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# Spherical Hypersurfaces with 2-Type Gauss Map

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**Abstract.** B. Y. Chen and P. Piccinni proved in [7] that spherical hypersurfaces have 1-type Gauss map if and only if they have constant mean curvature and constant scalar curvature. Moreover, they proved that there exists a codimension 2 compact spherical Einstein submanifold which has 2-type Gauss map and which also has constant mean curvature and constant length of second fundamental form. In contrast, we prove in this article that every compact spherical hypersurface with 2-type Gauss map is non-Einsteinian and has non-constant mean curvature and non-constant length of second fundamental form.

## 1. Introduction

Let $M$ be a compact Riemannian $n$-manifold and $\Delta$ denote the Laplacian operator of $M$. Let $\lambda_j$ denote the $j$-th nonzero eigenvalue of the Laplacian $\Delta$ on $M$. For a smooth map $x : M \to \mathbb{E}^m$ of $M$ in $\mathbb{E}^m$, let

$$x = x_0 + \sum_{t=p}^{q} x_t, \quad \Delta x_t = \lambda_t x_t, \quad \lambda_t \neq 0 \tag{1.1}$$

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denote the spectral decomposition of $x$, where $x_0$ is a constant map. The set

$$T(x) = \{t \in N_0 : x_t \neq 0\} \tag{1.2}$$

is called the order of the map. The smallest element $p$ in $T(x)$ is called the *lower order* of $x$ and the supremum $q$ of $T(x)$ is called the *upper order* of $x$. The map is said to be *of finite type* if the upper order $q$ is finite; and it is said to be of infinite type if the upper order $q$ is infinite. Moreover, the map is said to be *of $k$-type* if $T(x)$ contains exactly $k$ elements. In particular, when $x$ is an isometric immersion, the submanifold is said to be of finite type (respectively, of $k$-type) if the immersion is of finite type (respectively, of $k$-type).

Submanifolds of finite type were first introduced by B. Y. Chen in the late seventies (cf. [4]). Since then the theory of submanifolds of finite type has been investigated by many geometers and many interesting results have been obtained (see [6] for a recent survey).

For 2-type hypersurfaces of a hypersphere we mention the following (cf. Theorem 9.3 of [6]).

**Theorem A.** *Let $M$ be a compact hypersurface of a unit hypersphere $S^{n+1} \subset \mathbb{E}^{n+2}$. Then $M$ has nonzero constant mean curvature and constant scalar curvature if and only if $M$ is mass-symmetric and of 2-type, unless $M$ is a small hypersurface of $S^{n+1}$.*

Since isoparametric hypersurfaces of a hypersphere have constant mean curvature and constant scalar curvature, Theorem A implies that there exist abundant examples of 2-type spherical hypersurfaces with constant mean curvature and constant scalar curvature. Hence, by Gauss' equation, there exist abundant examples of 2-type spherical hypersurfaces with constant length of second fundamental form.

For an isometric immersion $x : M \to E^m$ of an oriented $n$-dimensional Riemannian manifold into Euclidean m-space, the Gauss map $\nu : M \to G(m-n, m)$ of $x$ is a smooth map which carries a point $u \in M$ into the oriented $(m-n)$-plane in $\mathbb{E}^m$ which is obtained from the parallel translation of the normal space of $M$ at $u$ in $\mathbb{E}^m$, where $G(m-n, m)$ denotes the Grassmannian manifold consisting of $(m-n)$-plane in $\mathbb{E}^m$. Since $G(m-n, m)$ is canonically embedded in $\bigwedge^{m-n} \mathbb{E}^m = \mathbb{E}^N$, $N = \binom{m}{m-n}$, the notion of type of the Gauss map is naturally defined.

In [7] B. Y. Chen and P. Piccinni had made a general study on submanifolds with finite type Gauss map in the frame work of finite type theory. Since then submanifolds with finite type Gauss map have been studied by many geometers (cf. for instance, [1], [6], [8]). In particular, Chen and Piccinni characterized and classified spherical hypersurfaces with 1-type Gauss map (see also [3]).

**Theorem B.** *A compact spherical hypersurface in $S^{n+1} \subset \mathbb{E}^{n+2}$ has 1-type Gauss map if and only if it has constant mean curvature and constant scalar curvature.*

**Theorem C.** *Let $M$ be a compact hypersurface of a hypersphere $S^{n+1}$ of $\mathbb{E}^{n+2}$. Then the Gauss map $\nu$ of $M$ is of 1-type if and only if $M$ is one of the following submanifolds:*

(1) *a mass-symmetric 2-type submanifold of $\mathbb{E}^{n+2}$;*

(2) *a small hypersphere in* $S^{n+1}$;
(3) *a minimal hypersurface of* $S^{n+1}$ *with constant scalar curvature.*

In [7] Chen and Piccinni also studied spherical submanifolds with 2-type Gauss map. In particular, they showed that there exists an isometric immersion of $S^2(\sqrt{3})$ in $S^4(1) \subset \mathbb{E}^5$ which has 2-type Gauss map and which also has constant mean curvature and constant length of second fundamental form.

In contrast, we prove in this paper that compact spherical hypersurfaces with 2-type Gauss map never have constant mean curvature or constant length of second fundamental form. Moreover, we prove that such hypersurfaces cannot be Einsteinian.

More precisely, we prove the following.

**Theorem 1.** *Let $M$ be a compact hypersurface of a hypersphere $S^{n+1} \subset \mathbb{E}^{n+2}$. If $M$ has 2-type Gauss map, then $M$ has non-constant mean curvature and non-constant length of second fundamental form.*

**Theorem 2.** *Let $M$ be a compact $n$-dimensional Einstein manifold (with constant scalar curvature $\tau$). Then $M$ admits no isometric immersion in $S^{n+1} \subset \mathbb{E}^{n+2}$ with 2-type Gauss map.*

**Theorem 3.** *Let $M$ be a compact oriented $n$-dimensional Riemannian manifold of class $C^\omega$ with constant scalar curvature. Then $M$ admits no isometric immersion in $S^{n+1} \subset \mathbb{E}^{n+2}$ with 2-type Gauss map unless $M$ is homeomorphic to an $n$-sphere.*

A submanifold $M$ of a Euclidean $m$-space is said to have *biharmonic Gauss map* if its Gauss map $\nu$ satisfies $\Delta^2\nu = 0$ identically. The following result shows the non-existence of spherical hypersurfaces with biharmonic Gauss map.

**Theorem 4.** *There do not exist hypersurfaces of $S^{n+1} \subset \mathbb{E}^{n+2}$ with biharmonic Gauss map. In particular, there do not exist hypersurfaces in $S^{n+1} \subset \mathbb{E}^{n+2}$ with harmonic Gauss map.*

## 2. Basic formulas for Gauss map

For simplicity, we denote by $S^{n+1}$ the unit hypersphere of $\mathbb{E}^{n+2}$ centered at the origin. Let $x : M \to S^{n+1} \subset \mathbb{E}^{n+2}$ be an isometric immersion of a compact $n$-dimensional Riemannian manifold into $S^{n+1}$. Without loss of generality, we may assume $M$ is orientable, since otherwise we may replace $M$ by its two-fold covering. Let $H$ and $H'$ denote the mean curvature vector fields of $M$ in $\mathbb{E}^{n+2}$ and in $S^{n+1}$, respectively. We have $H = H' - x$.

Since the position vector $x$ of $M$ in $\mathbb{E}^{n+2}$ is normal to $M$ as well as to $S^{n+1}$, we may choose an oriented orthonormal local frame field $\{e_1, e_2, \dots, e_n; e_{n+1}, e_{n+2}\}$ on $M$ such that $e_1, \dots, e_n$ are tangent to $M$ which give the orientation of $M$, $e_{n+1} = x$, and $e_{n+2}$ is normal to $M$ in $S^{n+1}$. In this way, we have $H' = \beta e_{n+2}$, where $\beta^2 = \langle H', H'\rangle$.

The Gauss map $\nu : M \to G(2, n+2) \subset \bigwedge^2 \mathbb{E}^{n+2} = \mathbb{E}^N$, $N = \binom{n+2}{2}$, is given by

$$\nu(u) = (e_{n+1} \wedge e_{n+2})(u), \quad u \in M. \tag{2.1}$$

Let $\nabla$ and $\widetilde{\nabla}$ be the Levi-Civita connections on $M$ and $\mathbb{E}^{n+2}$ and $A, h$ and $D$ the Weingarten map, the second fundamental form and the normal connection of $M$ in $\mathbb{E}^{n+2}$, respectively.

Since $x$ is the position vector of $M$ in $\mathbb{E}^{n+2}$, $Dx = 0$. Moreover, we have (cf. [5])

$$\widetilde{\nabla}_i e_{n+1} = \widetilde{\nabla}_i x = e_i, \quad \widetilde{\nabla}_i e_{n+2} = -A_{n+2} e_i, \tag{2.2}$$

$$\Delta e_{n+1} = \Delta x = -nH, \tag{2.3}$$

$$\sum_i (\nabla_i A_{n+2}) e_i = \sum_i \left(\nabla_i (A_{n+2} e_i) - A_{n+2}(\nabla_i e_i)\right) = n \nabla \beta, \tag{2.4}$$

where $\widetilde{\nabla}_i = \widetilde{\nabla}_{e_i}, \nabla_i = \nabla_{e_i}$ and $A_r = A_{e_r}, i = 1, 2, \ldots, n$.

By Weingarten's formulas, we get

$$\begin{aligned} \Delta e_{n+2} &= -\sum_i \widetilde{\nabla}_i \widetilde{\nabla}_i e_{n+2} + \sum \widetilde{\nabla}_{\nabla_i e_i} e_{n+2} \\ &= \sum_i (\nabla_i A_{n+1}) e_i + \sum_i h(A_{n+2} e_i, e_i) \\ &= n \nabla \beta + \|A_{n+2}\|^2 e_{n+2} - n\beta e_{n+1}, \end{aligned} \tag{2.5}$$

where $\|A_r\|^2 = \operatorname{trace} A_r^2$.

Regarding $\nu$ as an $\mathbb{E}^N$-valued function on $M$ with $N = \binom{n+2}{n}$, we obtain from (2.2)–(2.5) that

$$\Delta \nu = \|h\|^2 \nu + n \left(e_{n+1} \wedge \nabla \beta\right), \tag{2.6}$$

where $\|h\|^2 = n + \|A_{n+2}\|^2$.

We also need the general formula of $\Delta^2 \nu$ which can be computed as follows.

$$\begin{aligned} \Delta^2 \nu =& (\Delta \|h\|^2) \nu + \|h\|^2 \Delta \nu - 2 \sum_i e_i(\|h\|^2) e_i(\nu) + n \left(\Delta e_{n+1} \wedge \nabla \beta\right) \\ &+ n \left(e_{n+1} \wedge \Delta(\nabla \beta)\right) - 2n \sum_i \widetilde{\nabla}_i e_{n+1} \wedge \widetilde{\nabla}_i (\nabla \beta). \end{aligned} \tag{2.7}$$

For the terms of (2.7), we have the following.

$$\begin{aligned} -2 \sum_i e_i(\|h\|^2) e_i(\nu) &= -2 \sum_i e_i(\|h\|^2) [\widetilde{\nabla}_i e_{n+1} \wedge e_{n+2} + e_{n+1} \wedge \widetilde{\nabla}_i e_{n+2}] \\ &= -2 \nabla \|h\|^2 \wedge e_{n+2} - 2 A_{n+2} \left(\nabla \|h\|^2\right) \wedge e_{n+1}], \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
\Delta(\nabla\beta) &= \Delta^T(\nabla\beta) + \sum_i A_{h(e_i,\nabla\beta)}e_i - nD_{\nabla\beta}H - 2\sum_i h(e_i,\nabla_i(\nabla\beta)) \\
&= \Delta^T(\nabla\beta) + \left(A^2_{n+2}(\nabla\beta) + \nabla\beta\right) - n|\nabla\beta|^2 e_{n+2} \\
&\quad - 2\sum_i \left\{\langle A_{n+2}e_i, \nabla_i(\nabla\beta)\rangle\, e_{n+2} - \langle e_i, \nabla_i(\nabla\beta)\rangle\, e_{n+1}\right\},
\end{aligned} \tag{2.9}
$$

where $\Delta^T = -\sum_i[\nabla_i\nabla_i - \nabla_{\nabla_i e_i}]$ and

$$
\begin{aligned}
\sum_i \langle A_{n+2}e_i, \nabla_i(\nabla\beta)\rangle &= \sum_i [e_i\langle A_{n+2}(\nabla\beta), e_i\rangle - \langle\nabla_i(A_{n+2}e_i), \nabla\beta\rangle] \\
&= \sum_i [\langle\nabla_i(A_{n+2}(\nabla\beta)), e_i\rangle - \langle(\nabla_i A_{n+2})e_i, \nabla\beta\rangle] \\
&= \sum_i \langle\nabla_i(A_{n+2}(\nabla\beta)), e_i\rangle - n\,|\nabla\beta|^2.
\end{aligned} \tag{2.10}
$$

Combining (2.9)–(2.10), we obtain

$$
\begin{aligned}
ne_{n+1}\wedge\Delta(\nabla\beta) = &-n\,\left(\Delta^T(\nabla\beta) + A^2_{n+2}(\nabla\beta) + \nabla\beta\right)\wedge e_{n+1} \\
&- n\left(2\sum_i \langle\nabla_i(A_{n+2}(\nabla\beta)), e_i\rangle - n|\nabla\beta|^2\right)\nu,
\end{aligned} \tag{2.11}
$$

$$
\begin{aligned}
-2n\sum_i \widetilde{\nabla}_i e_{n+1}\wedge\widetilde{\nabla}_i(\nabla\beta) &= -2n\sum_i e_i\wedge[\nabla_i(\nabla\beta) + h(e_i,\nabla\beta)] \\
&= -2n\sum_i e_i\wedge h(e_i,\nabla\beta) \\
&= -2n\left(A_{n+2}(\nabla\beta)\wedge e_{n+2} - \nabla\beta\wedge e_{n+1}\right),
\end{aligned} \tag{2.12}
$$

where we use the fact that the term $\sum_i e_i\wedge\nabla_i(\nabla\beta)$ vanishes identically.

Substituting (2.3), (2.6), (2.8), (2.11) and (2.12) into (2.7), we obtain

$$
\begin{aligned}
\Delta^2\nu = &-\left\{n(\|h\|^2 + n - 1)\nabla\beta + 2A_{n+2}(\nabla\|h\|^2)\right. \\
&\qquad \left. + n\,\Delta^T(\nabla\beta) + nA^2_{n+2}(\nabla\beta)\right\}\wedge e_{n+1} \\
&+ \left\{-2\nabla\|h\|^2 + n^2\beta\nabla\beta - 2nA_{n+2}(\nabla\beta)\right\}\wedge e_{n+2} \\
&+ \left(\Delta\|h\|^2 - 2n\sum_i \langle\nabla_i(A_{n+2}(\nabla\beta)), e_i\rangle + \|h\|^4 + n^2|\nabla\beta|^2\right)\nu.
\end{aligned} \tag{2.13}
$$

## 3. Proofs of Theorems 1, 2, 3 and 4

Let $M$ be a compact hypersurface of a hypersphere $S^{n+1}$ in $\mathbb{E}^{n+2}$. Then the Gauss map $\nu$ of $M$ is mass-symmetric in the unit hypersphere $S^{N-1}$ (cf. [7]).

Assume the Gauss map $\nu$ of $M$ is of 2-type. Then the Gauss map admits the following spectral decomposition:

$$\nu = \nu_p + \nu_q, \quad \Delta\nu_p = \lambda_p\nu_p, \quad \Delta\nu_q = \lambda_q\nu_q, \quad \lambda_p < \lambda_q, \tag{3.1}$$

where $p, q$ are the lower and the upper orders of $\nu$, respectively. Therefore, the Gauss map satisfies (cf. [6]):

$$\Delta^2\nu = a\Delta\nu + b\nu, \quad a = \lambda_p + \lambda_q, \quad b = -\lambda_p\lambda_q. \tag{3.2}$$

Thus, applying (2.6) and (2.13) and (3.2), we obtain

$$n\,\Delta^T(\nabla\beta) + 2A_{n+2}(\nabla\|h\|^2) + n\,A_{n+2}^2(\nabla\beta) + n(\|h\|^2 + n - 1)\nabla\beta = na\nabla\beta, \tag{3.3}$$

$$2nA_{n+2}(\nabla\beta) + 2\nabla\|h\|^2 - n^2\beta\nabla\beta = 0, \tag{3.4}$$

$$\Delta\|h\|^2 - 2n\sum_i \langle\nabla_i(A_{n+2}(\nabla\beta)), e_i\rangle + \|h\|^4 + n^2|\nabla\beta|^2 = a\|h\|^2 + b. \tag{3.5}$$

Equality (3.4) is equivalent to

$$A_{n+2}(\nabla\beta) = -\frac{1}{n}\nabla\|h\|^2 + \frac{n}{2}\beta\nabla\beta. \tag{3.6}$$

Substituting (3.6) into (3.5), we find

$$\begin{aligned} -2n\sum_i \langle\nabla_i(A_{n+2}(\nabla\beta)), e_i\rangle &= \sum_i \left(2\left\langle\nabla_i(\nabla\|h\|^2), e_i\right\rangle - n^2\left\langle\nabla_i(\beta\nabla\beta), e_i\right\rangle\right) \\ &= -2\Delta\|h\|^2 + \frac{n^2}{2}\Delta\beta^2. \end{aligned} \tag{3.7}$$

Hence (3.5) becomes

$$\Delta\|h\|^2 - \frac{n^2}{2}\Delta\beta^2 - n^2|\nabla\beta|^2 - (\|h\|^4 - a\|h\|^2 - b) = 0. \tag{3.8}$$

By using the identity

$$\Delta\beta^2 = 2\beta\Delta\beta - 2|\nabla\beta|^2, \tag{3.9}$$

(3.8) reduces to

$$\Delta\|h\|^2 - n^2\beta\Delta\beta - (\|h\|^4 - a\|h\|^2 - b) = 0. \tag{3.10}$$

Similarly, applying (3.3) and (3.6), we have

$$\begin{aligned} &\Delta^T(\nabla\beta) + \frac{1}{n}A_{n+2}(\nabla\|h\|^2) - \frac{1}{2}\beta\nabla\|h\|^2 \\ &\quad + \left(\frac{n^2}{4}\beta^2 + \|h\|^2 + n - 1 - a\right)\nabla\beta = 0. \end{aligned} \tag{3.11}$$

If we take the inner product of (3.11) with $\nabla\beta$, the first term of (3.11) becomes

$$\begin{aligned} \left\langle \Delta^T(\nabla\beta), \nabla\beta \right\rangle &= \left\langle \sum_i (-\nabla_i\nabla_i + \nabla_{\nabla_i e_i})(\nabla\beta), \nabla\beta \right\rangle \\ &= \frac{1}{2}\Delta(|\nabla\beta|^2) + |\text{Hess }\beta|^2, \end{aligned} \tag{3.12}$$

where $|\text{Hess }\beta|^2 = \sum_i \langle \nabla_i(\nabla\beta), \nabla_i(\nabla\beta)\rangle$.

For a given smooth function $f$ on $M$ we have the following formula of Bochner-Lichnerowicz:

$$-\frac{1}{2}\Delta(|\nabla f|^2) = |\text{Hess } f|^2 - |\Delta f|^2 + \text{Ric}(\nabla f, \nabla f), \tag{3.13}$$

where Ric denotes the Ricci curvature of $M$ (cf. [2]). Make use of (3.12) in (3.11), we obtain

$$\left\langle \Delta^T(\nabla\beta), \nabla\beta \right\rangle = |\Delta\beta|^2 - \text{Ric}(\nabla\beta, \nabla\beta). \tag{3.14}$$

On the other hand, by applying Gauss' equation, we have

$$\begin{aligned} \text{Ric}(\nabla\beta, \nabla\beta) &= -\sum_i |h(\nabla\beta, e_i)|^2 + n\langle H, h(\nabla\beta, \nabla\beta)\rangle \\ &= -|A_{n+2}\nabla\beta|^2 - |\nabla\beta|^2 + n\langle A_{n+2}(\nabla\beta), \beta\nabla\beta\rangle + n|\nabla\beta|^2 \\ &= (n-1)|\nabla\beta|^2 - |A_{n+2}(\nabla\beta)|^2 + n\langle A_{n+2}(\nabla\beta), \beta\nabla\beta\rangle . \end{aligned} \tag{3.15}$$

Consequently, we have

$$\begin{aligned} \left\langle \Delta^T(\nabla\beta), \nabla\beta \right\rangle = |\Delta\beta|^2 - (n-1)|\nabla\beta|^2 + |A_{n+2}(\nabla\beta)|^2 \\ - n\langle A_{n+2}(\nabla\beta), \beta\nabla\beta\rangle . \end{aligned} \tag{3.16}$$

For the other terms of (3.11), with the help of (3.6) we have

$$\frac{1}{n}\langle A_{n+2}(\nabla\|h\|^2), \nabla\beta\rangle = \frac{n}{2}\langle A_{n+2}(\nabla\beta), \beta\nabla\beta\rangle - |A_{n+2}(\nabla\beta)|^2, \tag{3.17}$$

$$-\frac{1}{2}\left\langle \beta\nabla\|h\|^2, \nabla\beta\right\rangle = -\frac{n^2}{4}\beta^2|\nabla\beta|^2 + \frac{n}{2}\left\langle A_{n+2}(\nabla\beta), \beta\nabla\beta\right\rangle . \tag{3.18}$$

Combining (3.11) and (3.16)–(3.18), we find that the inner product of (3.11) with $\nabla\beta$ yields

$$|\Delta\beta|^2 + (\|h\|^2 - a)|\nabla\beta|^2 = 0. \tag{3.19}$$

We summarize the main formulas from above as the following.

**Lemma 1.** *Let $M$ be a compact hypersurface of a hypersphere $S^{n+1} \subset \mathbb{E}^{n+2}$. If the Gauss map $\nu$ of $M$ is of 2-type associated with $\lambda_p, \lambda_q$, then*

(1) $A_{n+2}(\nabla\beta) = -\frac{1}{n}\nabla\|h\|^2 + \frac{n}{2}\beta\nabla\beta$;

(2) $\Delta\|h\|^2 - n^2\beta\Delta\beta - \|h\|^4 + a\|h\|^2 + b = 0$;

(3) $|\Delta\beta|^2 + (\|h\|^2 - a)|\nabla\beta|^2 = 0$,

*where $a = \lambda_p + \lambda_q$, $b = -\lambda_p\lambda_q$.*

Theorem 1 can be proved as follows. If $M$ has constant mean curvature, then $\beta$ is constant. Therefore, by (1) of Lemma 1, it follows that $\nabla\|h\| = 0$ identically. Thus, $M$ has constant length of second fundamental form. Since the scalar curvature $\tau$ of $M$ is given by

$$\tau = n^2|H|^2 - \|h\|^2 \tag{3.20}$$

by Gauss equation, $M$ has constant scalar curvature, too. This is impossible according to Theorem B. Therefore, $M$ has non-constant mean curvature.

Now, suppose $M$ has constant length of second fundamental form. Then $\|h\|$ is constant. Since $M$ is assumed to have 2-type Gauss map, $M$ has non-constant mean curvature. Thus, $\beta^2$ has a positive maximum at some point $y \in M$. At $y$, we have $\nabla\beta = 0$. Thus, from (3) of Lemma 1 we obtain $\Delta\beta = 0$ at $y$. Since $\Delta\beta^2 = 2\beta\Delta\beta - 2|\nabla\beta|^2$, these yield $(\beta\Delta\beta)(y) = 0$. Substituting these into (2) of Lemma 1, we find

$$\|h\|^4 - a\|h\|^2 - b = \left(\|h\|^2 - \lambda_q\right)\left(\|h\|^2 - \lambda_p\right) = 0 \tag{3.21}$$

at $y$. Because $\|h\|$ is constant, (3.21) must hold at every point on $M$. Consequently, from (3.21), we have

$$\|h\|^2 = \lambda_q \quad \text{or} \quad \|h\|^2 = \lambda_p. \tag{3.22}$$

Now, from (2) of Lemma 1 and (3.21), we have $\Delta\beta = 0$ identically by continuity. Thus, by applying (3) of Lemma 1, we obtain $\left(\|h\|^2 - \lambda_p - \lambda_q\right)|\nabla\beta|^2 = 0$ which implies that $\beta$ is constant according to (3.22). Therefore, we obtain a contradiction. Hence $M$ cannot have constant length of second fundamental form, too. This completes the proof of Theorem 1.

Now, we prove Theorem 2. Let $M$ be a compact $n$-dimensional Einstein manifold with constant scalar curvature $\tau$. Assume that $x : M \to S^{n+1} \subset \mathbb{E}^{n+2}$ is an isometric immersion with 2-type Gauss map. Then, by Theorem 1, $M$ has non-constant mean curvature $\beta$. Let $U = \{u \in M : \nabla\beta^2 \neq 0 \text{ at } u\}$. From (3.20) we obtain $\nabla\|h\|^2 = 2n^2\beta\nabla\beta$. Substituting this into (1) of Lemma 1, we obtain $A_{n+2}(\nabla\beta) = -\frac{3n}{2}\beta\nabla\beta$. This implies that $\nabla\beta$ is an eigenvector of $A_{n+2}$ with eigenvalue $-\frac{3n}{2}\beta$ on $U$. Without loss of generality, we may let $e_1$ in the direction of $\nabla\beta$ on $U$. Denote by $\kappa_2, \dots, \kappa_n$ the other eigenvalue of $A_{n+2}$. Then we have on $U$

$$\kappa_2 + \cdots + \kappa_n = \frac{5n}{2}\beta. \tag{3.23}$$

Combining these with Gauss' equation, we obtain

$$Ric(e_1) = n - 1 - \frac{15n^2}{4}\beta^2. \tag{3.24}$$

Since $M$ is assumed to be Einsteinian with constant scalar curvature, (3.24) implies that $M$ has constant mean curvature by continuity. This is a contradiction. Thus, Theorem 2 is proved.

Now, we prove Theorem 3. Let $M$ be a compact oriented $n$-dimensional Riemannian manifold of class $C^\omega$ with constant scalar curvature. Then, from (3.20), we obtain

$$\Delta\|h\|^2 = n^2\Delta\beta^2. \tag{3.25}$$

Substituting (3.9) and (3.25) into (2) of Lemma 1, we obtain

$$n^2\Delta\beta^2 - 2n^2|\nabla\beta|^2 = 2\left(\|h\|^2 - \lambda_q\right)\left(\|h\|^2 - \lambda_p\right). \tag{3.26}$$

Assume that the function $\|h\|^2$ attends its absolute maximum at a point $u_1 \in M$. Then $\nabla\|h\|^2(u_1) = 0$. Since from (3.20) we have $\nabla\|h\|^2 = 2n^2\beta\nabla\beta$, thus from (3) of Lemma 1, we have $\Delta\beta(u_1) = 0$. Hence, by (3.9) and (3.25) we also have $\Delta\beta^2(u_1) = \Delta\|h\|^2(u_1) = 0$. Substituting these into (3.26), we obtain $\max(\|h\|^2) = \lambda_q$ or $\max(\|h\|^2) = \lambda_p$.

Similarly, if $\|h\|^2$ attends its absolute minimum at a point $u_2 \in M$, then by (3.26), we obtain $\min(\|h\|^2) = \lambda_q$ or $\min(\|h\|^2) = \lambda_p$. Because $M$ is assumed to have 2-type Gauss map, $M$ has non-constant length of second fundamental form according to Theorem 1. Thus, we must have

$$\max(\|h\|^2) = \|h\|^2(u_1) = \lambda_q \quad \text{and} \quad \min(\|h\|^2) = \|h\|^2(u_2) = \lambda_p. \tag{3.27}$$

Because $M$ is of class $C^\omega$ and $(\|h\|^2)$ is non-constant, the maximum and the minimum points of $(\|h\|^2)$ are isolated maximum and minimum points.

Now, we claim that the function $(\|h\|^2)$ has no other critical points in $M$. This can be seen as follows. If $u_3$ is another critical point of $(\|h\|^2)$, then we have $\nabla\beta(u_3) = \Delta\beta^2(u_3) = 0$ according to (3) of Lemma 1 and (3.9). This implies that $\|h\|^2(u_3) = \lambda_q$ or $\|h\|^2(u_3) = \lambda_p$ by virtue of (3.26). This is clearly impossible, since the maximum and the minimum points of $(\|h\|^2)$ are isolated. Consequently, $M$ admits a smooth functions with only two critical points. Hence, by a result in Morse's theory [9], we conclude that $M$ is homeomorphic to an $n$-sphere.

Finally, we prove Theorem 4. Assume that $M$ is a hypersurface of $S^{n+1} \subset \mathbb{E}^{n+2}$ with biharmonic Gauss map, then $\Delta^2 \nu = 0$. By applying arguments similar to the proof of Lemma 1, we obtain

$$A_{n+2}(\nabla\beta) = -\frac{1}{n}\nabla\|h\|^2 + \frac{n}{2}\beta\nabla\beta, \tag{3.28}$$

$$\Delta\|h\|^2 - n^2\beta\Delta\beta - \|h\|^4 = 0, \tag{3.29}$$

$$|\Delta\beta|^2 + \|h\|^2|\nabla\beta|^2 = 0. \tag{3.30}$$

Since $\|h\|^2 = |A_{n+2}|^2 + n \neq 0$, (3.30) implies that $\beta$ is a constant. Hence, by (3.28), $\|h\|^2$ is also a constant. Therefore, by (3.29) we obtain $h = 0$ which is a contradiction, since $\|h\|^2 \geq n$. This proves Theorem 4. $\square$

**Remark 1.** From (3.9), Hopf's lemma, and (2) of Lemma 1, we obtain immediately the following.

**Proposition 5.** *Let $M$ be a compact hypersurface of a hypersphere $S^{n+1} \subset \mathbb{E}^{n+2}$. If the Gauss map $\nu$ of $M$ is of 2-type associated with $\lambda_p, \lambda_q$, then*

$$n^2 \int_M |\nabla\beta|^2 dV = \int_M \left(\lambda_q - \|h\|^2\right)\left(\|h\|^2 - \lambda_p\right) dV. \tag{3.31}$$

**Remark 2.** The results obtained in this article support the following conjecture of the first author.

**Conjecture.** *The only hypersurfaces in a hypersphere $S^{n+1} \subset \mathbb{E}^{n+2}$ with finite type Gauss map are hypersurfaces with 1-type Gauss map.*

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