones and H-closed classes

In this section we generalize the concept of G-clones. Instead of groups $G \leq S_A$ we use semigroups $H \leq T_A$ with unit (monoids) or without unit. However, definition 2.1 cannot be generalized canonically to H-clones. There are several possibilities (all equivalent for groups $H \leq S_A$).

5.1. Notations. $\langle H \rangle_{T_A}$ denotes the subsemigroup of T_A generated by H. Here

$$T_A := \{h \mid h : A \to A\}$$

is the set of all unary mappings on A. It is a monoid w.r.t. the usual composition of mappings. Again (cf. 1.2) we write a^h for the image $h(a)$ of a, composition is to read from left to right: $a^{(hh')} := (a^h)^{h'}$. Let $e : x \mapsto x$ denote the identical mapping. We write $H \leq T_A$ if H is a subsemigroup of T_A and $H \leq_e T_A$ if H is a submonoid (i.e. $e \in H \leq T_A$).

5.2. Definitions. *Let $H \subseteq T_A$ and $F \subseteq O_A$ be a clone (closed class, resp.).*

(a) *For $f, g \in O_A^{(n)}$, $h \in H$ we write*

$$g \rightrightarrows^h f$$

if $\forall x_1, \ldots, x_n \in A : (g(x_1, \ldots, x_n))^h = f(x_1^h, \ldots, x_n^h)$, i.e. if the following diagram commutes

$$\begin{array}{ccc} A^n & \xrightarrow{h^{[n]}} & A^n \\ \Big\downarrow g & & \Big\downarrow f \\ A & \xrightarrow{h} & A \end{array}$$

(where $h^{[n]}$ is defined componentwise). We write $g \gtrsim^h f$ if there exists a chain $g \rightrightarrows^h g_1 \rightrightarrows^h \ldots \rightrightarrows^h g_{m-1} \rightrightarrows^h f$, and $g \gtrsim^h_F f$ if in addition all g_i $(i = 1, \ldots, m-1)$ belong to F. A function f is called h-homogeneous if $f \rightrightarrows^h f$ and weakly h-homogeneous (in F) if $f \gtrsim^h f$ ($f \gtrsim^h_F f$), respectively.

(b) *F is called right H-clone (H-closed class, resp.) if for all $h \in H$ we have*

$$\forall f \in F\ \exists g \in F : g \rightrightarrows^h f. \qquad (*)$$

For short, right H-clones will be called H-clones. F is called left H-clone (H-closed class, resp.) if for all $h \in H$ we have

$$\forall f \in F\ \exists g \in F : f \rightrightarrows^h g. \qquad (**)$$

(c) *F is called H-homogeneous (right) H-clone if*

$$\forall f \in F\, \exists g \in F\, \forall h \in H : g \rightrightarrows^h f.$$

Analogously one defines H-homogeneous left H-clones.[6]

(d) *Let $\mathcal{L}_{A,H}$ ($\mathcal{L}'_{A,H}$, resp.) denote the set of all H-clones $F \leq O_A$ (closed classes, resp.).*

5.3. Examples. 1) J_A is an H-homogeneous H-clone for every $H \leq T_A$.

2) For given $H \leq T_A$, the set $\{g \in O_A \mid \forall h \in H : g \rightrightarrows^h g\}$ is the largest clone F for which all $h \in H$ are endomorphisms (of the algebra $\langle A; F\rangle$) and it is an H-clone.

3) For $e \in H$, right and left H-homogeneous H-clones coincide (note $g \rightrightarrows^e f$ implies $g = f$), and F is an H-homogeneous H-clone iff $\forall f \in F\, \forall h \in H : f \rightrightarrows^h f$. In particular, for $|A| = k = 2$ and $H = T_2$ we have: F is a T_2–homogeneous (right=left) T_2–clone iff $F \leq \mathsf{D}_1$.

4) Note that $g \rightrightarrows^c f$ means for some constant c, that $c = f(c, \ldots, c)$. Thus, for $|A| = k = 2$ and $H = \{c_0, c_1\}$, a clone F is an H–homogeneous clone iff every $f \in F$ is idempotent ($f(0, \ldots, 0) = 0$ and $f(1, \ldots, 1) = 1$), i.e. $F \leq \mathsf{C}_4$. On the other hand, a clone F is a left H-homogeneous H-clone if, by definition,

$$\forall\, f \in F\, \exists\, g \in F : \; 0 = g(0, \ldots, 0) \,\wedge\, 1 = g(1, \ldots, 1).$$

Since $g = e_1^1 \in F$ fulfils this condition, every clone is an H- homogeneous left H-clone.

5.4. Remarks.

(a) For a permutation $s \in S_A$ with $g \rightrightarrows^s f$, the function f uniquely determines g (and vice versa), namely we have $g \rightrightarrows^s f \iff g^s = f$. Consequently, for $H = G \subseteq S_A$, both definitions, 5.2b) and 2.1 coincide and left H-clones agree with right H-clones, they are just the G-clones in the sense of 2.1. Moreover, in that case, a clone is left H-homogeneous iff it is right H-homogeneous iff it is G-homogeneous (cf. 2.1).

(b) Obviously, H-homogeneous H-clones are H-clones since the property in 5.2(c) implies (*) in 5.2(b).

(c) For an algebra $\mathcal{A}$ with F as clone of term operations, mappings $h \in T_A$ with the property 5.2(*) (in several modifications) were intensively studied by K. Głazek (cf. e.g. [9]) as semi-weak endomorphisms, cf. also [7, 3.4]).

(d) A simple but useful observation is: For $H' \subseteq H$, every H-closed class is H'-closed, too. Moreover, a class F is $H' \cup H''$-closed iff it is both, H'-closed and H''-closed. In particular, F *is an H-clone iff it is an $\{h\}$-clone for every $h \in H$.* In 5.6 we shall see that F is an H-clone iff it is an $\langle H\rangle_{T_A}$-clone.

(e) Let $g_0 \rightrightarrows^h g_1 \rightrightarrows^h \ldots \rightrightarrows^h g_m$. This situation implies the following properties:

(i) g_0 preserves $ker\, h$, $ker\, h^2$,…,$ker\, h^m$,

(ii) g_m preserves $Im\, h$, $Im\, h^2$,…,$Im\, h^m$.

In fact, e.g. $g_0(x_1, \ldots, x_n)^h = g_1(x_1^h, \ldots, x_n^h)$ implies $g_0 \in Pol\, ker\, h$ and $g_m(x_1^h, \ldots, x_n^h) = g_{m-1}(x_1, \ldots, x_n)^h$ implies $g_m \in Pol\, Im\, h$ (analogously for $h^2, \ldots, h^m$).

[6] For $|H| = 1$, right (left) H-homogeneous clones are just the right (left) H-clones.

(f) Let $F_1(H) := F_1 := \bigcap_{h\in H} Pol_A Im(h)$ be the set of all functions preserving every image $Im\,h$. Because of (ii) above, every H-clone is contained in $F_1(H)$. However, in general, F_1 is not an H-clone (cf. also 5.11 and 5.12).
To see this, let $A = \{0,1,2\}$, $H := \langle h_1, h_2, h\rangle$ where $0^{h_1} = 2^{h_1} = 0$, $1^{h_1} = 1$, $0^{h_2} = 2^{h_2} = 2$, $1^{h_2} = 1$, $0^h = 2^h = 1$, $1^h = 0$. Then the constant $f := c_1^1 \in O_A^{(1)}$ belongs to $F_1(H)$, but no $g \in F_1$ satisfies $g \rightrightarrows^h f$, because $g(1)^h = f(1^h) = f(0) = 1$ would imply $g(1) \in h^{-1}(1) = \{0,2\}$ in contradiction to $g(1) \in Im\,h_1 \cap Im\,h_2 = \{1\}$.

The following results are formulated for H-clones, but they can be used analogously for arbitrary closed classes. If not otherwise stated, H always denotes a subset of T_A for a finite set A.

5.5. Lemma. *Let $g_i, f_i \in O_A^{(n)}$ and $g_i \rightrightarrows^h f_i$ for $i = 1,2$. Then we have $(\sigma g_1) \rightrightarrows^h (\sigma f_1)$ for every operator $\sigma \in \{\tau, \zeta, \Delta, \nabla\}$ and $(g_1 * g_2) \rightrightarrows^h (f_1 * f_2)$.*

Proof. The first part immediately follows from the definitions. The second part follows from

$$\begin{aligned} g_1(g_2(x_1,\dots,x_m),\dots,x_{m+n-1})^h &= f_1(g_2(x_1,\dots,x_m)^h,\dots,x_{m+n-1}^h) \\ &= f_1(f_2(x_1^h,\dots,x_m^h),\dots,x_{m+n-1}^h). \end{aligned} \qquad \square$$

The following Theorem provides a criterion for H-clones based on generating sets and will be useful for the construction of H-clones.

5.6. Theorem. *Let $H = \langle H_0\rangle_{T_A} \le T_A$ and $F = \langle F_0\rangle_{O_A} \le O_A$ with generating sets H_0 and F_0. Then the clone $F \le O_A$ is an H-clone iff*

$$\forall\, h \in H_0\ \forall\, f \in F_0\ \exists\, g \in F:\ g \rightrightarrows^h f \qquad (^\circ)$$

In particular, F is an H-clone iff F is an H_0-clone.

Proof. Obviously we have to prove the "if"-part only. At first we proceed by induction on the generation of F from F_0. By Lemma 5.5 we conclude from $(^\circ)$ that

$$\forall\, h \in H_0\ \forall\, f \in \langle F_0\rangle\ \exists\, g \in F:\ g \rightrightarrows^h f \qquad (^{\circ\circ})$$

Secondly, let $h_1, h_2 \in H_0$, $f \in F$. Then, by $(^{\circ\circ})$ there exists $g_2 \in F$ with $g_2 \rightrightarrows^{h_2} f$, and for g_2 there exists $g_1 \in F$ such that $g_1 \rightrightarrows^{h_1} g_2$. Then

$$g_1(x_1,\dots,x_n)^{h_1h_2} = (g_2(x_1^{h_1},\dots,x_n^{h_1}))^{h_2} = f(x_1^{h_1h_2},\dots,x_n^{h_1h_2}),$$

i.e. $g_1 \rightrightarrows^{h_1h_2} f$ and the property $(^{\circ\circ})$ extends to the closure $\langle H_0\rangle = H$. Thus F is H-closed. $\square$

5.7. Remark. Every $H \le T_A$ splits into the "group part" $G := H \cap S_A$ and the "pure semigroup part" $H' := H \setminus G$ (note that $H' \le T_A$ for finite A). By 5.6 these parts can be treated separately (e.g. using generating systems).

The identity plays no role for H-clones. Since $f \rightrightarrows^e f$ we have:

5.8. Lemma. *A clone F is an H-clone iff it is an $(H \setminus \{e\})$-clone.*

Let us look now for intersections and unions of H-clones.

5.9. Lemma. *For $h \in T_A$, a set $H \subseteq T_A$ and a clone $F \leq O_A$ let*

$$\begin{aligned} \gamma_1(F) &:= \{f \in F \mid \exists g \in F: \ g \rightrightarrows^h f\}, \\ \gamma_{n+1}(F) &:= \gamma_1(\gamma_n(F)), \quad n = 1, 2, 3, \ldots, \\ \gamma_{\{h\}}(F) &:= \bigcap_{n=1}^{\infty} \gamma_n(F), \\ \gamma_H(F) &:= \bigcap_{h \in H} \gamma_{\{h\}}(F). \end{aligned}$$

Then

(a) *$\gamma_1(F)$ is a clone contained in F.*

(b) *$\gamma_{\{h\}}(F)$ is an $\{h\}$-clone contained in F.*

(c) *$\gamma_H(F)$ is an H-clone contained in F.*

Proof. (a) directly follows from Lemma 5.5. Thus also $\gamma_{\{h\}}(F)$ and therefore $\gamma_H(F)$ are clones. In order to prove (b) let $f \in \gamma_{\{h\}}(F)$ be an m-ary function. By definition $f \in \gamma_n(F)$, i.e., for every $n \in \mathbf{N}$ there exists $g_n \in \gamma_n(F)^{(m)}$ such that $g_n \rightrightarrows^h f$. Consider the sequence $(g_n)_{n \in \mathbf{N}}$. Since $O_A^{(m)}$ is finite there is some $n_0 \in \mathbf{N}$ such that g_{n_0} appears infinitely many often among $g_1, g_2, g_3, \ldots$. Therefore $g_{n_0} \in \gamma_n(F)$ for all $n \geq n_0$ and we have $g_{n_0} \in \gamma_{\{h\}}(F)$ (note that the $\gamma_n(F)$ form a descending chain). Now $g_{n_0} \rightrightarrows^h f$ shows that $\gamma_{\{h\}}(F)$ is an $\{h\}$-clone. (c) follows from the definitions and 5.4(d). □

5.10. Theorem. *Let $H \subseteq T_A$.*

(1) *Let $F \leq O_A$ be a clone. Then $\gamma_H(F)$ is the greatest H-clone contained in F.*

(2) *The join (in $\mathcal{L}_A$) $F \vee G = \langle F \cup G \rangle_{O_A}$ of H-clones F, G is again an H-clone (analogously for H-homogeneous H-clones).*

(3) *$\mathcal{L}_{A,H}$ is a lattice w.r.t. inclusion with least element J_A and greatest element $\gamma_H(O_A)$. The corresponding join $\vee$ and meet $\wedge_H$ are given by*

$$\begin{aligned} F \vee G &= \langle F \cup G \rangle_{O_A} \quad \textit{and} \\ F \wedge_H G &= \gamma_H(F \wedge G). \end{aligned}$$

Proof. (1): Let F' be an H-clone contained in F. Then $F' = \gamma_{\{h\}}(F') \subseteq \gamma_{\{h\}}(F)$ for $h \in H$, hence $F' \subseteq \gamma_H(F)$ and, by Lemma 5.9(c), $\gamma_H(F)$ is the largest H-clone contained in F.

(2): The proof of (2) immediately follows from 5.6.

(3): By (1), $F \wedge_H G$ is the largest H-clone contained in both, F and G. □

We ask for a more explicite description of $\gamma_H(F)$, in particular for the largest H-clone $\gamma_H(O_A)$. This will be answered by the next two propositions. Because of the definition of $\gamma_H(F)$ (5.9), it suffices to consider the case $H = \{h\}$ only.

5.11. Proposition. *Let $h \in T_A$ and $F \leq O_A$. Then we have*

$$\gamma_{\{h\}}(F) = \{f \in F \mid \exists g \in F : g \precsim_F^h g \textit{ and } g \precsim_F^h f\}.$$

Proof. Let R denote the right side of the above equation and let $f_0 := f$ be an element of $\gamma_{\{h\}}(F)$, the greatest $\{h\}$-clone contained in F. Thus there exist $f_1 \in \gamma_{\{h\}}(F)$ with $f_1 \overset{h}{\rightrightarrows} f_0$ and, by induction, there are $f_i \in \gamma_{\{h\}}(F)$ with $f_{i+1} \overset{h}{\rightrightarrows} f_i$ $(i \geq 0)$. Since all f_i are of the same arity as f, the set $\{f_i \mid i \in \mathbf{N}\}$ is finite and there must exist $r, m \in \mathbf{N}$ with $f_{r+m} = f_r =: g$. Thus $g \precsim^h_F g$ and $g \precsim^h_F f$, consequently $f \in R$.
Conversely, let $f \in R$, i.e., there is a chain

$$g = f_{r+m} \overset{h}{\rightrightarrows} f_{r+m-1} \overset{h}{\rightrightarrows} \dots \overset{h}{\rightrightarrows} f_r = g \overset{h}{\rightrightarrows} f_{r-1} \overset{h}{\rightrightarrows} \dots f_1 \overset{h}{\rightrightarrows} f_0 = f$$

(with $f_i \in F$). Then the infinite sequence $(f_i)_{i \in \mathbf{N}}$ with $f_{r+l+km} := f_{r+l}$ for $l \in \{0, 1, \dots, m-1\}$, $k \in \mathbf{N}$, satisfies $f_{i+1} \overset{h}{\rightrightarrows} f_i$ for all $i \in \mathbf{N}$. Consequently, $f_i \in \gamma_{\{h\}}(F)$, in particular $f = f_0 \in \gamma_{\{h\}}(F)$. □

Note that 5.11 characterizes in particular the largest $\{h\}$-clone $\gamma_{\{h\}}(O_A)$. Having this in mind we look for a characterization of weakly h-homogeneous functions (cf. 5.2(a)).

5.12. Proposition. (1) *A function $g \in O_A^{(n)}$ is weakly h-homogeneous (i.e. $g \precsim^h g$) iff there exists $m \in \mathbf{N}$ such that the following three conditions hold:*
(i) *g is h^m-homogeneous,*
(ii) *g preserves $ker\, h$, $ker\, h^2, \dots,$ $ker\, h^m$,*
(iii) *g preserves $Im\, h$, $Im\, h^2, \dots,$ $Im\, h^m$.*
(2) *Let $h \in T_A$ be idempotent. Then the largest $\{h\}$-clone is $\gamma_{\{h\}}(O_A) = Pol\, Im\, h$.*

Proof. (1): Let g be weakly h-homogeneous, i.e. we have $g \overset{h}{\rightrightarrows} g_1 \overset{h}{\rightrightarrows} \dots \overset{h}{\rightrightarrows} g_{m-1} \overset{h}{\rightrightarrows} g$ for suitable m and g_i. Then g satisfies (ii) and (iii) because of 5.4(e). Moreover, $g(x_1, \dots, x_n)^{h^m} = g_1(x_1^h, \dots, x_n^h)^{h^{m-1}} = \dots = g_{m-1}(x_1^{h^{m-1}}, \dots, x_n^{h^{m-1}})^h = g(x_1^{h^m}, \dots, x_n^{h^m})$, i.e. (i) also holds.
Conversely, let g satisfy (i) – (iii) and put $g_0 := g_m := g$. We define $g_1, \dots, g_{m-1}$ as follows:

$$g_{i+1}(x_1^{h^{i+1}}, \dots, x_n^{h^{i+1}}) := g_i(x_1^{h^i}, \dots, x_n^{h^i})^h, \quad i = 0, 1, , \dots, m-2.$$

Because of (ii), $g_1, g_2, \dots, g_{m-1}$ are well-defined on $Im\, h$, $Im\, h^2$, $\dots$, $Im\, h^{m-1}$, respectively. The remaining values are chosen arbitrary but according to the following conditions:

$$g_i(y_1, \dots, y_n) \in h^{-1}(g_{i+1}(y_1^h, \dots, y_n^h)) \text{ for } (y_1, \dots, y_n) \in A^n \setminus (Im\, h^i)^n,$$

$i = 1, \dots, m-1$. Such values really exist according to (iii). Then, by definition, we have $g_i \overset{h}{\rightrightarrows} g_{i+1}$ for $i = 0, 1, \dots, m-2$. Moreover, we also have $g_{m-1} \overset{h}{\rightrightarrows} g$.
In fact, $g_{m-1}(y_1, \dots, y_n)^h = g(y_1^h, \dots, y_n^h)$ for $(y_1, \dots, y_n) \in A^n \setminus (Im\, h^{m-1})^n$ by definition, and for $(y_1, \dots, y_n) = (x_1^{h^{m-1}}, \dots, x_n^{h^{m-1}}) \in (Im\, h^{m-1})^n$ we have

$$g_{m-1}(x_1^{h^{m-1}}, \dots, x_n^{h^{m-1}})^h = g_{m-2}(x_1^{h^{m-2}}, \dots, x_n^{h^{m-2}})^{h^2} = \dots$$
$$\dots = g_0(x_1, \dots, x_n)^{h^m} =_{(i)} g(x_1^{h^m}, \dots, x_n^{h^m}).$$

Thus $g \overset{h}{\rightrightarrows} g_1 \overset{h}{\rightrightarrows} \dots \overset{h}{\rightrightarrows} g_{m-1} \overset{h}{\rightrightarrows} g$, i.e. g is weakly h-homogeneous.
(2): By 5.4(e) and 5.11 we have $\gamma_{\{h\}}(O_A) \subseteq Pol\, Im\, h$. Let $f \in Pol\, Im\, h$. Because of $hh = h$ we have $x^h = x$ for $x \in Im\, h$, thus $(f(x_1^h, \dots, x_n^h))^h = f(x_1^h, \dots, x_n^h)$, i.e. $h^{[n]} f h = h^{[n]} f$ (cf. also 5.2). Consequently, for $g := h^{[n]} f$ we have $gh = h^{[n]} f h = h^{[n]} f = h^{[n]} h^{[n]} f$, i.e. $g \overset{h}{\rightrightarrows} g \overset{h}{\rightrightarrows} f$. Obviously, $g \in Pol\, Im\, h$, thus $f \in \gamma_{\{h\}}(O_A)$ by 5.11. □

In general, the intersection of H-clones might not be an H-clone. However, H-homogeneous clones are closed w.r.t. intersections:

5.13. Proposition. *Assume $e \in H$. Then we have:*

(a) *For an H-homogeneous H-clone F, every $f \in F$ is an H-homogeneous function.*

(b) *The intersection $F \wedge G$ of H-homogeneous H-clones F and G is an H-homogeneous H-clone.*

Proof. (a) directly follows from the definitions (5.2, note $g \rightrightarrows^e f \iff g = f$).
Now (b) follows from (a) (use $g = f$ in 5.2(c)). □

6. H-clones and H-closed classes on $A = \{0,1\}$ and $A = \{0,1,2\}$

A) The case $A = \{0,1\}$

6.1. On the 2-element set $\mathbf{2} = \{0,1\}$ there are 4 unary mappings $(e, \neg, c_0, c_1)$ and 9 non-empty subsemigroups of T_2, three of which do not contain the identity e and need not to be considered in connection with H-clones because of 5.8. The lattice of subsemigroups is given in Fig. 3.

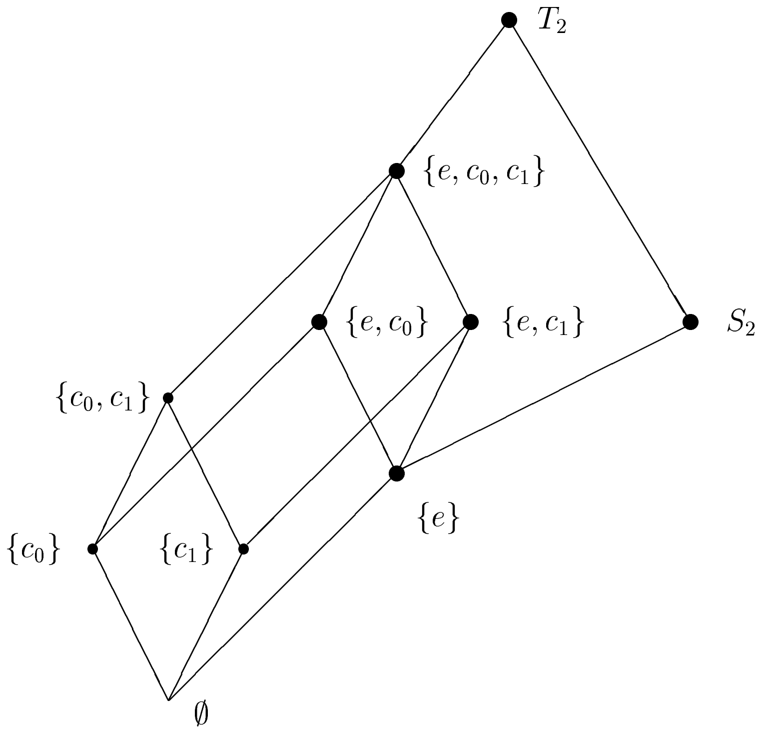

Figure 3: The subsemigroup lattice of T_2

We are going to determine the H-closed classes for all $H \leq T_2$. Obviously $H = \{e\}$ gives Post's lattice $\mathcal{L}_{2,H} = \mathcal{L}_2$ of all clones (which we do not want to give here (cf. e.g. [15], [25]). For $H = S_2$ the lattice $\mathcal{L}_{2,S_2}$ is given in Fig. 1 (page 188).

We shall indicate the case dealing with right H-clones by $\boxed{\mathbf{R}}$, the left H-clone case by $\boxed{\mathbf{L}}$, and analogously $\boxed{\mathbf{RH}}$ (and $\boxed{\mathbf{LH}}$) if we consider right (left) H-homogeneous clones.

6.2. The case $H = \{c_0\}$ (and $H = \{c_1\}$).
$\boxed{\mathbf{R}}$: For $H = \{c_0\}$ (or, equivalently $H = \{e, c_0\}$), cf. 5.8), Definition 5.2b immediately gives (note $g \rightrightarrows^{c_0} f \iff 0 = f(0, \dots, 0)$):

$$F \leq O_2 \text{ is } \{c_0\}\text{-closed iff } F \leq \mathsf{C}_3 := \{f \mid f(0, \dots, 0) = 0\}.$$

Analogously, $F \leq O_2$ is $\{c_1\}$-closed iff $F \leq \mathsf{C}_2 := \{f \mid f(1, \dots, 1) = 1\}$.
$\boxed{\mathbf{L}}$: By Definition 5.2b, $F \leq O_2$ is left $\{c_0\}$-closed iff $F \cap \mathsf{C}_3 \neq \emptyset$. Thus every clone is a left $\{c_0\}$-clone and every non-empty closed class except $\mathsf{O}_2 := \{c_1\}$ is left $\{c_0\}$-closed. (Analogous results hold for $H = \{c_1\}$ (with $\mathsf{O}_3 := \{c_1\}$ instead of O_2.)
$\boxed{\mathbf{RH}}$ $\boxed{\mathbf{LH}}$: As remarked above (footnote in 5.2) these cases coincide with $\boxed{\mathbf{R}}$ and $\boxed{\mathbf{L}}$.

6.3. The case $H = \{c_0, c_1\}$.
$\boxed{\mathbf{R}}$: By 6.2 (cf. also 5.3(4)) we have $F \leq O_2$ is $\{c_0, c_1\}$-closed (equivalently, $\{e, c_0, c_1\}$-closed) iff $F \leq \mathsf{C}_4 = \mathsf{C}_2 \cap \mathsf{C}_3$ (cf. 3.2).
$\boxed{\mathbf{L}}$: By 6.2 we have: $F \leq O_2$ is left $\{c_0, c_1\}$-closed iff $F \cap \mathsf{C}_3 \neq \emptyset$ and $F \cap \mathsf{C}_2 \neq \emptyset$. Thus every clone is a left $\{c_0, c_1\}$-clone and every non-empty closed class except O_2 and O_3 is left $\{c_0, c_1\}$-closed.
$\boxed{\mathbf{RH}}$: Right $\{c_0, c_1\}$-homogeneous closed classes are clones, since they contain with some idempotent function also the identity. Due to 5.3(4), a clone F is $\{c_0, c_1\}$-homogeneous iff $F \leq \mathsf{C}_4$.
$\boxed{\mathbf{LH}}$: By 6.2 we get: F is left $\{c_0, c_1\}$-homogeneous iff $F \cap \mathsf{C}_2 \cap \mathsf{C}_3 \neq \emptyset$. Thus again, every clone is left $\{c_0, c_1\}$-homogeneous (see also 5.3(4)) and every closed class except O_7, O_2 and O_3 is left $\{c_0, c_1\}$-homogeneous.

Thus the lattices $\mathcal{L}_{2,\{c_0\}}, \mathcal{L}_{2,\{c_1\}}, \mathcal{L}_{2,\{c_0,c_1\}}$ are special parts of Post's lattice $\mathcal{L}_2$. All these parts are infinite.

6.4. Remark. If one considers $H \cup \{e\}$ instead of H in the cases 6.2 and 6.3 then, according to 5.8, nothing changes for clones. Changes appear only in case of left H-homogeneous clones which coincide with the right one's (cf. 5.3(3)).

6.5. The case $H = T_2$.
$\boxed{\mathbf{R}}$: With 6.2 (and 5.4d) one easily checks: $F \leq O_2$ is T_2-closed iff F is S_2-closed and $F \leq \mathsf{C}_4$ (i.e. every function is idempotent). Thus $\mathcal{L}_{2,T_2}$ is the part of Fig. 1 (page 188) consisting of clones below C_4, i.e. the six T_2-clones $\mathsf{O}_1, \mathsf{D}_2, \mathsf{D}_1, \mathsf{L}_4, \mathsf{A}_4, \mathsf{C}_4$.
$\boxed{\mathbf{L}}$ The set of left T_2-clones differs from the set of the just described (right) T_2-clones. $F \leq O_2$ is a left T_2-clone (cf. 5.2b) iff F is a (left=right, cf. 5.4(a)) S_2-clone and a left $\{c_0, c_1\}$-clone. Thus, by 6.3, left T_2-clones and S_2-clones coincide.
By 5.3(3), $\boxed{\mathbf{RH}}$ =$\boxed{\mathbf{LH}}$, and the T_2-homogeneous clones F are those with $F \leq \mathsf{D}_1$.

All results are collected in Table 1 at the end of this paper.

B) The case $A = \{0, 1, 2\}$

We shall consider here only two subsemigroups H both containing S_3, namely $H_C := S_3 \cup \{c_0, c_1, c_2\}$ and T_3 (for $H = S_3$ see 3.5, 3.6). Moreover, we shall restrict to the case $\boxed{\mathbf{R}}$ of (right) H-clones only (however the cases $\boxed{\mathbf{L}}$, $\boxed{\mathbf{LH}}$ and $\boxed{\mathbf{RH}}$ are also included in Table 1).

6.6. H_C-clones.
F is an H_C-clone iff it is an S_3-clone with the additional property of being a $\{c_0, c_1, c_2\}$-clone, i.e. $\forall f \in F\ \exists g_0, g_1, g_2 \in F:\ g_i \rightrightarrows^{c_i} f$ (i.e. explicitly $c_i = f(c_i, \ldots, c_i)$, $i \in \{0, 1, 2\}$). Consequently, every $f \in F$ must be idempotent and we have (note that P_3^α denotes the clone of idempotent functions):

$$\mathcal{L}_{3,H_C} = \{F \mid F \in \mathcal{L}_{3,S_3} \text{ and } F \leq \mathsf{P}_3^\alpha\},$$

cf. Fig. 2 on page 190.

We are going to describe T_3-clones. Obviously, every T_3-clone F is an S_3-clone as well as an H_C-clone, in particular $F \leq \mathsf{P}_3^\alpha$ due to 6.6. Moreover, $T_3 = \langle S_3 \cup \{h_0\}\rangle_{T_3}$ for every $h_0 : A \to A$ with $|Im\, h_0| = |A| - 1 = 2$. Thus, by 5.6 we get:

6.7. Lemma. *A clone $F \leq O_3$ is a T_3-clone iff $F \leq \mathsf{P}_3^\alpha$ and F satisfies*

$$\forall f \in F\ \exists g \in F:\ g \rightrightarrows^{h_0} f \tag{*}$$

i.e. $(g(x_1, \ldots, x_n))^{h_0} = f(x_1^{h_0}, \ldots, x_n^{h_0})$ for all $x_1, \ldots, x_n \in A$, $n = ar\, f$ and some $h_0 \in T_3$, $|Im\, h_0| = 2$. □

With notations from 3.7 idempotency and condition (*) for all such h_0 imply $f \in \mathsf{P}_3\mathsf{C}_4$. Due to 5.10(2) we can restrict our attention to T_3-clones below $\mathsf{P}_3\mathsf{C}_4$ which are $\vee$-irreducible in the lattice of T_3-clones. Thus, in order to determine all T_3-clones we firstly determine the $\vee$-irreducible (in $\mathcal{L}_{3,S_3}$) S_3-clones F (from 3.7) with $F \leq \mathsf{P}_3\mathsf{C}_4$ and (*) (e.g. using $h_0 : x \mapsto x^2 \pmod 3$). It turns out that only one of the 9 $\vee$-irreducible S_3-clones $F \leq \mathsf{P}_3\mathsf{C}_4$ is not a T_3-clone (namely $\mathsf{S}_2\mathsf{L}_4$, cf. Fig. 2).

6.8. Lemma. *Among the $\vee$-irreducible S_3-clones there are exactly the following 8 T_3-clones $F \leq \mathsf{P}_3\mathsf{C}_4$ (notations see 5.3 and Fig. 2):*

$$\begin{array}{lll}
\mathsf{S}_2^*\mathsf{D}_2 & := & \mathsf{S}_2 \wedge \mathsf{P}_3\mathsf{S}^*\mathsf{D}_2 \\
\mathsf{S}^*\mathsf{D}_2 & := & \mathsf{S} \wedge \mathsf{P}_3\mathsf{S}^*\mathsf{D}_2 \\
\mathsf{P}_3\mathsf{S}^*\mathsf{D}_2 & & \\
\mathsf{S}_2\mathsf{O}_1 & := & \mathsf{S}_2 \wedge \mathsf{P}_3\mathsf{L}_4 \wedge \mathsf{P}_3\mathsf{A}_4 \\
\mathsf{S}\mathsf{O}_1 & := & \mathsf{S} \wedge \mathsf{P}_3\mathsf{L}_4 \wedge \mathsf{P}_3\mathsf{A}_4 \\
\mathsf{P}_3\mathsf{S}\mathsf{O}_1 & := & \mathsf{P}_3\mathsf{S}\mathsf{C}_4 \wedge \mathsf{P}_3\mathsf{L}_4 \wedge \mathsf{P}_3\mathsf{A}_4 \\
\mathsf{P}_3\mathsf{O}_1 & := & \mathsf{P}_3\mathsf{L}_4 \wedge \mathsf{P}_3\mathsf{A}_4.
\end{array}$$

We delete the technical proof.

Now there remain a few more clones which are candidates for a $\vee$-irreducible (now in $\mathcal{L}_{3,T_3}$!) T_3-clone, namely (with notations from Fig. 2) the clones SL_4, SD_1, $\mathsf{P}_3\mathsf{SL}_4$, SC_4, $\mathsf{P}_3\mathsf{SC}_4$, $\mathsf{P}_3\mathsf{SA}_4$, $\mathsf{P}_3\mathsf{A}_4$. However, among these there are only two T_3-clones, namely $\mathsf{P}_3\mathsf{SL}_4$ and $\mathsf{P}_3\mathsf{A}_4$ (the others are not T_3-clones).

Thus all possible unions of the T_3-clones in Lemma 6.8 together with $\mathsf{P}_3\mathsf{SL}_4$ and $\mathsf{P}_3\mathsf{SA}_4$ give the set of all T_3-clones. We describe them also in terms of $\wedge$-irreducible clones using the list in 3.7.

6.9. Proposition. *There are 18 T_3-clones, the 8 clones from Lemma 6.8 and the following 10 T_3-clones:*

$$
\begin{array}{lcll}
\mathsf{S_2D_2} & := & \mathsf{S_2O_1} \vee \mathsf{S_2^*D_2} & = \mathsf{S_2} \wedge \mathsf{P_3D_1} \wedge \mathsf{P_3A_4}, \\
\mathsf{SD_2} & := & \mathsf{SO_1} \vee \mathsf{S^*D_2} & = \mathsf{S} \wedge \mathsf{P_3D_1} \wedge \mathsf{P_3A_4} \\
\mathsf{P_3SD_2} & := & \mathsf{P_3SO_1} \vee \mathsf{P_3S^*D_2} & = \mathsf{P_3SC_4} \wedge \mathsf{P_3D_1} \wedge \mathsf{P_3A_4}, \\
\mathsf{P_3D_2} & := & \mathsf{P_3O_1} \vee \mathsf{P_3S^*D_2} & = \mathsf{P_3D_1} \wedge \mathsf{P_3A_4}, \\
\mathsf{P_3SL_4} & := & & = \mathsf{P_3SC_4} \wedge \mathsf{P_3L_4}, \\
\mathsf{P_3L_4} & := & \mathsf{P_3SL_4} \vee \mathsf{P_3O_1}, & \\
\mathsf{P_3SD_1} & := & \mathsf{P_3SL_4} \vee \mathsf{P_3S^*D_2} & = \mathsf{P_3SC_4} \wedge \mathsf{P_3D_1}, \\
\mathsf{P_3D_1} & = & \mathsf{P_3SL_4} \vee \mathsf{P_3S^*D_2} \vee \mathsf{P_3O_1}, & \\
\mathsf{P_3A_4}, & & & \\
\mathsf{P_3C_4} & = & \mathsf{P_3SL_4} \vee \mathsf{P_3A_4}. &
\end{array}
$$

That part of Fig. 2 which represents the lattice $\mathcal{L}_{3,T_3}$ of T_3-clones is given in Fig. 4. Table 1 below contains the results of this section with some additional information.

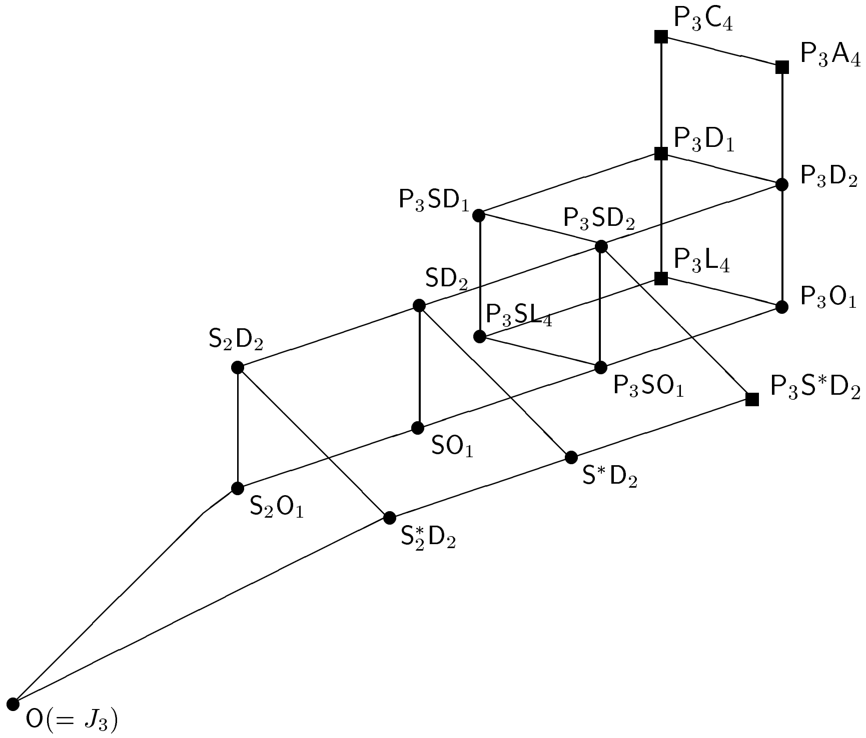

Figure 4: The lattice $\mathcal{L}_{3,T_3}$ of T_3-clones

H	$\boxed{\mathbf{R}}$: (right) H-clones F	$\boxed{\mathbf{L}}$: left H-clones F	$\boxed{\mathbf{RH}}$: right H-homogeneous clones F	$\boxed{\mathbf{LH}}$: left H-homogeneous clones F
$k=2$				
$\{c_0\}$	$F \leq \mathsf{C}_3$	$\mathcal{L}_2$ (all clones)	$F \leq \mathsf{C}_3$	$\mathcal{L}_2$ (all clones)
$\{c_1\}$	$F \leq \mathsf{C}_2$	$\mathcal{L}_2$ (all clones)	$F \leq \mathsf{C}_2$	$\mathcal{L}_2$ (all clones)
$\{c_0, c_1\}$	$F \leq \mathsf{C}_4$	$\mathcal{L}_2$ (all clones)	$F \leq \mathsf{C}_4$	$\mathcal{L}_2$ (all clones)
$\{e, c_0\}$	$F \leq \mathsf{C}_3$	$\mathcal{L}_2$ (all clones)	$F \leq \mathsf{C}_3$	$F \leq \mathsf{C}_3$
$\{e, c_1\}$	$F \leq \mathsf{C}_2$	$\mathcal{L}_2$ (all clones)	$F \leq \mathsf{C}_2$	$F \leq \mathsf{C}_2$
$\{e, c_0, c_1\}$	$F \leq \mathsf{C}_4$	$\mathcal{L}_2$ (all clones)	$F \leq \mathsf{C}_4$	$F \leq \mathsf{C}_4$
S_2 (Fig. 1)	$\mathcal{L}_{2,S_2}$	$\mathcal{L}_{2,S_2}$	$[\mathsf{O}_1, \mathsf{D}_3]_{\mathcal{L}_{2,S_2}}$	$[\mathsf{O}_1, \mathsf{D}_3]_{\mathcal{L}_{2,S_2}}$
T_2 (Fig. 1)	$\mathcal{L}_{2,T_2}$	$\mathcal{L}_{2,T_2}$	$[\mathsf{O}_1, \mathsf{D}_1]_{\mathcal{L}_{2,T_2}}$	$[\mathsf{O}_1, \mathsf{D}_1]_{\mathcal{L}_{2,T_2}}$
$k=3$				
S_3 (Fig. 2)	$\mathcal{L}_{3,S_3}$	$\mathcal{L}_{3,S_3}$	$[J_3, \mathsf{S}_2]_{\mathcal{L}_{3,S_3}}$	$[J_3, \mathsf{S}_2]_{\mathcal{L}_{3,S_3}}$
H_C (Fig. 2)	$[J_3, \mathsf{P}_3^{\alpha}]_{\mathcal{L}_{3,S_3}}$	$\mathcal{L}_{3,S_3}$	$[J_3, \mathsf{S}_2]_{\mathcal{L}_{3,S_3}}$	$[J_3, \mathsf{S}_2]_{\mathcal{L}_{3,S_3}}$
T_3 (Fig. 4, 2)	$\mathcal{L}_{3,T_3}$	$[J_3, \mathsf{Q}_{35}]_{\mathcal{L}_{3,S_3}}$	J_3	J_3

Table 1: H-clones for all $H \leq T_2$ and some $H \leq T_3$

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