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Generators of Order two for $\mathrm{B_n}$ and three of its Double Covers

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1. Introduction

In this paper the minimum number of involutions required to generate either the Hyperoctrahederal Group B_n or three of its seven double covers are calculated. In addition explicit multiplications for the three particular double covers are introduced, along with specific generators, of order two, for each of the groups.

Definition 1.1. *Denote by $i(G)$ the minimum number of involutions required to generate a group G.*

As a starting point we observe that if $i(G) = 1$ then G is cyclic and if $i(G) = 2$ then G is dihedral. Hence for any group of order greater than two that is not isomorphic to a dihedral group we have $i(G) \geq 3$.

To considering our particular double covers of B_n we require the double cover of S_n that lifts a transposition of S_n to an element of order two. This has the following standard presentation. Generators are $z, s_1, s_2, \ldots, s_{n-1}$ with relations

$$\begin{aligned} z^2 &= 1, \quad zs_i = s_i z,\ s_i^2 = 1 \quad \text{for} \quad 1 \leq i \leq n-1, \\ (s_j s_{j+1})^3 &= 1 \quad \text{for} \quad 1 \leq j \leq n-2, \\ s_k s_h &= z s_h s_k \quad \text{for} \quad \mid h-k \mid > 1 \quad \text{and} \quad 1 \leq h,k \leq n-1. \end{aligned}$$

We will consider B_n as the permutation wreath product $C_2 pwr S_n$ (see chapter 4 [**3**]). The Schur multiplier of B_n, for $n > 3$, is $C_2 \times C_2 \times C_2$ [**2**] and it will be convenient to introduce here a notation that encompasses all seven double covers. We follow the usual indexing of the triples $(\zeta, \lambda, \mu) = (\pm 1, \pm 1, \pm 1)$ to distinguish between the seven double covers. Then we have the following presentation for the double covers of B_n, dependent on a particular triple.

Generators are $b_1, b_2, \ldots, b_n$ with the relations,

$$\begin{array}{rcl} b_i^2 & = & 1 \quad \text{for} \quad 1 \leq i \leq n, \quad (b_j b_{j+1})^3 = 1 \quad \text{for} \quad 1 \leq j \leq n-2, \\ (b_i b_j)^2 & = & \zeta \quad \text{for} \quad 1 \leq i, j \leq n-1 \quad \text{and} \quad \mid i - j \mid > 1, \\ (b_i b_n)^2 & = & \lambda \quad \text{for} \quad 1 \leq i \leq n-2, \quad \text{and} \quad (b_{n-1} b_n)^4 = \mu. \end{array}$$

2. $\mathbf{i(B_n{}^{(1,1,1)})}$

We shall refer to the elements of $C_2 pwr S_n$ as wreath permutations.

Definition 2.1. *For* $\underline{c} = (c_1, c_2, \ldots, c_n) \in (C_2)^n$ *and* $\pi \in S_n$, *let*

$$(\underline{c})_\pi = \underline{c}_\pi = (c_{\pi^{-1}(1)}, c_{\pi^{-1}(2)}, \ldots, c_{\pi^{-1}(n)}).$$

We now have elements of the form

$$(\underline{c}\,;\pi), \quad (\underline{d}\,;\kappa) \quad \text{for} \quad \underline{c}, \underline{d} \in (C_2)^n \quad \text{and} \quad \pi, \kappa \in S_n,$$

with multiplication being defined as $(\underline{c}\,;\pi) \cdot (\underline{d}\,;\kappa) = (\underline{c} \cdot \underline{d}_\pi\,;\pi\kappa)$. The identity element of $C_2 pwr S_n$ is then $(\underline{1}\,;1)$, where $\underline{1} = (1, 1, \ldots, 1)$ and an inverse for a typical wreath permutation $(\underline{c}\,;\pi)$ is $(\underline{c}\,;\pi)^{-1} = ((\underline{c}^{-1})_{\pi^{-1}}\,;\pi^{-1})$, where $(\underline{c}^{-1})_{\pi^{-1}} = (\underline{c}_{\pi^{-1}})^{-1}$. Associativity follows by noting that, for $\pi,\ \kappa \in S_n$,

$$(\underline{c}_\pi)_\kappa \quad = \quad \underline{c}_{\kappa\pi} \qquad \text{and} \qquad (\underline{c} \cdot \underline{d})_\pi \quad = \quad \underline{c}_\pi \cdot \underline{d}_\pi.$$

Lemma 2.2. *A wreath permutation* $(\underline{c}\,;\pi) \neq (\underline{1}\,;1) \in \mathrm{B}_n$ *is an involution if and only if* $\pi^2 = 1$ *and* $\underline{c}_\pi = \underline{c}$.

Proof. Let $\pi^2 = 1$ and $\underline{c}_\pi = \underline{c}$, then $(\underline{c}\,;\pi)^2 = (\underline{c} \cdot \underline{c}_\pi\,;\pi^2) = (\underline{1}\,;1)$. □

As we shall be considering the decomposition of the permutation π it will be convenient at this point to introduce some notation.

Definition 2.3. *The following product of* $\delta + 1$ *disjoint transpositions*

$$(\alpha, \beta)(\alpha+1, \beta-1)(\alpha+2, \beta-2)\ldots(\alpha+\delta, \beta-\delta),$$

will be denoted by $T(\alpha, \alpha+\delta, \alpha+\beta)$. *We will also use* $[\quad]_I$ *to denote the integer part of a number.*

Theorem 2.4. *For* $n > 3$, *the following three involutions generate* B_n.

$$\begin{array}{rcl} a & = & ((-1, 1, \ldots, 1);\ T(2, [n/2]_I, n+1)), \\ b & = & ((-1, 1, \ldots, 1);\ T(2, [(n+1)/2]_I, n+2)) \quad \textit{and} \\ c & = & (\underline{1}\,; (1, 2)). \end{array}$$

Moreover, $\quad i(\mathrm{B}_n) = 3$.

Proof. The permutation $T(2,[n/2]_I,n+1)$ fixes $(-1,1,\ldots,1)$ thus

$$ab=(\underline{1}\,;(2,3,\ldots,n)),\quad \text{hence}\quad abc=(\underline{1}\,;(1,2,\ldots,n))\ ,$$

and it is clear that $\langle abc,c\rangle$ is isomorphic to S_n. Thus $\langle abc,c\rangle$ will generate all the generators b_i, for $1\le i\le n-1$, from our presentation for B_n. It only remains to show that a, b and c also generate b_n.

So consider the following wreath permutations, which are clearly elements of $\langle abc,c\rangle$, $f=(\underline{1};T(2,[(n+1)/2]_I,n+2))$ and $g=(\underline{1};(1,n))$. Then we may generate $(gbf)^2bf=((1,\ldots,1,-1);1)=b_n$. □

3. $\mathbf{i}(\mathbf{B_n}^{(\mathbf{1},\mathbf{1},-\mathbf{1})})$

To investigate involutions in ${\mathrm{B}_n}^{(1,1,-1)}$, $n\ge 4$, we consider an alternative presentation for ${\mathrm{B}_n}^{(\zeta,\lambda,\mu)}$ [**1**]. Here we have generators $w_1,w_2,\ldots,w_n$ and $r_1,r_2,\ldots,r_{n-1}$ with relations,

$$\begin{aligned}
r_i^2 &= 1 \quad\text{for}\quad 1\le i\le n-1,\quad w_i^2 = 1 \quad\text{for}\quad 1\le i\le n,\\
(r_ir_j)^2 &= \zeta \quad\text{for}\quad 1\le i,j\le n-1 \quad\text{and}\quad |\,i-j\,|>1,\\
(r_ir_{i+1})^3 &= 1 \quad\text{for}\quad 1\le i\le n-2,\\
w_iw_j &= \mu w_jw_i \quad\text{for}\quad 1\le i,j\le n,\quad i\ne j,\\
w_ir_i &= \lambda r_iw_{i+1} \quad\text{for}\quad 1\le i\le n-1,\\
w_jr_i &= \lambda r_iw_j \quad\text{for}\quad 1\le i,j\le n,\quad\text{and}\quad j\ne i,i+1.
\end{aligned}$$

Now consider the subgroup of ${\mathrm{B}_n}^{(1,1,-1)}$, for $n>3$, which is generated by w_i, for $1\le i\le n$. This group, which we denote $\mathcal{N}$, has the following presentation.

$$\langle w_1,\ldots,w_n : w_i^2=1,\quad 1\le i\le n\quad\text{and}\quad (w_iw_j)^2=-1\quad\text{for}\quad i\ne j\rangle$$

The fact that $\mathcal{N}$ is normal in ${\mathrm{B}_n}^{(1,1,-1)}$ follows readily from the two relations

$$\begin{aligned}
r_i^{-1}w_ir_i &= w_{i+1} \quad\text{for}\quad 1\le i\le n-1,\\
r_i^{-1}w_jr_i &= w_j \quad\text{for}\quad 1\le i,j\le n \quad\text{and}\quad j\ne i,i+1.
\end{aligned}$$

Also the subgroup generated by $r_1,r_2,\ldots,r_{n-1}$ is just S_n (e.g. let r_i be the transposition $(i,i+1)$). Hence as $\mathcal{N}\bigcap S_n=1$, ${\mathrm{B}_n}^{(1,1,-1)}$, for $n\ge 4$, is the semi-direct product of $\mathcal{N}$ by S_n.

Definition 3.1. *Let π be any permutation in S_n and $\mathcal{N}$ the subgroup of $\mathrm{B}_n^{(1,1,-1)}$ as defined above. Also let θ_π be the automorphism of $\mathcal{N}$, where for*

$$\underline{w}=w_iw_j\ldots w_k\in\mathcal{N},\quad \theta_\pi(\underline{w})=w_{\pi^{-1}(i)}w_{\pi^{-1}(j)}\ldots w_{\pi^{-1}(k)}.$$

Then we denote by ϕ_π the homomorphism from S_n to the automorphism group of $\mathcal{N}$ defined by

$$\phi_\pi(\underline{w})=\phi(\theta_\pi(\underline{w}))=w_{\pi^{-1}(i)}w_{\pi^{-1}(j)}\ldots w_{\pi^{-1}(k)}.$$

As we are multiplying permutations in S_n from the left, the fact that ϕ_π is a homomorphism follows as $\phi_{\pi'\pi''} = \phi_{\pi''}\phi_{\pi'}$.

Hence $\mathrm{B}_n{}^{(1,1,-1)}$, for $n \geq 4$, may be considered as the semi-direct product of $\mathcal{N}$ by S_n with, for $\underline{w}_1, \underline{w}_2 \in \mathcal{N}$ and $\pi_1, \pi_2 \in S_n$, multiplication given by

$$(\underline{w}_1\,;\pi_1)\cdot(\underline{w}_2\,;\pi_2) = (\underline{w}_1\cdot\phi_{\pi_1}(\underline{w}_2)\,;\pi_1\pi_2).$$

An inverse for $(\underline{w}_1\,;\pi_1)$ is $(\phi_{\pi_1^{-1}}(\underline{w}_1^{-1})\,;\pi_1^{-1})$ and the identity element is $(1\,;1)$.

Lemma 3.2. *Let $g = (w_i\,;\pi)$ be any element of $\mathrm{B}_n^{(1,1,-1)}$, $n > 3$. Then g is an involution if either π is the identity, or $\phi_\pi(w_i) = w_i$ and $\pi^2 = 1$.*

Proof. $g^2 = (w_i\phi_\pi(w_i)\,;\pi^2) = (1\,;1)$. □

Theorem 3.3. *For* $n \geq 4$, $i(\mathrm{B}_n^{(1,1,-1)}) = 3$.

Proof. As one or two involutions will not generate $\mathrm{B}_n{}^{(1,1,-1)}$, $n > 3$, it suffices to show that three involutions do in fact generate the group.

First consider the cycle $(1,3,4,\ldots,n) \in S_n$, we may write this as the product $\pi'.\pi''$, where

1. for $n = 4$, let $\pi' = (1,3)$ and $\pi'' = (1,4)$,
2. for $n = 5$, let $\pi' = (1,4)$ and $\pi'' = (1,5)(3,4)$,
3. for $n \geq 6$ let

$$\begin{aligned}\pi' &= (1,n-1).T(3,[n/2]_I,n+1) \quad \text{and}\\ \pi'' &= (1,n).T(3,[(n+1)/2]_I,n+2).\end{aligned}$$

Now we prove that the following three group elements, each of which have order two, generate $\mathrm{B}_n{}^{(1,1,-1)}$.

$$a = (1\,;(12)), \quad b = (w_2\,;\pi') \quad \text{and} \quad c = (w_2\,;\pi'').$$

We have $abc = (1\,;(1,2,3,4,\ldots,n))$. Thus a and abc generate a subgroup isomorphic S_n. Hence we may generate all r_i, for $1 \leq i \leq n-1$, from our presentation, along with

$$b' = (1\,;\pi') \quad \text{and} \quad d = (1\,;(2,n)), \quad \text{so that} \quad d.b.b'.d = (w_n\,;1).$$

Now all the generators w_i, for $1 \leq i \leq n$, may be obtained as follows

$$(w_i\,;1) = (1\,;(i,n))\cdot(w_n\,;(i,n)).$$

□

4. $\mathbf{i(B_n{}^{(1,-1,1)})}$

For the double cover $\mathrm{B}_n^{(1,-1,1)}$, we have the following presentation.

Generators are $b_1, b_2, \ldots, b_n$ and relations,

$$\begin{aligned} b_i^2 &= 1 \quad \text{for} \quad 1 \leq i \leq n, \quad (b_jb_{j+1})^3 = 1 \quad \text{for} \quad 1 \leq j \leq n-2,\\ (b_ib_j)^2 &= 1 \quad \text{for} \quad 1 \leq i,j \leq n-1 \quad \text{and} \quad |\,i-j\,| > 1,\\ (b_ib_n)^2 &= -1 \quad \text{for} \quad 1 \leq i \leq n-2, \quad \text{and} \quad (b_{n-1}b_n)^4 = 1.\end{aligned}$$

We use a multiplication similar to that used in Section 2 for $\mathrm{B}_n^{(1,1,1)}$, but here we will be dependent on the product of two homomorphisms.

Definition 4.1. *Let ψ be the usual sign homomorphism from S_n to $\mathbf{Z}_2$ which maps even permutations to 0 and odd permutations to 1.*

Definition 4.2. *Let ρ be the mapping from $(C_2)^n$ to $\mathbf{Z}_2$ which takes the values, for $\underline{c} \in (C_2)^n$,*

$\rho(\underline{c}) = 0$ *if $\underline{c}$ has an even number of negative entries, or*
$\rho(\underline{c}) = 1$ *if $\underline{c}$ has an odd number of negative entries.*

The fact that the above mapping ρ is a homomorphism follows readily by noting that $\rho(\underline{c}) + \rho(\underline{d}) = 1$ if and only if $\rho(\underline{c} \cdot \underline{d}) = 1$.

Note also that as a permutation, π say, acting on $\underline{c}$ does not alter the number of negative coordinates of $\underline{c}$, we have $\rho(\underline{c}) = \rho(\underline{c}_\pi)$.

Proposition 4.3. *Let $C_2\, wr_\lambda S_n$ be the set of elements of the form*

$$(-1)^s(\underline{c}\,;\pi), \quad \text{for} \quad \underline{c} \in (C_2)^n,\ \pi \in S_n \quad \text{and} \quad s = 1 \quad \text{or} \quad 0.$$

Then $C_2\, wr_\lambda S_n$ is a group with multiplication defined by

$$(-1)^{s_i}(\underline{c}\,;\pi) \cdot (-1)^{s_j}(\underline{d}\,;\kappa) = (-1)^{s_i+s_j+\psi(\pi)\rho(\underline{d})}(\underline{c} \cdot \underline{d}_\pi\,;\pi\kappa).$$

Moreover $C_2\, wr_\lambda S_n$ is isomorphic to $\mathrm{B}_n^{(1,-1,1)}$.

Proof. Clearly closure holds, the identity element is $(1;1)$ and a typical element

$$(-1)^s(\underline{c}\,;\pi) \quad \text{has an inverse} \quad (-1)^{s+\psi(\pi^{-1})\rho(\underline{c}_\pi^{-1})}(\underline{c}_{\pi^{-1}}^{-1}\,;\pi^{-1}).$$

Associativity follows as $(\underline{d} \cdot \underline{e}_\kappa)_\pi = \underline{d}_\pi \cdot (\underline{e}_\kappa)_\pi$, $(\underline{e}_\kappa)_\pi = \underline{e}_{\pi\kappa}$ and $\rho(\underline{c}) = \rho(\underline{c}_\pi)$. That $\mathrm{B}_n^{(1,-1,1)}$ is embedded in $C_2\, wr_\lambda S_n$ follows by letting

$$b_1 = (\underline{1}\,;(1,2)), \quad b_2 = (\underline{1}\,;(2,3)), \ldots, b_{n-1} = (\underline{1}\,;(n-1,n)),$$

$$b_n = ((1,1,\ldots,1,-1);1) \quad \text{and} \quad \lambda = -(\underline{1}\,;1).$$

The fact that we have an isomorphism now follows by noting that the order of $\mathrm{B}_n^{(1,-1,1)}$ is $2^{n+1}.n!$, which is the same as that of $C_2\, wr_\lambda S_n$. □

Henceforth we shall refer to the elements of $\mathrm{B}_n^{(1,-1,1)}$ as signed wreath permutations in $C_2\, wr_\lambda\, S_n$. Thus we have a group multiplication similar to that of B_n but we must take account of the sign associated with the wreath permutation when determining involutions in $\mathrm{B}_n^{(1,-1,1)}$.

Lemma 4.4. *A signed wreath permutation*

$$\pm(\underline{c}\,;\pi) \neq (\underline{1}\,;1) \in \mathrm{B}_n^{(1,-1,1)},$$

is an involution if and only if $\pi^2 = 1$, $\underline{c}_\pi = \underline{c}$ and either $\psi(\pi) = 0$ or $\rho(\underline{c}) = 0$.

Proof. Let $\pi^2 = 1$, $\underline{c}_\pi = \underline{c}$ and either $\psi(\pi) = 0$ or $\rho(\underline{c}) = 0$. then

$$\begin{aligned}((-1)^s(\underline{c}\,;\pi))^2 &= (-1)^s(\underline{c}\,;\pi)\cdot(-1)^s(\underline{c}\,;\pi)\\ &= (-1)^{2s+\psi(\pi)\rho(\underline{c})}(\underline{c}\cdot\underline{c}_\pi\,;\pi^2)\\ &= (-1)^{2s+0}(\underline{1}\,;1) \quad = \quad (\underline{1}\,;1).\end{aligned}$$ □

It will be convenient at this point to introduce some more notation. Let $\underline{c}_j \in (C_2)^n$ be such that the j^{th} component, c_j of $\underline{c}_j$, is negative, with c_i strictly positive for all $i \neq j$.

Proposition 4.5. *The following four involutions generate* $\mathrm{B}_4^{(1,-1,1)}$.

$$b_1 = (\underline{1}\,;(1,2)), \quad b_2 = (\underline{1}\,;(2,3)), \quad b_3 = (\underline{1}\,;(3,4)) \quad b_4 = (\underline{c}_4\,;1)\ .$$

Moreover $\quad i(\mathrm{B}_4^{(1,-1,1)}) = 4$.

Proof. As the four given involutions satisfy our presentation for $\mathrm{B}_n^{(1,1,-1)}$, we have $i(\mathrm{B}_n^{(1,1,-1)}) \leq 4$. The fact that $i(\mathrm{B}_n^{(1,1,-1)}) \geq 4$ follows as at least one of our generators must contain an odd number of negative c_i and, as we require the generator to have order two, this generator must have $\pi = 1$. Now each of our remaining generators must contain only permutations that have order two and also generate S_4, but $i(S_4) = 3$. □

Proposition 4.6. $\mathrm{B}_5^{(1,-1,1)}$ *is generated by the following three involutions.*

$$a = (\underline{c}_1\,;(2,5)(3,4)), \quad b = (\underline{1}\,;(3,5)), \quad c = (\underline{1}\,;(1,2))\ .$$

Proof. We only need show that a, b and c generate all the signed wreath permutations of the form $(\underline{1}\,;\pi)$ for any $\pi \in S_5$. As then we may generate

$$\begin{aligned}(\underline{1}\,;(1,5))\ a\ (\underline{1}\,;(2,5)(3,4))(\underline{1}\,;(1,5)) &= (\underline{c}_5\,;1))\\ \text{and} \qquad ((\underline{c}_5\,;1)(\underline{1}\,;(1,2)))^2 &= -(\underline{1}\,;1)\end{aligned}$$

thus all of $\mathrm{B}_5^{(1,-1,1)}$. But as $aba = (\underline{1}\,;(2,4)(3,5))$ it is readily verified that b, aba and c generate all the signed wreath permutations of the form $(\underline{1}\,;\pi)$. □

Proposition 4.7. *For* $n \geq 4$, *if* $n \equiv 2$ *modulo* 4 *then the following three involutions generate* $\mathrm{B}_n^{(1,-1,1)}$.

$$\begin{aligned}a &= (\underline{c}_1\,;T(2,[n/2]_I,n+1)),\\ b &= (\underline{c}_1\,;T(2,[(n+1)/2]_I,n+2)) \quad \textit{and}\\ c &= (\underline{1}\ ;(1,2)).\end{aligned}$$

Proof. From the proof of Theorem 2.4 it is clear that a, b and c will generate at least one, up to parity of sign, of each type of signed wreath permutation of the group. Thus we need only show that $\lambda = -(\underline{1}\ ;1)$ has also been generated as multiplication by λ will generate the signed wreath permutations of opposite parity. But we have generated either $\pm(\underline{c}_n\ ;(1,2))$ which squares to give λ. □

Proposition 4.8. *When $n = 7, 8$ and 9 the following three involutions, dependent on n, generate $\mathrm{B}_n^{(1,-1,1)}$.*

$n = 7$, $\quad a = (\underline{1}\ ; (4,5)(3,6))$, $b = (\underline{c}_7\ ; (2,5)(3,4))$ *and* $c = (\underline{1}\ ; (1,2)(4,7)(3,6))$.

$n = 8$, $\quad a = (\underline{1}\ ; (1,2)(3,5)(7,8))$, $b = (\underline{c}_6\ ; (2,4)(1,5))$ *and* $c = (\underline{1}\ ; (3,6)(5,7))$.

$n = 9$, $\quad a = (\underline{1}\ ; (1,2)(3,6)(4,5)(7,8))$, $b = (\underline{c}_9\ ; (1,3)(4,6))$ *and* $c = (\underline{1}\ ; (6,7)(8,9)(3,4))$.

Proof. For $n = 7$ we have $\quad (ab)^2 = \pm(\underline{1}\,; (2,3,4,5,6))$,

$$(ac)^3 = \pm(\underline{1}\,; (1,2)) \quad \text{and} \quad (ab)^2.(ac)^3 = \pm(\underline{1}\,; (1,2,3,4,5,6)).$$

So we may generate $d = \pm(\underline{1}\,; (1,4))$ and $d' = c.d.c = \pm(\underline{c}_1\ ; (1,7))$ from which we obtain $\pm(\underline{1}\,; (1,2,3,4,5,6,7))$. The proof now follows using steps similar to those of Proposition 4.6.

When $n = 8$ the proof is similar to that of $n = 7$ and is omitted.

The proof for $n = 9$ requires more involved calculations, as we show. Here

$$(ab)^2 = \pm(\underline{1}\,; (1,3,5)(2,4,6)) \quad \text{and} \quad (ac)^7 = \pm(\underline{1}\,; (1,2)).$$

Thus

$$\begin{aligned}
(ab)^2.\pm(\underline{1}\,; (1,2)).(ab)^{-2} &= \pm(\underline{1}\,; (5,6))\\
(ab)^2.\pm(\underline{1}\,; (5,6)).(ab)^{-2} &= \pm(\underline{1}\,; (3,4))\\
b.\pm(\underline{1}\,; (5,6)).b &= \pm(\underline{1}\,; (4,5))\\
(ab)^2.\pm(\underline{1}\,; (4,5)).(ab)^{-2} &= \pm(\underline{1}\,; (2,3))
\end{aligned}$$

and we have the required generators for the subgroup of $\mathrm{B}_9^{(1,-1,1)}$ isomorphic to S_6. Thus we may generate

$$x = \pm(\underline{1}\,; (1,4)), \quad y = \pm(\underline{1}\,; (1,6)) \quad \text{and} \quad z = \pm(\underline{1}\,; (1,3)).$$

Also we have

$$\begin{aligned}
x' &= (ac).\pm(\underline{1}\,; (5,6)).(ac)^{-1} &&= \pm(\underline{1}\,; (4,8))\\
y' &= (ac).\pm(\underline{1}\,; (4,8)).(ac)^{-1} &&= \pm(\underline{1}\,; (6,9))\\
z' &= (ab)^2.\pm(\underline{1}\,; (4,8)).(ab)^{-2} &&= \pm(\underline{1}\,; (3,7))
\end{aligned}$$

which as

$$x'.x.x' = \pm(\underline{1}\,; (1,8)), \quad y'.y.y' = \pm(\underline{1}\,; (1,9)), \quad z'.z.z' = \pm(\underline{1}\,; (1,7)),$$

means that we may generate all the subgroup of $\mathrm{B}_9^{(1,-1,1)}$ isomorphic to S_9 from which the proof follows as in the previous cases. □

Proposition 4.9. *For $n \geq 11$, the following three involutions, dependent on the value of n modulo 4, generate* $\mathrm{B}_n^{(1,-1,1)}$.

$n \equiv 0 \pmod 4$ $\quad a = (\underline{1}\ ; T(3, n/2, n+1))$,
$b = (\underline{c}_n\ ; (2,3)T(4, n/2, n+2))$ *and*
$c = (\underline{1}\ ; (1,2)(3,n)(4,n-1)T(5, n/2, n+1))$.

$n \equiv 1 \pmod 4$ $\quad a = (\underline{1}\ ; T(3, (n-1)/2, n))$,
$b = (\underline{c}_n\ ; (2,3)T(4, (n-1)/2, n+1))$ *and*
$c = (\underline{1}\ ; (1,2)(3,n)(4,n-1)(5,n-2)T(6, (n-1)/2, n))$.

$n \equiv 3 \pmod 4$ $\quad a = (\underline{1}\ ; T(3, (n+1)/2, n+2))$,
$b = (\underline{c}_n\ ; (2,3)T(4, (n+1)/2, n+3))$ *and*
$c = (\underline{1}\ ; (1,2)(3,n)T(4, (n+1)/2, n+2))$.

Proof. For each n, the permutations in a and b fix $\underline{c}_n$ and

$$a.b = \pm(\underline{c}_n\ ; (2,3,\ldots,k))$$

thus $a.b$ contains a permutation of length $k-1$ where

$$k = \begin{cases} n-2 & \text{when} \quad n \equiv 0 \pmod 4, \\ n-3 & \text{when} \quad n \equiv 1 \pmod 4, \\ n-1 & \text{when} \quad n \equiv 3 \pmod 4. \end{cases}$$

Thus as

$$(ab)^2 = \pm(\underline{1}\ ; (2,4,\ldots,k,3,\ldots,k-1))$$

it follows that

$$(ab)^k = \pm(\underline{1}\ ; (2,3,\ldots,k)).$$

Also, for each n,

$$(ac)^3 = \pm(\underline{1}\ ; (1,2)).$$

Thus consider when $n \equiv 0$, as in the previous proposition, we may generate either $\pm(\underline{c}_n; (1,2))$ which squares to give λ. So we may generate all the group $\mathrm{B}_{n-2}^{(1,-1,1)}$. Hence we may generate $d = \pm(\underline{1}\ ; (2,3))$ and $e = \pm(\underline{1}\ ; (2,4))$ to obtain

$$d' = d.c.d = \pm(\underline{1}\ ; (1,n)) \quad \text{and} \quad e' = e.d.e = \pm(\underline{1}\ ; (1,n-1))$$

from which all $\mathrm{B}_n^{(1,-1,1)}$ may be generated.

The proofs for $n \equiv 1$ and 3 modulo 4 follow similar steps. □

Theorem 4.10. *For* $\quad n \geq 5, \quad i(\mathrm{B}_n^{(1,-1,1)}) = 3.$

Proof. This is immediate from the previous propositions. □

5. $\mathbf{I(B_n^{(1,-1,-1)})}$

We adapt the multiplication of Sections 3 and 4 for $\mathrm{B}_n^{(1,-1,-1)}$, $n > 3$. It will be convenient to consider the following presentation of the group $\mathrm{B}_n^{(1,-1,-1)}$. Take as generators $w_1, w_2, \ldots, w_n$ and $r_1, r_2, \ldots, r_{n-1}$ with relations,

$$\begin{aligned}
r_i^2 &= 1 \quad \text{for} \quad 1 \leq i \leq n-1, \quad w_i^2 = 1 \quad \text{for} \quad 1 \leq i \leq n, \\
(r_i r_j)^2 &= 1 \quad \text{for} \quad 1 \leq i, j \leq n-1 \quad \text{and} \quad \mid i-j \mid > 1, \\
(r_i r_{i+1})^3 &= 1 \quad \text{for} \quad 1 \leq i \leq n-2, \\
w_i w_j &= -w_j w_i \quad \text{for} \quad 1 \leq i, j \leq n, \\
w_i r_i &= -r_i w_{i+1} \quad \text{for} \quad 1 \leq i \leq n-1, \\
w_j r_i &= -r_i w_j \quad \text{for} \quad 1 \leq i, j \leq n, \quad \text{and} \quad j \neq i, i+1.
\end{aligned}$$

In addition to the subgroup S_n of $\mathrm{B}_n^{(1,-1,-1)}$ we also have a subgroup $\mathcal{A}$ generated by $w_1, w_2, \ldots, w_n$. This subgroup has the following presentation.

$$\langle w_1, w_2, \ldots, w_n : w_i^2 = 1, \leq i \leq n \quad \text{and} \quad (w_i w_j)^2 = -1 \quad \text{for} \quad i \neq j \rangle$$

Definition 5.1. *Let*

$$\underline{w} = \pm w_{a_1}^{s_{a_1}} w_{a_2}^{s_{a_2}} \ldots w_{a_n}^{s_{a_n}}, \quad \textit{for} \quad s_{a_i} = 0 \quad \textit{or} \quad 1,$$

where a_i is any positive integer from 1 to n, be any element of $\mathcal{A}$. Then denote by k the number of non-zero s_{a_i} of $\underline{w}$ and define by τ a map from $\mathcal{A}$ to $\mathbf{Z}_2$ which takes the values, for $\underline{w} \in \mathcal{A}$,

$$\tau(\underline{w}) = 0 \textit{ if } k \textit{ is even} \quad \textit{and} \quad \tau(\underline{w}) = 1 \textit{ if } k \textit{ is odd.}$$

As the mapping τ from $\mathcal{A}$ to $\mathbf{Z}_2$, defined above, takes only the values 0 or 1 it is a straight forward exercise to show that τ is a homomorphism by noting that $\tau(\underline{w}) + \tau(\underline{w}') = 1$ if and only if $\tau(\underline{w}\underline{w}') = 1$.

Definition 5.2. *Let $\underline{w} = w_i w_j \ldots w_k$ be any word in $\mathcal{A}$ and π a permutation in S_n. Then we denote by ϕ_π the homomorphism from S_n to AUT $\mathcal{A}$, the automorphism group of $\mathcal{A}$, defined by*

$$\phi_\pi(\underline{w}) = w_{\pi^{-1}(i)} w_{\pi^{-1}(j)} \ldots w_{\pi^{-1}(k)}.$$

Note that a permutation acting on $\underline{w}$ does not alter the value of τ as only the positioning of the $w_i \in \underline{w}$ may be altered, not the number of non-zero w_i.

Proposition 5.3. *Let $\mathcal{A}\, wr_\lambda S_n$ be the set of elements of the form*

$$(\underline{w}\,; \pi), \quad \textit{for} \quad \underline{w} \in \mathcal{A}, \quad \pi \in S_n.$$

Then $\mathcal{A}\, wr_\lambda S_n$ is a group with multiplication defined by

$$(\underline{w}_i\,; \pi) \cdot (\underline{w}_j\,; \kappa) = ((-1)^{\psi(\pi)\tau(\underline{w}_j)} \underline{w}_i \cdot \phi_\pi(\underline{w}_j)\,; \pi\kappa).$$

Moreover for $n \geq 4$, $\mathrm{B}_n^{(1,-1,-1)}$ is isomorphic to $\mathcal{A}\, wr_\lambda S_n$

Proof. Clearly closure holds. The identity element is $(1;1)$ and a typical element

$$(\underline{w}\,;\pi) \quad \text{has an inverse} \quad ((-1)^{\psi(\pi^{-1})\tau(\underline{w}^{-1})}\phi_{\pi^{-1}}(\underline{w}^{-1})\,;\pi^{-1}).$$

For associativity, as $\phi_{\pi\kappa} = \phi_\pi\phi_\kappa$ and $\tau(\phi_{\pi_j}(\underline{w}_k)) = \tau(\underline{w}_k)$ we have

$$\begin{aligned}
&((\underline{w}_i\,;\pi)\cdot(\underline{w}_j\,;\kappa))\cdot(\underline{w}_k\,;\delta)\\
&= ((-1)^{\psi(\pi)\tau(\underline{w}_j)}\underline{w}_i\cdot\phi_\pi(\underline{w}_j)\,;\pi\kappa)\cdot(\underline{w}_k\,;\delta)\\
&= ((-1)^{\psi(\pi\kappa)\tau(\underline{w}_k)}(-1)^{\psi(\pi)\tau(\underline{w}_j)}\underline{w}_i\phi_\pi(\underline{w}_j)\cdot\phi_{\pi\kappa}(\underline{w}_k)\,;\pi\kappa\delta)\\
&= ((-1)^{\psi(\pi)\tau(\underline{w}_k)+\psi(\kappa)\tau(\underline{w}_k)+\psi(\pi)\tau(\underline{w}_j)}\underline{w}_i\phi_\pi(\underline{w}_j)\cdot\phi_{\pi\kappa}(\underline{w}_k)\,;\pi\kappa\delta)\\
&= ((-1)^{\psi(\pi)\tau(\underline{w}_j)+\psi(\pi)\tau(\phi_\kappa(\underline{w}_k))+\psi(\kappa)\tau(\underline{w}_k)}\underline{w}_i\phi_\pi(\underline{w}_j)\cdot\phi_{\pi\kappa}(\underline{w}_k)\,;\pi\kappa\delta)\\
&= ((-1)^{\psi(\pi)\tau(\underline{w}_j\phi_\kappa(\underline{w}_k))+\psi(\kappa)\tau(\underline{w}_k)}\underline{w}_i\phi_\pi(\underline{w}_j\cdot\phi_\kappa(\underline{w}_k))\,;\pi\kappa\delta)\\
&= ((-1)^{\psi(\pi)\tau((-1)^{\psi(\kappa)\tau(\underline{w}_k)}\underline{w}_j\phi_\kappa(\underline{w}_k))}\underline{w}_i\phi_\pi((-1)^{\psi(\kappa)\tau(\underline{w}_k)}\underline{w}_j\cdot\phi_\kappa(\underline{w}_k))\,;\pi\kappa\delta)\\
&= (\underline{w}_i\,;\pi)\cdot((-1)^{\psi(\kappa)\tau(\underline{w}_k)}\underline{w}_j\cdot\phi_\kappa(\underline{w}_k)\,;\kappa\delta))\\
&= (\underline{w}_i\,;\pi)\cdot((\underline{w}_j\,;\kappa)\cdot(\underline{w}_k\,;\delta))
\end{aligned}$$

Clearly the orders of both groups are the same as the number of elements of the form $(\underline{w}\,;\pi)$ is $2^{n+1}.n!$, as in $\mathrm{B}_n^{(1,-1,1)}$. Thus $\mathcal{A}\,wr_\lambda S_n$ has $2^{n+1}.n!$ group elements, which is also the order of $\mathrm{B}_n^{(1,-1,-1)}$. To see that $\mathrm{B}_n^{(1,-1,-1)}$ is embedded in $\mathcal{A}\,wr_\lambda S_n$, identify -1 with $(-1\,;1)$, r_i with $(1\,;(i,i+1))$ and y_j with $(w_j\,;1)$. □

Hence we consider the elements of $\mathrm{B}_n^{(1,-1,-1)}$ as being of the form

$$(\underline{w}\,;\pi), \quad \text{for} \quad \underline{w}\in\mathcal{A}, \quad \pi\in S_n.$$

Lemma 5.4. *Let $(w_i\,;\pi)$ be any element of $\mathrm{B}_n^{(1,-1,-1)}$, $n>3$. Then $(w_i\,;\pi)$ is an involution when either π is the identity, or $\phi_\pi(w_i)=w_i$ and also π has cycle type 2^r with r congruent to 0 or 2 modulo 4.*

Proof. The statement of this proposition is similar to that of Lemma 3.2, the only difference being that we have imposed the extra condition that as $\tau(\underline{w})\neq 0$ we have an even number of transpositions so that $\psi(\pi)=0$. □

Proposition 5.5. *The following four involutions generate $\mathrm{B}_4^{(1,-1,-1)}$.*

$$b_1 = (\underline{1}\,;(1,2)), \quad b_2 = (\underline{1}\,;(2,3)), \quad b_3 = (\underline{1}\,;(3,4)), \quad b_4 = (w_4\,;1).$$

Moreover $i(\mathrm{B}_4^{(1,-1,-1)}) = 4$.

Proof. This proof just follows the steps of Proposition 4.5 with $\underline{c}_4$ replaced by w_4. □

Proposition 5.6. $\mathrm{B}_5^{(1,-1,-1)}$ *is generated by the following three involutions.*

$$a = (w_1\,;(2,5)(3,4)), \quad b = (\underline{1}\,;(3,5)), \quad c = (\underline{1}\,;(1,2))\,.$$

Proof. Again we need only refer to the proof of Proposition 4.6 with $\underline{c}_1$ replaced by w_1 and the result follows. □

Proposition 5.7. *For $n \geq 4$, if $n \equiv 2$ modulo 4 then the following three involutions generate $B_n^{(1,-1,-1)}$.*

$$\begin{aligned} a &= (w_1\,; T(2,[n/2]_I, n+1)),\\ b &= (w_1\,; T(2,[(n+1)/2]_I, n+2)) \quad \textit{and} \quad c = (1\,;(1,2)). \end{aligned}$$

Proof. Note first that for each $n \equiv 2$ modulo 4 we have

$$\psi(T(2,[n/2]_I, n+1)) = \psi(T(2,[(n+1)/2]_I, n+2)) = 0.$$

Thus it is clear by Theorem 2.4, with $(-1,1,\ldots,1)$ replaced by w_1, that a, b and c will generate a subgroup of $B_n^{(1,-1,-1)}$ isomorphic to S_n. Hence we may generate

$$b' = (1\,; T(2,[(n+1)/2]_I, n+2)) \quad \text{so that} \quad (c.b.b')^4 = (-1\,;1).$$

Then as, for any $1 \leq i \leq n$,

$$(w_i\,;1) = (-1\,;1)\cdot(1\,(1,i))\cdot(w_1\,;1)\cdot(1\,(1,i))$$

we may generate all the group. □

Proposition 5.8. *When $n = 7, 8$ and 9 the following three involutions, dependent on n, generate $B_n^{(1,-1,-1)}$.*

$n = 7$ $\quad a = (1\,;(4,5)(3,6))$, $b = (w_7\,;(2,5)(3,4))$ *and* $c = (1\,;(1,2)(4,7)(3,6))$.

$n = 8$ $\quad a = (1\,;(1,2)(3,5)(7,8))$, $b = (w_6\,;(2,4)(1,5))$ *and* $c = (1\,;(3,6)(5,7))$.

$n = 9$ $\quad a = (1\,;(1,2)(3,6)(4,5)(7,8))$, $b = (w_9\,;(1,3)(4,6))$ *and* $c = (1\,;(6,7)(8,9)(3,4))$.

Proof. If we replace $\underline{c}_i$ by w_i in each generator c, then Proposition 4.8 shows that for each n we may generate a subgroup of $B_n^{(1,-1,-1)}$ isomorphic to S_n. The remainder of the proof now follows as in Proposition 5.7. □

Proposition 5.9. *For $n \geq 11$, the following three involutions, dependent on the value of n modulo 4, generate $B_n^{(1,-1,-1)}$.*

$n \equiv 0 \pmod 4$ $\quad a = (1\,;T(3, n/2, n+1))$,
$b = (w_n\,;(2,3)T(4, n/2, n+2))$ *and*
$c = (1\,;(1,2)(3,n)(4,n-1)T(5, n/2, n+1))$.

$n \equiv 1 \pmod 4$ $\quad a = (1\,;T(3,(n-1)/2, n))$,
$b = (w_n\,;(2,3)T(4,(n-1)/2, n+1))$ *and*
$c = (1\,;(1,2)(3,n)(4,n-1)(5,n-2)T(6,(n-1)/2, n))$.

$n \equiv 3 \pmod 4$ $\quad a = (1\,;T(3,(n+1)/2, n+2))$,
$b = (w_n\,;(2,3)T(4,(n+1)/2, n+3))$ *and*
$c = (1\,;(1,2)(3,n)T(4,(n+1)/2, n+2))$.

Proof. First we note that, for each n, the permutation π in b is such that $\phi_\pi(w_n) = w_n$ and $\psi(\pi) = 0$, thus a, b and c are involutions.

As in Proposition 5.5 it suffices to show that for each n we may generate

$$(1\,;(1,2)) \quad \text{and} \quad (1\,;(1,2,\ldots,n)).$$

But this follows easily from Proposition 4.9, with $\underline{c}_n$ replaced by w_n. □

Theorem 5.10. *For* $\quad n \geq 5, \quad i(\mathrm{B}_n^{(1,-1,-1)}) = 3.$

Proof. This is immediate from the previous propositions. □

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