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A Non-Standard Approach to the Euclidian Study of Plane Curves

Petru Dragoş
Faculty of Sciences, University of Oradea
str. Armatei Romana, No. 5, Oradea, Romania

Abstract. Starting from the field $\mathbf{R}$ of the real numbers as a model for the formal system of the theory of the real ordered closed fields, complete in the topological sense and Archimedian. A. Robinson [13] has found the extension $^*\mathbf{R}$ which is ordered, nonarchimedian, not complete and, moreover, the models of $^*\mathbf{R}$ are not isomorphic. We work on a model [6] based on a δ-incomplete ultrafilter, fixed and built as a consequence of the Lowenheim-Skolem theorem [5] which states that any infinite structure has an elementary extension. We use the three principles: of transfer, of idealization and of standardization, and the rules stated in [12].

Using relevant examples, we point out the efficiency of the infinitesimal methods resulting from the non-standard analysis in the study of geometry. The fundamental object is the point, which is supposed to be non-standard and finite [9]. The point defines a standard frame and we consider that through the point is passing a differentiable manifold, a in this case standard differentiable curve. The point defines also the euclidian invariant as the curvature (see [9] and [14]).
MSC: 03H05, 53A99
Keywords: non-standard curve, non-standard plane, circle of curvature, osculating circle, monad, infinitesimal element.

1. Non-standard differentiable curves

We propose here a geometrical approach to the infinitesimal.

Definition 1. *Let* $({}^*\mathcal{E}_2, {}^*\vec{E}, {}^*\varphi)$ *be the non-standard Euclidian plane,* $\mathcal{R} = (O; B = (\vec{i}, \vec{j}))$ *an orthonormal frame and* ${}^*h_{\mathcal{R}}\colon {}^*\mathcal{E}_2 \to {}^*\mathbf{R}^2$ *the standard system of Cartesian coordinates defined by R. Let* ${}^*I \subset {}^*\mathbf{R}$ *be a closed interval and* $c\colon {}^*I \to {}^*\mathcal{E}_2$ *an application. We consider the linear space* ${}^*\mathbf{R}^2$ *and the standard real, vector valued function* $h_{\mathcal{R}} \circ c \stackrel{not}{=} \vec{r}\colon {}^*I \to {}^*\mathbf{R}^2$. *For any* $t \in {}^*I$ *we have* $\vec{r}(t) = (x(t), y(t)) \in {}^*\mathbf{R}^2$, *where* $(x(t), y(t))$ *are the coordinates of the point* $c(t) \in \gamma = c({}^*I)$. *There consequently appear two standard real functions of real variable, written* $x, y\colon {}^*I \to {}^*\mathbf{R}$ *and defined by:*

$$\begin{cases} t \to x(t) \\ t \to y(t) \end{cases}, \forall t \in {}^*I \tag{1}$$

By this method, on the standard curves in ${}^*\mathcal{E}_2$ appear non-standard points.

We consider on ${}^*\mathbf{R}$ the topology of the monads ([13]) and on ${}^*I$ the induced topology. The vector function $\vec{r}$ is called of class C^k on ${}^*I$, $k = 0, 1, \ldots, \infty$ if its component functions x, y are of class C^k.

Definition 2. *Any application $c: {}^*I \to {}^*\mathcal{E}_2$, with $h_{\mathcal{R}} \circ c \overset{not}{=} \vec{r}$: ${}^*I \to {}^*\mathbf{R}^2$ of class C^k, for any orthonormal frame R, is called parametrised plane curve of class C^k, $k = 0, 1, \ldots, \infty$.*

2. The circle of curvature

We consider the curve γ and an ordinary point $M \in \gamma$. Let $M' \in \gamma \cap \mu(M)$ ($\mu(M)$ the monad of M, $\mu(M) = \cap U$ s.t. $M \in U$, U a neighborhood containing standard interval and external elements) and $T_M\gamma, T_{M'}\gamma$ the tangents at the curve γ in the points M and M'.

Proposition 1. *The angle $(T_M\gamma, T_{M'}\gamma)$ is a first order infinitesimal element.*

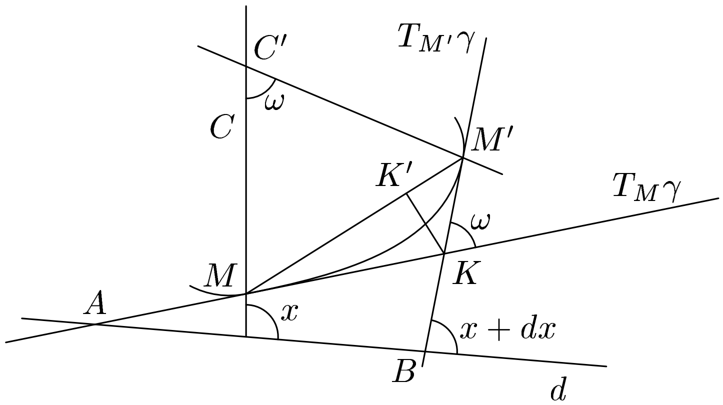

Figure 1

This angle may be regarded as the arithmetical average of all the measure of the angles of the tangents from the monad $\mu(M)$ (that are infinitesimal) and this justified the name "circle of curvature" that appears in Definition 4. There are satisfied the following relations: (these relations are seen intuitively in fig. 1, that is the monade of the point M)

$$1 < \frac{arcMM'}{|MM'|} < \frac{|MK| + |KM'|}{|MK'| + |K'M|} \xrightarrow[M \to M']{} 1 \Rightarrow \frac{arcMM'}{|MM'|} \xrightarrow[M \to M']{} 1$$

Definition 3. *The angle $(T_M\gamma, T_{M'}\gamma)$ is called the total curvature of the infinitesimal $arcMM'$.*

Remark 1. If the point M is singular then from the computation of the limit it follows that the angle is an infinitesimal of higher order, and its neglecting would be the conclusion that, locally the curve is identified with the tangent.

Remark 2. Let d be a fixed direction, non perpendicular to $T_M\gamma$, and we denote by x and $x+dx$ the angle that d makes with $T_M\gamma$ and $T_{M'}\gamma$ respectively. Writing $\omega = (T_M\gamma, T_{M'}\gamma)$, from the triangle ABK it results that $x + dx = x + \omega$, and then $\omega = dx$.

Proposition 2. *The angle formed by the chord $|MM'|$ with the tangent $T_M\gamma$ is an infinitesimal of the first order.*

Proof. Considering the triangle MKM', as ω is an infinitesimal, then the angle $M'MT_M\gamma$ is also infinitesimal of the same order.

Remark 3. Because ω and $arcMM'$ are infinitesimal of the same order, then the $\lim\limits_{M'\to M} \dfrac{\omega}{\text{length}(arcMM')}$ exists and this limit has the standard part equal to the usual curvature of the curve γ in M; we call it the *curvature in the point M*. If the curve γ is a circle, then $\text{length}(arcMM') = R\omega \Rightarrow \dfrac{\omega}{\text{length}(arcMM')} = \dfrac{1}{R} = c = constant$, independently of the fact M, M' are not in the same monad.

Definition 4. *Let γ be a planar curve, M be an ordinary point of γ and $\mu(M)$ be the monad of M. The circle which contains the points $M \in \gamma$ and $M' \in \mu(M)$ and satisfies that γ and Γ have the same tangent at M, is called the circle of curvature of γ at M.*

Proposition 3. *The circle of curvature at an ordinary point $M \in \gamma$ of the curve γ has the radius $R = \lim\limits_{M'\to M} \dfrac{\text{length}(arcMM')}{\omega}$, and its center C is situated on the normal at M of γ, with $\|MC\| = R$.*

Proof. Let C, C' be the centers of curvature for M and M' in the same monad and $M'A \perp MC$.

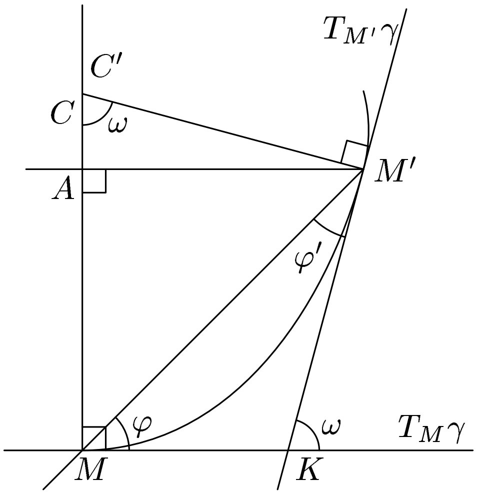

Figure 2

In the triangle AMM' we have $|AM'| \equiv |MM'| \cdot \sin A\hat{M}M'$. In the triangle $C'AM'$ we have $|AM'| \equiv |C'M'| \cdot \sin M\hat{C}'M'$. Because the quadrilateral $MKM'C'$ has its vertices on the circle we have:

$$M\hat{C}'M' = \omega \Rightarrow |MM'| \cdot \cos\varphi \equiv |C'M'| \cdot \sin\omega \Rightarrow$$
$$|C'M'| \equiv \frac{|MM'| \cdot \cos\varphi}{\sin\omega} \equiv \frac{|MM'|}{arcMM'} \cdot \frac{arcMM'}{\omega} \cdot \frac{\omega}{\sin\omega} \cdot \cos\varphi \Rightarrow$$
$$\lim_{M' \to M} \|C'M'\| = \lim_{M' \to M} \frac{\text{length}(arcMM')}{\omega} = R.$$

Remark 4. The limit with respect to $M' \to M$ of the intersection point between the normal passing through point $M \in \gamma$ and the normal of γ at $M' \in \gamma \cap \mu(M)$, is the center of the circle of curvature of γ at M.

3. The osculating circle

Let γ be a curve, $M \in \gamma$ an ordinary point of the curve and $T_M\gamma$ the tangent at M. We consider the circle $\mathcal{C}(M, M', T_M\gamma)$ passing through $M \in \gamma$ and $M' \in \gamma \cap \mu(M)$, and such that $T_M\gamma$ is a tangent at the point M.

Proposition 4. *The circle $\mathcal{C}(M, M', T_M\gamma)$ tends to the circle of curvature when $M' \to M$.*

Proof. The following relations are true:

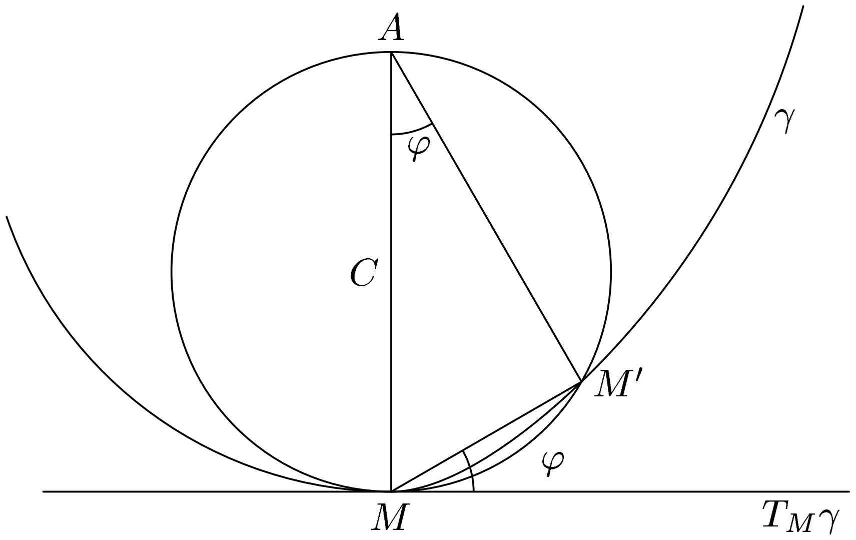

Figure 3

$$|MM'| \equiv |AM| \cdot \sin\varphi \equiv 2|MC| \cdot \sin\varphi \Rightarrow |MC| \equiv \frac{|MM'|}{2\sin\varphi} \Rightarrow$$

$$\lim_{M' \to M} |MC| = \lim_{M' \to M} \frac{|MM'|}{arcMM'} \cdot \frac{arcMM'}{2\varphi} \cdot \frac{\varphi}{\sin\varphi} = 1 \cdot R \cdot 1 = R.$$

Now we consider the circle, tangent to $T_M\gamma$ at the point M (M an ordinary point) and containing the points M', M'' of $(\gamma)\cap\mu(M)$, one on each side of M. This circle is uniquely determined by the three points M, M', M'' and it is called the *osculating circle* of the curve γ at the point M.

Theorem 1. *The osculating circle at a curve in an ordinary point is the standard part of the circle of curvature.*

Proof.

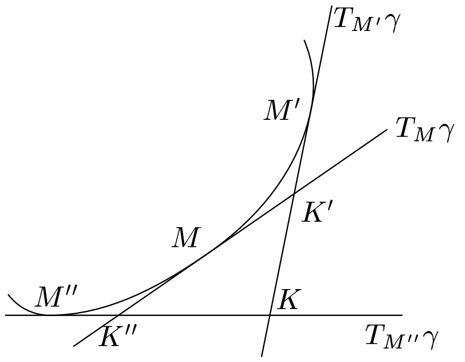

Figure 4

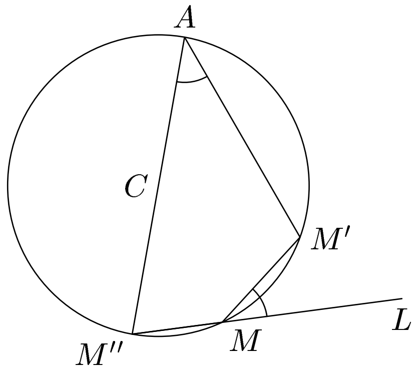

Figure 5

We have:

$$T_{M''}\hat{K}K' = T_{M'}\hat{K}''K' + K''\hat{K}'K.$$

From $M''\hat{M}K'' + K'\hat{M}M' = \Pi - M''\hat{M}M'$ results that $M''\hat{M}M'$ is the supplement of the half of the total curvature of the arc $M''MM'$ for any M situated between M' and M''. Consequently $L\hat{M}M' = \frac{1}{2}$ (the total curvature of the arc $M''MM'$). By using $|M''C| \equiv \dfrac{|M''M'|}{2\sin L\hat{M}M'}$ and passing to the limit in the ratio $\dfrac{M''M'}{\frac{1}{2}\,(\text{total curvature})}$ the conclusion of the theorem results.

4. Infinitesimal elements of order I, II, III associated to a curve in the monad of a point of the curve

Proposition 5. *The following entities*

(i) *the angle φ formed by the tangent $T_M\gamma$ and the chord MM', where $M \in \gamma$ is an ordinary point and $M' \in \gamma\cap\mu(M)$,*
(ii) $\|MK\|, \|M'K'\|$ *(called the length of the tangents at M and M'),*
(iii) *the projection of the chord $|MM'|$ on a direction not perpendicular to the tangent at M of th curve ($M' \in \gamma\cap\mu(M)$)*

are infinitesimal elements of order I.

Proof. We have:

$$2|MC|\cdot\sin\varphi \equiv |MM'| \Rightarrow \frac{\sin\varphi}{|MM'|} = \frac{1}{2|MC|} \Rightarrow$$

$$\frac{\varphi}{\text{length}(arcMM')} = \frac{\varphi}{\sin\varphi}\cdot\frac{\sin\varphi}{|MM'|}\cdot\frac{|MM'|}{\text{length}(arcMM')} \Rightarrow$$

$$\lim_{M'\to M}\frac{\varphi}{\text{length}(arcMM')} = \lim_{M'\to M}\frac{\varphi}{\sin\varphi}\cdot\frac{1}{2|MC|}\cdot\frac{|MM'|}{\text{length}(arcMM')} = 1\cdot\frac{1}{2R}\cdot 1 = \frac{1}{2R}.$$

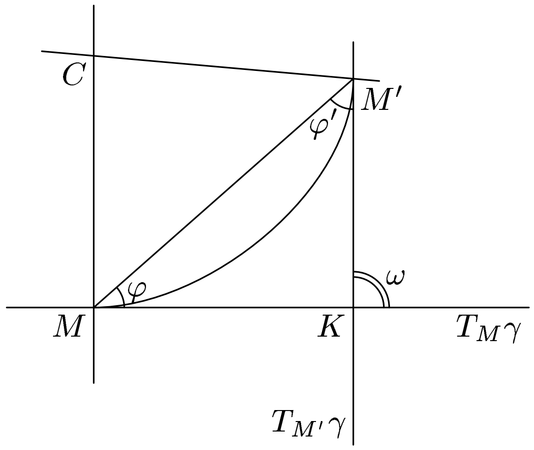

Figure 6

The other statements are analogously proved.

Proposition 6. *Suppose that M is an ordinary point of γ and let $T_M\gamma$ the tangent to γ at M. The following entities*

(i) *the distance from M' tp $P'' \in T_M$ satisfying $\|M'P''\| \sim \frac{s^2}{2R}$ (here M is an ordinary point and s is the length of the arc),*

(ii) *$\|MK\| - \|M'K\|$ (with the previous notations),*

(iii) *the distance between two points of $\mu(M)$ and situated on the curves γ and γ' where the tangent to γ' at M is $T_M\gamma$ too*

are infinitesimal elements of order II.

Proof. (i) We have

$$\lim_{M'\to M}\frac{|M'P'|}{|MM'|^2} = \lim_{M'\to M}\frac{|MM'|\cdot\sin P'\hat{M}M'}{|MM'|^2} = \lim_{M'\to M}\frac{\sin P'\hat{M}M'}{\|MM'\|} =$$

$$\lim_{M'\to M}\frac{\sin P'\hat{M}M'}{P'\hat{M}M'}\cdot\frac{P'\hat{M}M'}{|MM'|} = \frac{1}{2R}$$

$$\frac{|M'P''|}{\sin M'\hat{M}P''} = \frac{|MM'|}{\sin M'\hat{P}''M} \Rightarrow \frac{|M'P''|}{|MM'|} = \frac{\sin P''\hat{M}M'|}{\sin M'\hat{P}''M} \Rightarrow$$

$$\frac{\|M'P''\|}{\|MM'\|^2} = \frac{\sin P''\hat{M}M'}{P''\hat{M}M'}\cdot\frac{P''\hat{M}M'}{\|MM'\|}\cdot\frac{1}{\sin M'\hat{P}''M}$$

and by passing to the limit there results:

$$\|M'P''\| \sim \frac{(\text{length}(arcMM'))^2}{2R} \Rightarrow \|M'P''\| \sim \frac{s^2}{2R}.$$

The other statements are analogously proved.

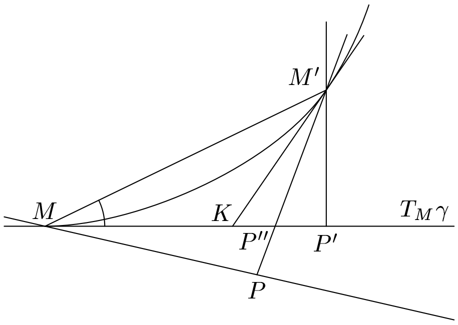

Figure 7

Proposition 7. *The following entities*

(i) *the difference* $(\|MK\| + \|M'K\|) - \|MM'\|$,
(ii) *the difference* $(\|MK\| + \|M'K\|) - \text{length}(arcMM')$,
(iii) $\text{length}(arcMM') - \|MM'\| \sim \frac{s^3}{24R}$,
(iv) $\|MC\| - \|M'C\|$ *(with the previous notation for* $C = C'$*)*,
(v) *the distance from* M' *to* $T_M\gamma$ *when* M' *is an inflexion point*

are infinitesimal elements of order III.

Proof. (iii) We have:

$$\left.\begin{array}{r}\|MM'\| = 2\rho \cdot \sin\dfrac{\omega}{2}\\ \text{length}(arcMM') = \rho \cdot \omega\end{array}\right\} \Rightarrow$$

$$\text{length}(arcMM') - \|MM'\| = 2\rho\left(\frac{\omega}{2} - \sin\frac{\omega}{2}\right) \sim 2 \cdot \frac{1}{6} \cdot \left(\frac{\omega}{3}\right)^3 \cdot \rho = \frac{\omega^3}{24} \cdot \rho \Rightarrow$$

$$\text{length}(arcMM') - \|MM'\| \sim \rho \cdot \frac{\omega^3}{24}, \text{but } \rho\omega = s \Rightarrow \text{length}(arcMM') - \|MM'\| \sim \frac{s^3}{24R}.$$

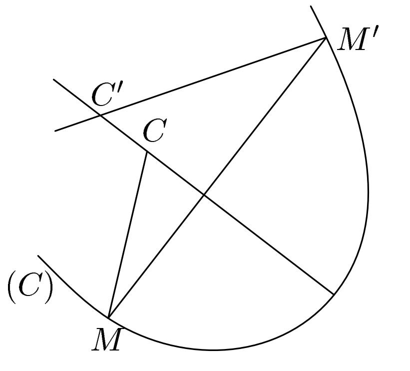

Figure 8

(iv) Let $M \in \gamma$ be an inflexion point. We find $|M'P| \equiv |MM'| \cdot \sin\varphi$, and because φ is an infinitesimal element of order I, it results that $\|M'P\|$ is an infinitesimal element of order III.

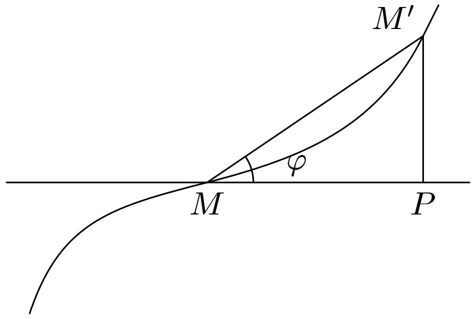

Figure 9

(v)
$$\frac{|MC|}{|M'C|} = \frac{\cos\varphi'}{\cos\varphi} \Rightarrow \frac{|MC| - |M'C|}{|M'C|} = \frac{\cos\varphi' - \cos\varphi}{\cos\varphi} \Rightarrow$$

$$|MC| - |M'C| \equiv \frac{|M'C|}{\cos\varphi} \cdot 2\sin\frac{\varphi' - \varphi}{2} \cdot \sin\frac{\varphi' + \varphi}{2},$$

$\sin\dfrac{\varphi' - \varphi}{2}$ is an infinitesimal element of order II, and then $\|MC\| - \|M'C\| \sim \varepsilon_3$ is an infinitesimal element of order III.

Remark 5.

(i) Suppose that the curve γ and the tangent $T_M\gamma$ have a contact of order I. Then

$$\frac{|M'P|}{|MP|^2} = \frac{|MM'|\sin\varphi}{|MM'|^2 \cdot \cos^2\varphi} \sim \frac{\sin\varphi}{|MM'|} = \lambda \neq 0 \Rightarrow \|M'P\| \sim \lambda\|MP\|^2.$$

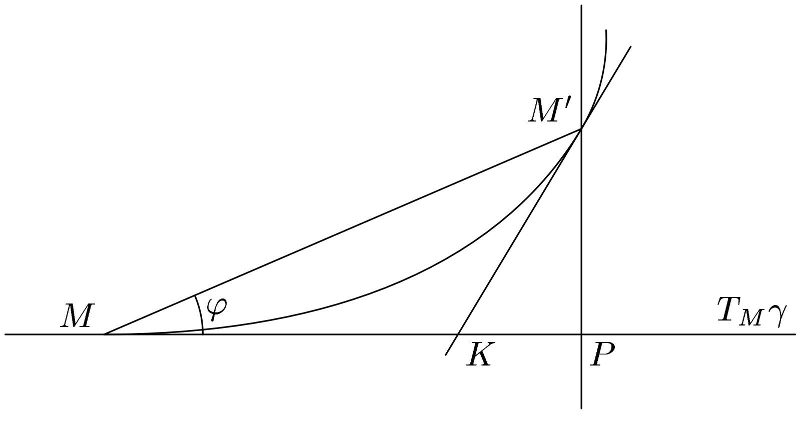

Figure 10

(ii) If $\|M'P\| \sim \lambda \cdot \|MP\|^{n+1}$, it is said that the curve γ and the tangent have a *contact of order* n and we obtain by computations that $\omega \sim (n+1)P\hat{M}M'$. In this case

$\|MK\|, \|KM'\|$ have not the same order, not by passing to the limit (like in the case that M is ordinary), and $\|M'P\|$ is an infinitesimal element of order $n+1$.

We remark that in the monad of μ if $M' \to M$ the angle φ is naturally stationary and $\varphi' \sim n\varphi$, and if $M \to M'$ the roles of φ and φ' are interchanged.

5. The relation between the circle of curvature and the osculating circle. Applications.

Proposition 8. *Let $\mathcal{C}(C,R)$ be the osculating circle of the curve γ at the ordinary point M of γ and $N \in \mu(M) \cap T_M\gamma, NC \cap (\gamma) = M'$. Then the relation $R \sim \frac{\|MN\|^2}{2\|NM'\|}$ is true.*

Proof. There are satisfied:
$\|MN\|^2 = \|NM''\| \cdot (\|NM''\| + 2R)$;
$\|NM''\|$ is an infinitesimal element of order II;
$\|NM''\|^2$ is an infinitesimal element of order IV and we neglect it.

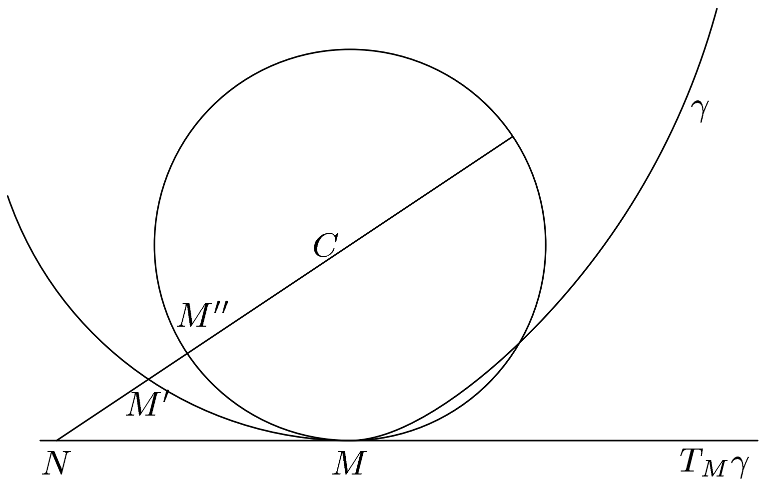

Figure 11

Thus

$$\|MN\|^2 \sim \|NM\| \cdot R \Rightarrow R \sim \frac{\|MN\|^2}{2\|NM''\|}, \text{ because } \|NM''\| \sim \|NM'\| \Rightarrow R \sim \frac{\|MN\|^2}{2\|NM'\|}.$$

Proposition 9. *Let $P \in \mu(M) \cap \mathcal{C}(C,R)$ with $PN \perp T_M\gamma$, and let $M' \in PN \cap \gamma$. Then the relation $R \sim \dfrac{\|MN\|^2}{2\|M'N\|}$ holds.*

Proof. We have

$$\|PQ\|^2 = \|MQ\| \cdot \|QS\| = \|MQ\| \cdot (2R - \|MQ\|) \Rightarrow \|NM\|^2 = \|NP\| \cdot (2R - \|NP\|).$$

As $\|NP\|$ is an infinitesimal element of order II then $\|NP\|^2$ is of order IV and we neglect it and $R \sim \dfrac{\|MN\|^2}{2\|NP\|}$. As $\|NP\| \sim \|NM'\|$ then $R \sim \dfrac{\|MN\|^2}{2\|M'N\|}$.

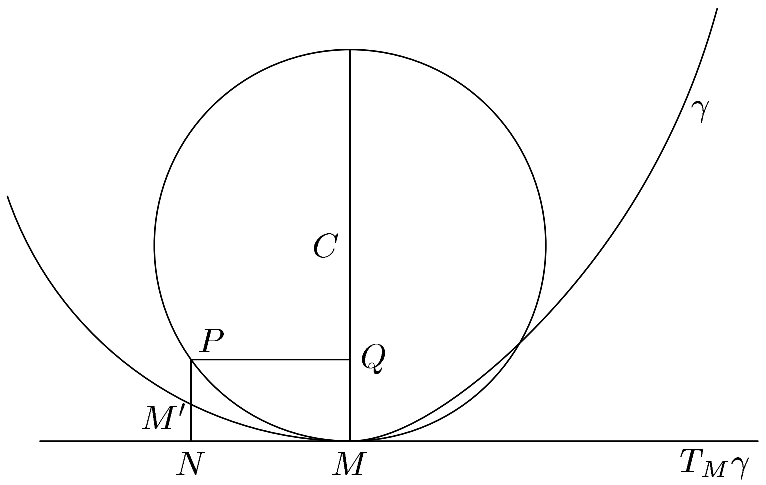

Figure 12

Proposition 10. *Let $\mathcal{C}(C,R)$ be the circle of curvature which has the same tangent with the curve γ at the ordinary point M and containing also a point of $\mu(M) \cap (\gamma)$. If $m \in \mu(M) \cap \mathcal{C}(C,R)$, then $mP \perp T_M\gamma$ and $M \in mP \cap \gamma$. If ρ is the radius of curvature of the curve at M' and R is the radius of curvature of the curve at M, then*

$$\|mM'\| \sim \frac{\|MP\|^2}{2} \cdot \left(\frac{1}{\rho} - \frac{1}{R}\right).$$

Proof. $\rho \sim \dfrac{\|MP\|^2}{2\|mP\|}, R \sim \dfrac{\|MP\|^2}{2\|M'P\|} \Rightarrow \|mM'\| \sim \dfrac{\|MP\|^2}{2} \cdot \left(\dfrac{1}{\rho} - \dfrac{1}{R}\right)$.

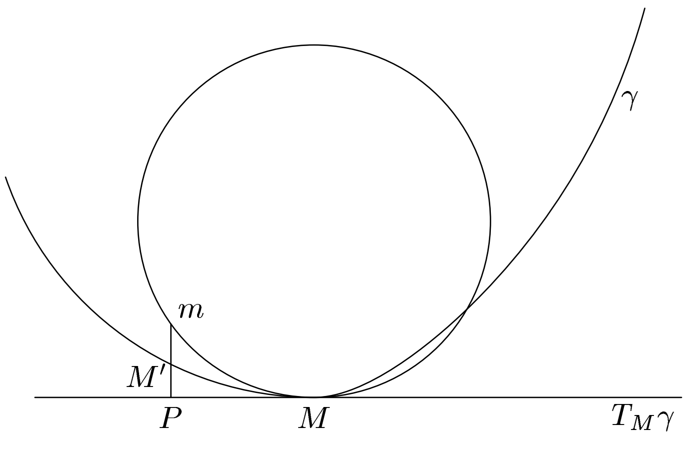

Figure 13

Remark 6.

a) If $\rho \neq R$, the circle of the curvature is not the osculating circle (satisfying weaker conditions), and $\|mM'\|$ is an infinitesimal element of order II.

b) If $\rho = R$, the circle of curvature is an osculating circle, and the distance $\|mM'\|$ is an infinitesimal element of order at least III.

Corollary. *Between all the circles which are tangent to the curve γ at M, the osculating circle at $M \in \gamma$ is the closest to the curve γ, because the contact determines uniquely the circle. Because $\left(\frac{1}{\rho} - \frac{1}{R}\right) \cdot \frac{\|MP\|^2}{2}$ changes the sign at M, the osculating circle crosses the curve at the point M.*

Application. Study of the curve (γ), given by $r(\varphi) = \dfrac{3\sin\varphi\cos\varphi}{\cos^3\varphi + \sin^3\varphi}$, in double points, the point O is the point of intersection.

(1) If $\varphi \sim \frac{\pi}{2}$ the $\cos\varphi = \sin\left(\varphi + \frac{\pi}{2}\right)$, $\sin\varphi \sim 1$ and $\cos^3\varphi + \sin^3\varphi \sim -1$ so that $r(\varphi) \sim 3\left(\frac{\pi}{2} + \varphi\right)$ and by passing to the limit in cartesian coordinates, we get $x \sim \frac{y^2}{3}$. For $\varphi \to -\frac{\pi}{2}$, the curve lays on the parabola $x \sim \frac{y^2}{3}$.

(2) $\varphi \to \pi$, the curve lays on the parabola $y \sim \frac{x^2}{3}$. We analogously prove for $\varphi \to 0$, that $y \sim \frac{x^2}{3}$, and for $\varphi \to \frac{\pi}{2}$, that $x \sim \frac{y^2}{3}$.

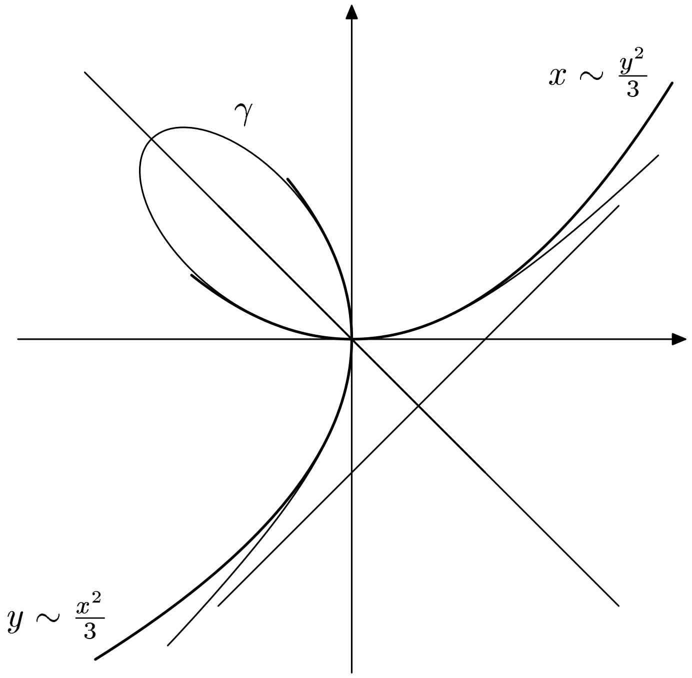

Figure 14

The efficiency of the non-standard method is obvious in the case of this application too.

6. Cover curves

a) Cover curves

Let Γ_a be a family of curves. Denote by Γ_{k-1}, Γ_k two infinitely close curves of the family and let M_k be their intersection. Then the geometrical locus of the standard points m_k, corresponding to the monads of M_k, is a curve Γ called the *envelope* of the curves Γ_a. This curve is tangent to each of the curves of the family Γ_a at the point m_k. This result is obvious if we remark that $M_{k-1}M_k$ tends to the tangent to Γ_k at m_k.

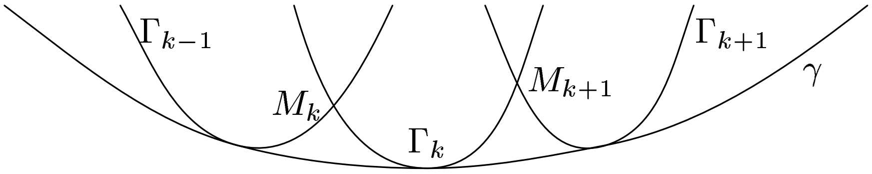

Figure 15

b) The evolute and the involute of the curve γ

(1) A curve γ, which is the envelope of the normals of a given curve Γ, is called the evolute of the curve Γ. The standard point $m_k \in \gamma$ of the normal at the point M_k to Γ is the center of curvature μ, at the point M_k, being the intersection of the normal at M_k with the infinitely close normal.

(2) The curve Γ is called the involute of the curve γ.

Proposition 11. *The length of an arc $\mu'\mu$ of the evolute is equal to the difference of the radii of curvatures $\|\mu N\|$, $\|\mu' N'\|$ of the involute.*

Proof. Let $|N\mu|$, $|N'\mu'|$ be two normals of the curve γ in the monad of the same point. We have $\|N'\mu'\| - \|N\mu\| = \|N'P| - \|NP| + (\|P\mu'\| + \|P\mu\|)$, and because $|N'P\| - \|NP\|$ is infinitesimal of order III it may by neglected. So $\|N'\mu'\| - \|N\mu\|$ may be replaced with $\|P\mu'\| + \|P\mu\|$.

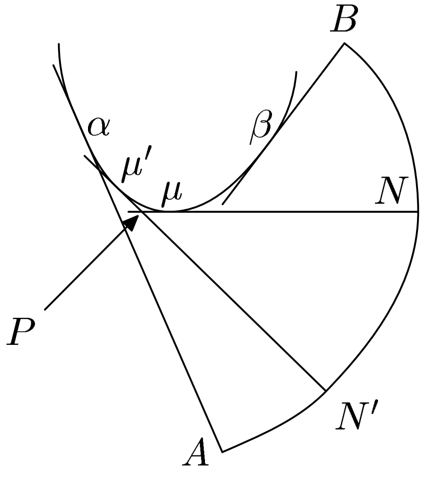

Figure 16

But length$(arc\mu\mu') - (\|P\mu'\| + \|P\mu\|)$ is an infinitesimal of order III, so that $\|N'\mu'\| - \|N\mu\| \sim$ length$(arc\mu\mu')$. It results $d(N\mu) = d(\mu)$, where d is the differentiation operator. Thus the variation of the normal $(N\mu)$ to the curve (γ) is equivalent to the variation of the arc of the evolute Γ. By integrating from A to N we obtain $\|A\alpha\| - \|N\mu\| =$ length$(arc\alpha\mu)$, i.e., the statement of the proposition.

7. The variation of arcs of curves delimited by two fixed tangents

Let the tangents to the curve (A) at the points A and A' belong to the same monad (fig. 17).

Proposition 12. *The growth of the arc of curve (B) delimited by two tangents at points in the monad of a point A of the curve (A) is equal to the product of the total curvature of the arc AA' on the curve (A) and the length bB of the normal segment, where b is the point of intersection of the normal with the normal at A to the curve (A).*

Proof. We consider the tangents $T_A, T_{A'}$ to the curve (A) at points from the same monad. We write $|Aa$ for the normal to (A), $|Bb$ for the normal to (B), $|Cc$ for the normal to (C), a for the center of curvature, SB for the tangent to (B), SC for the tangent to (C).

The tangents T_A, $T_{A'}$ determine on the curve (B) the points B and B'. Let $\{M\} = T_A \cap T_{A'}$ and we denote like in the figure the angles $\alpha, \omega, \beta, \gamma, \varphi$, where $\omega, (BB')$, and φ are infinitesimal of order I.

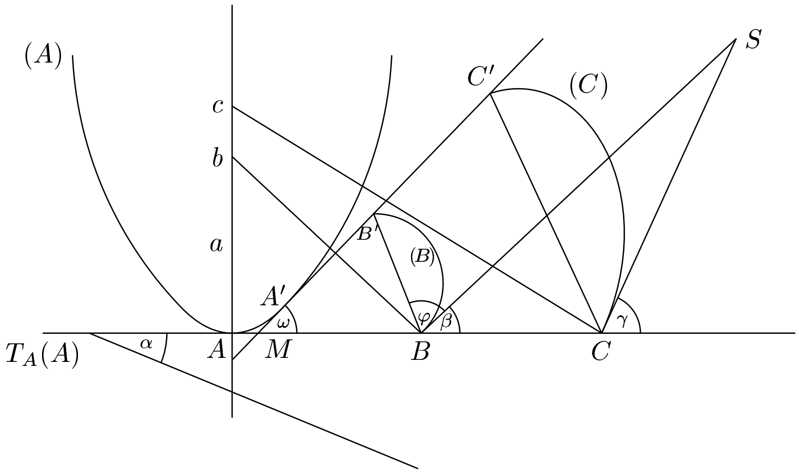

Figure 17

From $\Delta MBB'$ it follows $\dfrac{\|BB'\|}{\sin\omega} = \dfrac{\|BM\|}{\sin(M\hat{B}'B)}$. Consequently $|BB'|\sin\hat{B}' \equiv |BM|\sin\omega$. But length$(arcBB') - \|BB'\|$ takes out to be an infinitesimal of order I, $\sim \varepsilon_3$, and $\|BM\| = \|AB\| - \|AM| \sim \|AB\| - \varepsilon_1$; we have also $\omega + \hat{B}' = \beta + \varphi \Rightarrow \hat{B}' = \hat{B} + \varphi - \omega$, which implies
$\sin\hat{B}' \sim \sin\beta + \varepsilon_1' \Rightarrow$ (length$(arcBB') - \varepsilon_3) \cdot (\sin\beta + \varepsilon_1') \sim (\|AB\| - \varepsilon_1)\sin\omega \Rightarrow$ length$(arcBB') \sim \dfrac{\|AB\|}{\sin\beta}\omega$.

Denoting by α the angle of the tangent at A with a fixed direction and using the previous result $\omega = d\alpha \Rightarrow \text{length}(arcBB') \sim \dfrac{\|AB\|}{\sin\beta} d\alpha$.

Let $d(B)$ be the variation of the arc of the curve (B). It follows $d(B) = \dfrac{\|AB\|}{\sin\beta} d\alpha$. From the right triangle ABb we find

$$\|AB\| = \|bB\| \sin A\hat{b}B \Rightarrow \|AB\| = \|bB\| \sin\beta \Rightarrow \frac{\|AB\|}{\sin\beta} = \|bB| \Rightarrow d(B) = \|bB\| d\alpha. \ \square$$

Applications. By considering another curve (C), we get the relations

(1) $\dfrac{d(B)}{d(C)} = \dfrac{|bB|}{|cC|}$;

(2) $\dfrac{d(B)}{d(C)} = \dfrac{|AB| \sin\gamma}{|AC| \sin\beta}$;

(3) $\dfrac{d(B)}{d(C)} = \dfrac{|AB|}{|AC|} \cdot \dfrac{|SB|}{|SC|}$, since $\dfrac{|SB|}{|SC|} = \dfrac{\sin\gamma}{\sin\beta}$.

8. The variation of a line segment

With the notations and the notions from the previous paragraph there takes place the next statement.

Proposition 13. *The variation $d(|AB|)$ of the segment $|AB|$ is given by the product between the length $\|ab\|$ of the segment $|ab|$ and the total curvature $d\alpha$ of the $arcAA'$.*

Proof. We have $d|AB| = \|A'B'\| - \|AB\|$, where $|A'B'|$ is in the same monad with $|AB|$. By projecting the polygonal line $MB'B$ on the tangent $|AB|$, we get

$\|MB'\| \cos\omega = \|MB\| + \|BB'\| \cos B'\hat{B}C \Rightarrow$

$(\|MA'\| + \|A'B'\|) \cos\omega \sim \|AB\| - \|AM\| + (\text{length}(arcBB') - \varepsilon_3) \cos(\varphi + \beta)$,

where $\|BB'\|$ and φ are infinitesimal of order I. Also

$\|MA'\| - \|A'B'\| \sim \|AB\| - \|AM\| + (\text{length}(arcBB') - \varepsilon_3)(\cos\varphi\cos\beta - \sin\varphi\sin\beta) \Rightarrow$

$\|MA'\| + \|A'B'\| \sim \|AB\| - \|AM\| + \text{length}(arcBB') \cos\beta \Rightarrow$

$\|A'B'\| - \|AB\| \sim \text{length}(arcBB') \cos\beta - (\|MA'\| + \|AM\|) \Rightarrow$

$\|A'B'\| - \|AB\| \sim \text{length}(arcBB') \cos\beta - \text{length}(arcAA') \Rightarrow d|AB| = d(b)\cos\beta - d(a)$.

Let $|Aa|$ be the radius of curvature at A and because $d(B) = \|bB\| d\alpha$ it follows $\|Aa\| = \dfrac{d(A)}{d\alpha} \Rightarrow d(A) = \|Aa\| \cdot d\alpha \Rightarrow d|AB| = \|bB\| \cos\beta d\alpha - \|Aa\| - d\alpha$. From the right triangle ABb we have $|bB| \cos\beta = |Ab| \Rightarrow d|AB| = \|Ab\| d\alpha - \|Aa\| d\alpha \Rightarrow d|AB| = \|ab\| d\alpha$.

Remark 8. For $\beta = 90°$ there results that (B) is the involute of the curve (A), b is identified with A, $\|ab\| = \|aA\| = R$ (the radius of curvature) whence $d|AB| = R d\alpha$, but

$Rd\alpha = d(A) \Rightarrow d|AB| = d(A)\cdot(\|AB\|)$, just the radius of the curvature at B of the curve (B), where $d(A)$ is the variation of the arc on the evolute. We get the difference of two radii of curvature of the involute is equal with the corresponding arc of the evolute.

Remark 9. If $|AB|$ has constant length, then $d|AB| = 0$ implies $\|ab\| = 0$, i.e. the normal at B to the curve (A) passes through the center of curvature of the curve (A) at A.

Remark 10. If $T_A \cap (C) = \{C\}$, then $d|AC| = d(C)\cos\gamma - d(A)$. Consequently $d|BC| = d|AC| - d|AB| = d(C)\cos\gamma - d(B)\cos\beta$.

Because $d|AB| = \|ab\|d\alpha$ and $d|AC| = \|ac\|d\alpha$, we obtain $d|BC| = \|bc\|d\alpha$. For $BC = c$-constant, we get $d|BC| = 0 \Rightarrow \|bc\| = 0$, i.e. the normals to curves that are descibed by the ends of a constant segment meet on the normal to the envelope of the segment $|BC|$. From this are obtained the involutes of the evolute of the given curve. The curves (B) and (C) are called parallel.

Remark 11. If $M \in |BC|$ and $|BM| \equiv k|BC|, k = \mathit{constant}$, then $d|BM| = k\|BC\| \Rightarrow |bm| \equiv k|bc| \Rightarrow |bm|/|bc| = |BM|/|BC|$. Consequently, if the point M divides the segment $|BC|$ in a constant ratio, the normal to the envelope of these segments meets the normal to the curves $(B),(M),(C)$ preserving the ratio; the special case when M is in the middle of $|BC|$ is very important.

9. The variation of an angle

Let A be a point of the curve (A) and $|AM|,|AN|$ the tangents to the curves (M) and (N) respectively. If m and n are the centers of curvature of $(M),(N)$ respectively and if $\alpha = M\hat{A}N$ then there is true the next proposition

Proposition 14. *The variation of the angle $M\hat{A}N$ is given by the relation*
$d\alpha = d(A)\left(\frac{1}{\|Am\|} - \frac{1}{\|An\|}\right)$.

Proof. Let A, A' be points on (A) from the same monad. From A' we draw the tangents $|A'M|, |A'N'|$ and we write $d\alpha = \alpha - \alpha' = B\hat{T}A - B\hat{T}'A \Rightarrow d\alpha = d\theta - d\theta'$. But

$$\left.\begin{array}{l} d(A) = \|Am\|d\theta \\ d(A) = \|An\|d\theta \end{array}\right\} \text{ imply } d\alpha = d(A)\left(\frac{1}{\|Am\|} - \frac{1}{\|An\|}\right). \qquad \square$$

Remark 12. Let (L) be the point where the bisector of the angle $M\hat{A}N$ meets the envelope (L) of the bisector and l be the point where the normal at (L) intersects the normal to (A) at A.

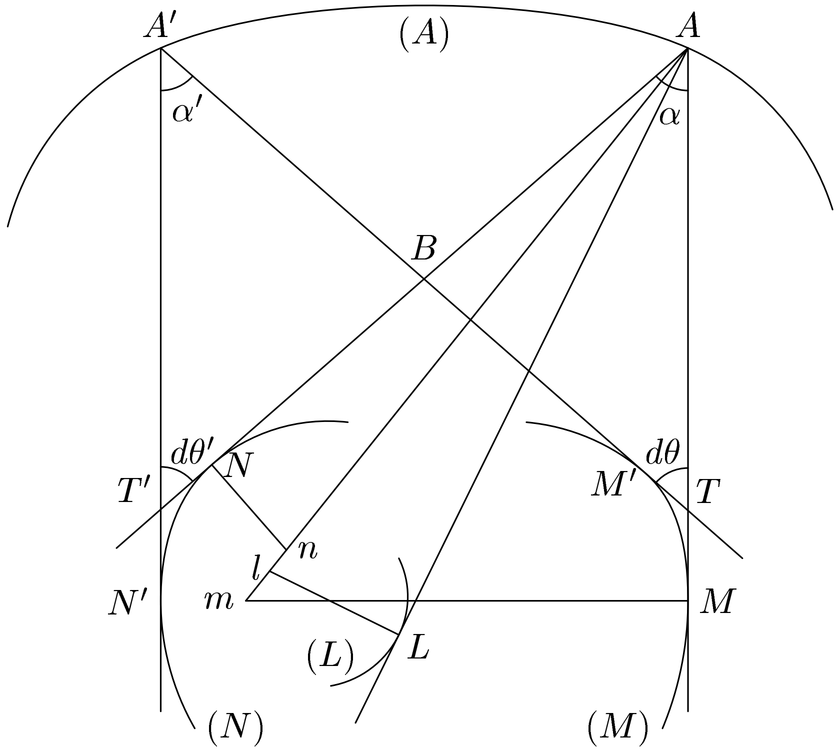

Figure 18

We obtain $d(M\hat{A}L) = d(A)\left(\dfrac{1}{\|Am\|} - \dfrac{1}{\|An\|}\right) : d(N\hat{A}L) = d(A)\left(\dfrac{1}{\|Al\|} - \dfrac{1}{\|An\|}\right)$. Consequently $\dfrac{2}{\|Al\|} = \dfrac{1}{\|Am\|} + \dfrac{1}{\|An\|}$. Thus the point l is the harmonic conjugate of A relative to m and n.

Remark 13. If the bisector $|AL|$ is also normal to the curve (A), the point l is the center of curvature to (A) at A, and m, n divide harmonically the radius of curvature $|Aa|$.

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