Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 39 (1998), No. 1, 121–133.

Solvable Collineation Groups of Smooth Projective Planes

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Abstract. Let Δ be a closed solvable subgroup of the collineation group of one of the four classical Moufang planes $\mathcal{P}_2\mathbb{F}$ over the domain $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, where $\mathbb{H}$ denotes the quaternions and $\mathbb{O}$ the Cayley numbers. Then $\dim \Delta \leq 4, 9, 17, 30$ if $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. For arbitrary compact connected projective planes $\mathcal{P}$ of finite dimension n, M. Lüneburg derived upper bounds for the dimension of Δ depending on n and the configuration of the fixed elements of Δ, see [17]. In this paper we consider $\dim \Delta$ in the case of smooth projective planes. Compared to the results of [17], the upper bounds of $\dim \Delta$ can be lowered in almost every case; except for one single case we even derive sharp bounds.

0. Introduction

Stable planes originate from Salzmann's paper *Topological planes* [24] and have been studied systematically by R. Löwen and by M. Stroppel, see for example [14], [15], [26], and [27]. We started the investigation of *smooth* stable planes in [1]. In this paper we continue the study of collineation groups of a smooth stable plane $\mathcal{S}$ (see [2] and [3]). This time we will restrict our attention to closed solvable subgroups Δ of the group $\Gamma = \mathrm{Aut}(\mathcal{S})$ of *continuous* collineations of $\mathcal{S}$. As in [3], we want to derive upper bounds for the dimensions of Δ. Our proofs rely on the fact that up to duality a solvable collineation group of a compact connected topological projective plane has a global fixed point p, i.e. $\Delta = \Delta_p$ holds. This, of course, is not valid for stable planes, since one can remove the fixed point p thus obtaining a smooth stable subplane on which Δ acts without any fixed points. Hence, we consider smooth *projective* planes instead of smooth stable planes.

1. Definitions and auxiliary results

Throughout we will use the notation and the definitions of [1] and [2].

Definition 1.1. *A projective plane $\mathcal{P} = (P, \mathcal{L})$ is called smooth if the set P of points and the set $\mathcal{L}$ of lines are smooth (= C^∞) manifolds such that the geometric operations $\vee$ of join and $\wedge$ of intersection are smooth mappings.*

It is convenient to identify every line $L \in \mathcal{L}$ with the set of points incident with L. If we look at a line this way, we will refer to L as a *point row*. Dually, the set $\mathcal{L}_p$ of all lines through some point p is called a *line pencil*. The integer $n = \dim P$ which is called the *dimension* of the projective plane $\mathcal{P}$ is always a power 2^k of 2, where $k = 1, 2, 3, 4$, see [25], §54. Moreover, for $n = 2l$ we have $\dim L = \dim \mathcal{L}_p = l$ for any point row L and any line pencil $\mathcal{L}_p$.

By [1], the tangent spaces T_pP $(p \in P)$ and $T_L\mathcal{L}$ $(L \in \mathcal{L})$ together with the tangent spreads $\mathcal{S}_p = \{T_pK \mid K \in \mathcal{L}_p\}$ and $\mathcal{S}_L = \{T_L\mathcal{L}_q \mid q \in L\}$ induced by the line pencil $\mathcal{L}_p$ and the point row L are locally compact affine translation planes $\mathcal{A}_p$ and $\mathcal{A}_L$, respectively. Their projective closures are denoted by $\mathcal{P}_p$ and $\mathcal{P}_L$ and in both cases we put L_∞ to be the line at infinity (which is also the translation line). According to [2], (2.3), every continuous collineation of $\mathcal{P}$ is in fact smooth. This enables us to compare the results in the topological situation with the results in the smooth case, see the end of this paper. The group Γ of continuous (or, equivalently, of smooth) collineations γ of $\mathcal{P}$ is a Lie transformation group (with respect to the compact-open topology) on both the set P of points and the set $\mathcal{L}$ of lines, see [2], (2.4). In particular, any collineation γ is a diffeomorphism of P onto P and of $\mathcal{L}$ onto $\mathcal{L}$. The stabilizer Γ_p of some point $p \in P$ induces an action on the tangent translation plane $\mathcal{A}_p$ via the *derivation mapping*

$$D_p : \Gamma_p \to \Sigma_0 := \mathrm{Aut}(\mathcal{A}_p)_0 : \gamma \mapsto D\gamma(p).$$

By [2], (3.3), the map D_p is a continuous homomorphism and $\ker D_p = \Gamma_{[p,p]}$ is the subgroup of all elations of Γ having p as their center.

We use the small inductive dimension ind rather than the (more common) covering dimension dim because of its local nature. In most cases, however, both of these notions of dimension coincide, for instance this is true for locally compact groups and for manifolds, of course. Thus, we will write dim even if we think of the small inductive dimension. Without further mentioning we will always use Halder's dimension formula, [12] and also [25], 96.10, as well as the following lemma.

Lemma 1.2. *If Δ is a locally compact connected collineation group of a smooth projective plane $\mathcal{P}$ which fixes some element $x \in P \cup \mathcal{L}$, then*

$$\dim \Delta = \dim \ker D_x + \dim D_x\Delta \quad \textit{and} \quad \dim D_x\Delta \leq \dim \mathrm{Aut}(\mathcal{A}_x)_0.$$

Proof. By Theorem (3.3) of [2], the group Δ acts on the Lie group $\Sigma_0 = \mathrm{Aut}(\mathcal{A}_x)_0$ via composition $(\Delta, \gamma) \mapsto D_x(\Delta)\gamma$. By Halder [12], this implies that

$$\dim \Delta/\Delta_1 = \dim 1^\Delta = \dim D_x\Delta \leq \dim \Sigma_0,$$

where the last inequality follows from the monotonicity of the small inductive dimension. From $\Delta_1 = \ker \mathrm{D}_x$ we conclude that

$$\dim \Delta/_{\ker \mathrm{D}_x} = \dim \mathrm{D}_x \Delta,$$

and hence $\dim \Delta = \dim \ker \mathrm{D}_x + \dim \mathrm{D}_x \Delta$ follows from Nagami [20], (2.1). □

Proposition 1.3. *Let $\mathcal{P}$ be a compact connected translation plane of dimension 2^k, where $1 \leq k \leq 4$, and let L_∞ be some translation axis of $\mathcal{P}$. Let $K, L \neq L_\infty$ be two different lines such that $o = K \wedge L$ does not lie on L_∞. If Δ is a closed connected solvable subgroup of the stabilizer ∇ of the triangle defined by K, L, L_∞, then $\dim \Delta \leq 2 + \operatorname{rk} \Delta$, where $\operatorname{rk} \Delta \leq k$ is the dimension of a maximal torus subgroup of Δ. In the case $k = 1$ we have $\operatorname{rk} \Delta = 0$.*

Proof. By H. Hähl [9], (4.2), the connected component ∇^1 of ∇ possesses a maximal compact subgroup K of codimension at most 2. Since a compact connected solvable Lie group is isomorphic to some torus group $\mathbb{T}^r$, we infer that a closed solvable subgroup Δ of ∇^1 has dimension at most $r + 2$. According to M. Lüneburg, [17], II, Satz 2 (see also [25], 55.37), we have $r \leq k$. If $k = 1$, then $\mathcal{P}$ is isomorphic to the projective plane over the reals (see H. Salzmann, [24], (7.24)), and hence $\operatorname{rk} \Delta = 0$ follows, because in this case Δ is a subgroup of the group of all regular real diagonal matrices. □

As usual, we write $\Gamma_{[c]}$ for the group of all central collineations with center c, and we write $\Gamma_{[c,A]}$ for the subgroup of $\Gamma_{[c]}$ consisting of all collineations with center c and axis A. Moreover, the group of all elations with center c is denoted by $\Gamma_{[c,c]}$. The following result deals with the structure of groups of central collineations. The second part of the next proposition is also true in the case of compact connected projective planes (see Hähl, [10]). Nevertheless we will give a (simpler) proof which uses the special situation of smoothness.

Proposition 1.4. *Let $\mathcal{P}$ be a smooth projective plane and let $p \in P$. Let $\Delta \neq \mathbb{1}$ be a connected locally compact collineation group of central collineations of $\mathcal{P}$ with center p. Then the derivation mapping D_p maps Δ onto a closed subgroup Υ of $\operatorname{Aut}(\mathcal{P}_p)$, and $\Delta = \Delta_{[p,A]} \ltimes \Delta_{[p,p]} \cong \Upsilon \ltimes \Delta_{[p,p]}$ for some line $A \notin \mathcal{L}_p$.*

Proof. By [2], (4.12), the image $\Upsilon = \mathrm{D}_p \Delta$ is a closed (connected) subgroup of $\mathbb{H}^\times$. If $\Upsilon \neq \mathbb{1}$, then Υ contains a nontrivial central element ω. If $\Delta_{[p,p]} = \ker \mathrm{D}_p = \mathbb{1}$, then D_p maps Δ isomorphically (as a topological group) onto Υ and Δ consists entirely of homologies. Moreover, the collineation $\delta := \mathrm{D}_p^{-1}(\omega)$ is a homology of $\mathcal{P}$ with some line A as its axis. Because δ is central in Δ, we infer that δ fixes every line of the orbit A^Δ. From $\Delta = \Delta_{[p]}$ we infer that the orbit A^Δ does not contain any line incident with the center p. This implies that $A^\Delta = A$, whence A is the axis of every homology of Δ, i.e. we have $\Delta = \Delta_{[p,A]}$. Now let $\Delta_{[p,p]} \neq \mathbb{1}$, and assume that $\Delta \neq \Delta_{[p,p]}$ (otherwise there is nothing left to prove). Fix some collineation δ in the inverse image $\mathrm{D}_p^{\leftarrow}(\omega)$. Then δ is a homology, since $D_p(\delta) \neq \mathbb{1}$. Again, let A be the axis of δ. Because the point p does not lie on the line A, we infer from $\Delta_{[p,p]} \neq \mathbb{1}$ that the line A is moved by Δ. By Hähl, [10], dual theorem of (1.3), this implies that $A^\Delta = A^{\Delta_{[p,p]}}$. Now let $\gamma \in \Upsilon$ and choose $\varepsilon \in \mathrm{D}_p^{\leftarrow}(\gamma)$.

Then there is some (uniquely determined) elation $\tau \in \Delta_{[p,p]}$ such that $A^\varepsilon = A^\tau$, whence we have $\varepsilon\tau^{-1} \in \Delta_{[p,A]}$. Thus we have proved that for every $\gamma \in \Upsilon$ there exists exactly one collineation in $\varepsilon \in \mathrm{D}_p^{\leftarrow}(\gamma) \cap \Delta_{[p,A]}$. In particular, the restriction $\mathrm{D}_p : \Delta_{[p,A]} \to \Upsilon$ is a (closed) isomorphism, i.e. the groups $\Delta_{[p,A]}$ and Υ are isomorphic as topological groups. This proves the proposition. □

Corollary 1.5. *Let $\mathcal{P}$ be a smooth projective plane and let $p \in P$. Then the connected component of the group $\Delta_{[p]}$ is a linear group.*

Proof. Let Υ denote the connected component of $\Delta_{[p]}$. According to the last proposition, we can write $\Upsilon = \Upsilon_{[p,A]} \ltimes \Upsilon_{[p,p]}$. The normal subgroup $\Upsilon_{[p,p]}$ is simply connected by [2], (4.16), and the complement $\Upsilon_{[p,A]}$ is a subgroup of the multiplicative group of the quaternion numbers by [2], (3.14). In particular, this complement is a reductive group. By [13], Chapter XVIII, Thereom 4.3, this implies the linearity of Υ. □

Remark. Similar results concerning the linearity of certain subgroups of Γ can be found in [2], especially Theorem (4.19) and Corollary (4.20).

Theorem 1.6. *Let $\mathcal{P}$ be a smooth projective plane of dimension 2^k, let K, L be two different lines, and put $p = K \wedge L$. If Δ is a closed connected solvable subgroup of the stabilizer of K and L, then*

$$\dim \Delta \leq \dim \Delta_{[p,p]} + \operatorname{rk} \Delta + 2.$$

Moreover, if one of the following conditions

(i) $\dim \mathrm{D}_p\Delta \geq \operatorname{rk} \Delta + 2$,

(ii) $\mathcal{A}_p$ *is classical and* $\dim \mathrm{D}_p\Delta \geq 3, 4, 6$ *(for $k = 2, 3, 4$)*

(iii) $k = 1$ *and* $\dim \mathrm{D}_p\Delta \geq 2$,

is satisfied, then $\dim \Delta_{[p]}/\Delta_{[p,p]} \geq 1$ *holds.*

Proof. We consider the derivation map $\mathrm{D}_p : \Delta \to \Sigma_0 = \mathrm{Aut}(\mathcal{A}_p)_0$. Using Corollary (3.9) of [2], Proposition 1.3 together with Lemma 1.2 yields

$$\dim \Delta = \dim \ker \mathrm{D}_p + \dim \mathrm{D}_p\Delta \leq \dim \Delta_{[p,p]} + \operatorname{rk} \Delta + 2.$$

From Hähl, [9], (4.2) or [25], 81.8, we know that $\mathrm{D}_p\Delta$ is a subgroup of some group $\mathrm{R}\cdot\mathrm{H}_k\cdot\mathrm{T}_k$, where $\mathrm{R} \cong \mathbb{R}$, H_k is a closed solvable subgroup of the homology group fixing the origin 0 and the line at infinity L_∞, $\mathrm{T}_0 = \mathbb{1}$, and $\mathrm{T}_k \cong \mathbb{T}^k$ if $k > 0$, see also M. Lüneburg, [17], II, Satz 1. In particular, we infer that $\mathrm{D}_p\Delta$ contains a one-parameter subgroup of homologies with center 0 if $\dim \mathrm{D}_p\Delta \geq \operatorname{rk} \Delta + 2$. Thus, Proposition 1.4 proves the second claim. Now we assume that $\mathcal{A}_p$ is a classical plane. By Salzmann [24], (7.24), this is always true for $k = 1$. Since $\mathrm{D}_p\Delta$ fixes T_pL and T_pK, we infer that $\mathrm{D}_p\Delta$ consists of diagonal real matrices if $k = 1$. Thus, we conclude from $\dim \mathrm{D}_p\Delta \geq 2$ that $\mathrm{D}_p\Delta$ contains the homology group $\{\lambda \cdot \mathrm{id} \mid \lambda > 0\}$ as a subgroup. Again, Proposition 1.4 shows that $\dim \Delta_{[p]}/\Delta_{[p,p]} \geq 1$. If $k \in \{2, 3\}$, then $\mathrm{H}_k \cong \mathbb{C}^\times$ and thus $\mathrm{D}_p\Delta$ contains a one-parameter subgroup of homologies with center 0 if $\dim \mathrm{D}_p\Delta \geq k + 1$. Thus it remains to study the case $k = 4$. Since we have

$\operatorname{rk} \mathrm{D}_p\Delta \leq k = 4$, this case is already covered by condition (i). Thus we have proved all parts of the theorem. □

Lemma 1.7. *Let $\mathcal{P}$ be a smooth projective plane of dimension $2l$, and let $p \in P$ and $L \in \mathcal{L}_p$. If $t = \dim \Gamma_{[p,L]}$, then $\dim \Gamma_{[p,p]} \leq l + t$.*

Proof. Choose some point x on L distinct from p, and select another point y not on L. Then the stabilizer $\left(\Gamma_{[p,p]}\right)_{x,y}$ is trivial and thus we have

$$\dim \Gamma_{[p,p]} = \dim x^{\Gamma_{[p,p]}} + \dim y^{(\Gamma_{[p,p]})_x} \leq l + t. \qquad \square$$

The last result of this section says that a solvable Lie group cannot act transitively on an n-sphere if $n \geq 2$. For a proof see Montgomery and Samelson, [18], cp. also [25], 96.20, 96.23, and 96.28.

Lemma 1.8. *Let G be a solvable Lie group acting on an n-sphere S. If $n \geq 2$, then G cannot act transitively on S.*

2. The dimensions of solvable groups

We are going to derive upper bounds for the dimensions of solvable groups Δ acting on a smooth projective plane $\mathcal{P} = (P, \mathcal{L})$ of finite positive dimension $n = 2^k = 2l$. By M. Lüneburg, [17], (G2), p. 22, the following four cases cover all possible situations which can occur for a closed connected solvable collineation group Δ of $\mathcal{P}$.

(I) Δ fixes some flag (p, L) and $\dim \Delta_{[p,L]} \geq 1$.
(II) Δ fixes some antiflag.
(III) Δ fixes some Baer subplane of $\mathcal{P}$, i.e. a subplane of dimension l.
(IV) Every minimal connected commutative normal subgroup N of Δ has some orbit x^{N}, where $x \in P \cap \mathcal{L}$, which generates a subplane of $\mathcal{P}$.

The following theorems will treat each of these cases separately.

Theorem 2.1. *Let $\mathcal{P}$ be a smooth projective plane of dimension $n = 2^k = 2l$. Assume that the solvable group Δ fixes some flag (p, L) such that $t := \dim \Delta_{[p,L]} \geq 1$ holds. Put $d_I(1) = 3$, $d_I(2) = 8$, $d_I(3) = \min\{16, 13 + t\}$, and $d_I(4) = \min\{29, 22 + t\}$. Then either $\dim \Delta$ is at most $d_I(k)$, or $\mathcal{P}$ is isomorphic to the classical plane of dimension 2^k.*

Proof. Let $K \in \mathcal{L}_p \setminus \{L\}$. If $\dim \Delta_{[p,p]} = 2l$, then $\mathcal{P}$ is a dual translation plane, whence $\mathcal{P}$ is a classical Moufang plane according to a theorem of J. Otte, [21] and [22]. Thus we may assume that $\dim \Delta_{[p,p]} < 2l$. Let $\mathrm{D}_p : \Delta_K \to \Sigma_0$ be the derivation mapping. By Lemma

1.7 and Corollary 1.6 we have

$$\begin{aligned}
\dim \Delta &= \dim K^{\Delta} + \dim \Delta_K \\
&= \dim K^{\Delta} + \dim \ker \mathrm{D}_p + \dim \mathrm{D}_p(\Delta_K) \\
&\le l + \dim \Delta_{[p,p]} + \operatorname{rk}\Delta + 2 \le l + (l+t) + \operatorname{rk}\Delta + 2 \\
&< 3l + k + 2 \le \begin{cases} 5 & \text{if } k=1 \\ 10 & \text{if } k=2 \\ 17 & \text{if } k=3 \\ 30 & \text{if } k=4 \end{cases}
\end{aligned} \tag{$*$}$$

Furthermore, we obtain from inequality $(*)$

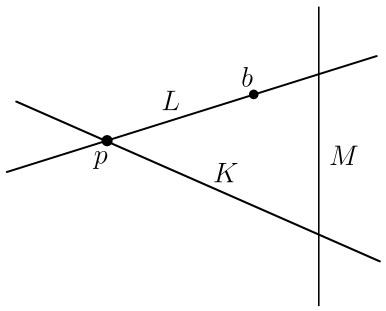
$$\dim \Delta \le 2l + t + k + 2 \le \begin{cases} 13+t & \text{if } k=3 \\ 22+t & \text{if } k=4 \end{cases}$$

which proves the theorem for $k \ge 3$. Now let us have a look at the cases $k = 1, 2$. Choose some point $b \ne p$ on L. Recalling that Δ fixes the point p, we get $\Delta_{[b,b]} = \Delta_{[b,L]}$. Thus

$$\dim \Delta = \dim b^{\Delta} + \dim \Delta_b \le l + \dim \ker \mathrm{D}_b + \dim \mathrm{D}_b(\Delta_b) \le l + \dim \Delta_{[b,L]} + \dim \mathrm{D}_b(\Delta_b)$$

holds.

Case $k = 1$. According to M. Lüneburg [17], III, Satz 1, there is a minimal connected closed abelian normal subgroup N of Δ with $\mathsf{N} = \mathsf{N}_{[p,L]}$. In particular, we have $\dim \mathsf{N} = 1$, and hence Δ is (p, L)-transitive (note that $L \setminus \{p\}$ is connected). If $\dim \Delta_{[b,L]} = 1$, then L is a translation axis of $\mathcal{P}$ and $\mathcal{P}$ is classical by Salzmann, [24], (7.25). Thus we may assume that $\dim \Delta_{[b,L]} = 0$, whence $\dim \Delta \le l + \dim \mathrm{D}_b(\Delta_b)$. If we suppose that $\dim \Delta = 4$, the last inequality shows that $\dim \mathrm{D}_b(\Delta_b) \ge 3$. Since $\mathrm{D}_b(\Delta_b)$ is a solvable subgroup of $\Sigma_0 \le \mathrm{GL}_2\mathbb{R}$, we infer that $\mathrm{D}_b(\Delta_b)$ contains the group $\{\lambda \cdot \mathrm{id} \mid \lambda > 0\}$ which consists of homologies (with center 0 and axis L_∞). By Lemma 1.4 the plane $\mathcal{P}$ has a one-dimensional group H consisting of homologies with center b. Since H is commutative, there is a line K through p such that $\mathsf{H} = \mathsf{H}_{[b,K]}$, cp. [25], 61.25. Since $\dim \Delta_{[p,p]} = 1$ and $\dim \Delta = 4$, the same argument shows that there is a homology δ having p as its center and some line M as its axis. Since we may choose the point b arbitrarily on $L \setminus \{p\}$, we may assume that b does not lie on $M \ne L$. Then $(\Delta_{[b,K]})^{\delta} = \Delta_{[b^{\delta},K]}$ with $b^{\delta} \ne b$.



Applying [25], Lemma 33.3 to H and $\Delta_{[b^\delta,K]} \neq \mathbb{1}$, we get a (p,K)-transitive translation group $\Delta_{[p,K]}$. Consequently, we have $\dim\Delta_{[p,p]} = 2$, contrary to our assumptions. This shows that $\dim\Delta \leq 3$ if $\mathcal{P}$ is not isomorphic to $\mathcal{P}_2\mathbb{R}$.

Case $k = 2$. If $\dim\Delta_{[p,p]} \leq 2$, then $\dim\Delta \leq 8$ follows immediately by $(*)$. Hence we may suppose that $\dim\Delta_{[p,p]} = 3$ and thus Δ is (p,K)-transitive for some line $K \in \mathcal{L}_p$, see [25], dual statement of 61.11(c) together with 61.12. Assume moreover that $\dim\Delta = 9$. From

$$9 = \dim\Delta = \dim\ker \mathrm{D}_p + \dim \mathrm{D}_p\Delta = \dim\Delta_{[p,p]} + \dim \mathrm{D}_p\Delta = 3 + \dim \mathrm{D}_p\Delta$$

we conclude that $\dim \mathrm{D}_p\Delta = 6$. Thus the tangent translation plane $\mathcal{A}_p$ has an automorphism group of dimension at least $\dim \mathrm{D}_p\Delta + \dim\mathsf{T} = 10$, where $\mathsf{T} \cong \mathbb{R}^4$ is the translation group of $\mathcal{A}_p$. According to the classification of 4-dimensional compact translation planes (see [25], Chapt. 7) this implies that $\mathcal{A}_p$ is isomorphic to the affine complex plane $\mathcal{A}_2\mathbb{C}$. In particular, we have $\mathrm{D}_p\Delta \leq \mathrm{GL}_2\mathbb{C}$. Since $\mathrm{D}_p\Delta$ fixes the spread element T_pL, we have up to conjugacy

$$\mathrm{D}_p\Delta \leq \Upsilon := \left\{ \begin{pmatrix} a & b \\ & c \end{pmatrix} \,\middle|\, a,b,c \in \mathbb{C}, ac \neq 0 \right\} \leq \mathrm{GL}_2\mathbb{C}$$

and from $\dim \mathrm{D}_p\Delta = 6 = \dim\Upsilon$ we conclude that $\mathrm{D}_p\Delta = \Upsilon$, since both $\mathrm{D}_p\Delta$ and Υ are connected. Thus

$$(\mathrm{D}_p\Delta)_{[p]} = \left\{ \begin{pmatrix} a & \\ & a \end{pmatrix} \,\middle|\, a \in \mathbb{C}^\times \right\}$$

is two-dimensional, and hence $\dim \Delta_{[p]}/\Delta_{[p,p]} \geq 2$. By Lemma 1.4 this implies that Δ contains a (commutative) two-dimensional group H consisting of homologies with center p and some axis M. Since $K \setminus \{p, K \wedge M\}$ is connected, this shows that H is (p,M)-transitive. Using the dual statement of Pickert, [23], 3.1.11, p. 67, we infer that $\mathcal{P}$ is $(K \wedge M, M)$-transitive. Since $p \neq K \wedge M$ this implies that $\mathcal{P}$ is (K,K)-transitive, see Pickert, [23], 3.1.12.1, p. 68. In particular, the plane $\mathcal{P}$ is a translation plane. Hence $\mathcal{P}$ is classical by Otte [22] or by T. Grundhöfer, H. Hähl, [8]. This finishes the proof of the theorem. □

Lemma 2.2. *Let $\mathcal{P}$ be a smooth projective planes of dimension $n = 2l = 2^k$ with $k \geq 2$. If the solvable group Δ fixes some point p, and if $\dim\Delta_{[p,p]} = 0$, then*

$$\dim\Delta \leq l + \mathrm{rk}\,\Delta + 2 \leq l + k + 2 \leq \begin{cases} 6 & \textit{if } k = 2 \\ 9 & \textit{if } k = 3 \\ 14 & \textit{if } k = 4 \end{cases}.$$

Proof. By Lemma 1.8, the group Δ acts not transitively on $\mathcal{L}_p$. Hence either the group Δ fixes some line $K \in \mathcal{L}_p$ or there is a non-trivial orbit K^Δ of dimension at most $l - 1$. If $K^\Delta = K$, we may apply Corollary 1.6 in order to get

$$\dim\Delta = \dim\Delta_K = \dim K'^{\Delta_K} + \dim\Delta_{K,K'} \leq l + k + 2$$

for any line $K' \in \mathcal{L}_p \setminus \{K\}$. Now let K^Δ be a non-trivial orbit of dimension at most $l-1$. Since Δ is solvable, there are normal subgroups P and Π of Δ with $\Pi < \mathsf{P}$ and $K^\Pi = K \neq K^\mathsf{P}$ such that P/Π is a minimal normal subgroup of Δ/Π. Thus, the group Δ acts irreducibly on $\Upsilon := \mathsf{P}/\Pi$ and Υ is isomorphic to one of the abelian groups $\mathbb{T}$, $\mathbb{R}$, or $\mathbb{R}^2$. If $\dim \Upsilon \leq 1$, then Δ induces on Υ a group Δ^Υ of dimension at most 1. If $\Upsilon \cong \mathbb{R}^2$, then the induced group Δ^Υ contains a torus subgroup for Δ acts irreducibly on Υ. This implies that $\dim \Delta^\Upsilon \leq 2$. Hence, in any case, the centralizer

$$\Xi = \left\{ \xi \in \Delta \,\middle|\, \forall\, \rho \in \mathsf{P} : \Pi\rho^\xi = \Pi\rho \right\}$$

has codimension at most 2 in Δ and contains the group Π. For $\xi \in \Xi$ with $K^\xi = K$ we have $K^{\rho\xi} = K^{\xi^{-1}\rho\xi} \in K^{\Pi\rho} = K^\rho$, whence Ξ_K acts trivially on the orbit K^P. In particular, the stabilizer Ξ_K fixes a second line $K' \in \mathcal{L}_p$. Applying Proposition 1.3 and Corollary 1.6 to $\Xi_K = \Xi_{K,K'}$, we get

$$\dim \Delta \leq \dim \Xi + 2 \leq \dim K^\Xi + \dim \Xi_K + 2 \leq (l-1) + (r+2) + 2 \leq l + k + 3.$$

If $\dim K^\Xi \leq l-2$ or if $r \leq k-1$, the same computation shows that $\dim \Delta \leq l+k+2$ holds, and the lemma is proved. Hence we may assume $r = k$ as well as $\dim K^\Xi = l-1$. Since the kernel of the tangent translation plane $\mathcal{A}_c$ is isomorphic to some closed connected subgroup of the multiplicative group of the quaternions (see Buchanan, Hähl, [4]), and because $r = k \geq 2$, there is a torus group $\mathsf{T} \cong \mathbb{T}$ of Ξ_K that acts faithfully on $\mathcal{L}_p$. By E. Floyd, [5], Th. 5.2, the dimension of the fixed point set of an element $\delta \in \mathsf{T}$ of order 3 is of codimension at least 2. In particular, the collineation δ acts non-trivially on K^Ξ, because the latter set has codimension 1. If we choose $K^* \in K^\Xi$ in such a way that $K^{*\delta} \neq K^*$, then $\dim K^{*T} = 1$. Consequently, the torus rank of Ξ_{K^*} is at most $r-1$ (recall that r is the torus rank of Ξ). Let $\xi \in \Xi$ with $K^\xi = K^*$. Since

$$\Xi_{K^*} = \Xi_{K^\xi} = (\Xi_K)^\xi = (\Xi_{K,K'})^\xi = \Xi_{K^*,K'^\xi},$$

we can apply Proposition 1.3 to Ξ_{K^*} in order to obtain

$$\dim \Xi_{K^*} \leq (r-1) + 2 \leq (k-1) + 2 = k+1.$$

This finally implies that

$$\dim \Delta \leq \dim \Xi + 2 \leq \dim K'^\Xi + \dim \Xi_{K'} + 2 \leq (l-1) + (k+1) + 2 = l + k + 2,$$

and the proof is completed. □

We turn to the antiflag case.

Theorem 2.3. *Let $\mathcal{P}$ be a smooth projective plane of dimension $n = 2^k = 2l$. Assume that the solvable group Δ fixes some antiflag (p, L). Put $d_{II}(1) = 3$, $d_{II}(2) = 6$, $d_{II}(3) = 9$, and $d_{II}(4) = 14$. Then the dimension of Δ is at most $d_{II}(k)$.*

Proof: Using Corollary 1.6 and noting that $\Delta_{[p,p]} = \mathbb{1}$ we get

$$\dim \Delta = \dim \ker \mathrm{D}_p + \dim \mathrm{D}_p\Delta = \dim \Delta_{[p,p]} + \dim \mathrm{D}_p\Delta = \dim \mathrm{D}_p\Delta.$$

Case $k = 1$. Since $\mathrm{D}_p\Delta$ is a solvable subgroup of $\Sigma_0 \leq \mathrm{GL}_2\mathbb{R}$, we get $\dim \mathrm{D}_p\Delta \leq 3$.

Case $k \geq 2$. For $k \leq 3$ we could give a similar proof as in the case $k = 1$ using the classification of 4-dimensional and 8-dimensional translation planes. However, the classification of 16-dimensional compact translation planes does not provide usable results for $k = 4$. Instead, we use Lemma 2.2 to finish the proof. □

The next theorem deals with case (IV).

Theorem 2.4. *Let $\mathcal{P}$ be a smooth projective plane of dimension $n = 2^k = 2l$. Put $d_{IV}(1) = 2$, $d_{IV}(2) = 3$, $d_{IV}(3) = 7$, and $d_{IV}(4) = 14$. If every minimal connected commutative closed normal subgroup N of Δ has some orbit x^{N} $(x \in P \cup \mathcal{L})$ which generates a subplane $\mathcal{E}$ of $\mathcal{P}$, then the dimension of the solvable group Δ is at most $d_{IV}(k)$.*

Proof: By M. Lüneburg, [17], Anhang B, Satz 3, the group Δ fixes some point or some line. Up to duality we may assume that Δ fixes some point p. Consider the derivation map $\mathrm{D}_p : \Delta \to \Sigma_0$. Then the following assertions (a) and (b) hold:

(a) $\dim \ker \mathrm{D}_p = 0$: If $\dim \ker \mathrm{D}_p > 0$, then there is a minimal connected commutative closed normal subgroup M in $\ker \mathrm{D}_p$ (namely a suitable commutator subgroup of $(\ker \mathrm{D}_p)^1$, since $\ker \mathrm{D}_p$ is a solvable group). By [2], (3.9), we have $\ker \mathrm{D}_p = \Delta_{[p,p]}$. Hence the normal subgroup M cannot have an orbit (neither in P nor in $\mathcal{L}$) that generates a subplane of $\mathcal{P}$. This contradicts the assumptions of the theorem, and thus we have $\dim \ker \mathrm{D}_p = 0$.

From (a) and Lemma 1.2 we get

$$\dim \Delta = \dim \mathrm{D}_p\Delta.$$

(b) $\dim \Delta_{[p]} = 0$: Clearly, $\Delta_{[p]}$ is a normal subgroup of Δ. Since the connected component of the identity $\Delta^1_{[p]}$ is a characteristic subgroup of $\Delta_{[p]}$, the connected group $\Delta^1_{[p]}$ is a normal subgroup of Δ, too. Assume that $\Delta^1_{[p]} \neq \mathbb{1}$. According to part (a), the group $\Delta^1_{[p]}$ contains no elations, i.e. $\Delta^1_{[p]}$ consists of homologies only. Since Δ is solvable and because locally compact groups of homologies are isomorphic to some closed connected subgroup of $\mathbb{H}^\times$ (see [2], (3.12)), we conclude that $\Delta^1_{[p]}$ is commutative (in fact, it is a subgroup of $\mathbb{C}^\times$). Thus, the group Δ has a minimal connected commutative closed normal subgroup M consisting entirely of homologies. As in part (a), we infer that M has no orbit that generates a subplane of $\mathcal{P}$. This contradiction shows that $\dim \Delta_{[p]} = 0$.

From $\dim \Delta_{[p]} = 0$ we conclude that $\mathrm{D}_p\Delta$ has no nontrivial connected subgroup consisting of homologies with center 0. In the following we put $\Sigma := \mathrm{Aut}(\mathcal{A}_p)^1$.

Case $k = 1$. We know that $\mathrm{D}_p\Delta$ is a subgroup of $\mathrm{GL}_2\mathbb{R}$. If $\mathrm{rk}\, \mathrm{D}_p\Delta = 1$, this implies that $\dim \mathrm{D}_p\Delta \leq 2$. Hence, we may assume that $\mathrm{rk}\, \mathrm{D}_p\Delta = 0$. Since $\mathrm{D}_p\Delta$ is solvable, we conclude that $\mathrm{D}_p\Delta$ is conjugate to some subgroup of the 3-dimensional group ∇ of all (upper, say) triangular matrices of $\mathrm{GL}_2\mathbb{R}$. If $\dim \mathrm{D}_p\Delta = 3$, this implies that $\mathrm{D}_p\Delta$ is conjugate to ∇^1 (recall that Δ is connected). In particular, the group $\mathrm{D}_p\Delta$ contains the connected set $\Lambda := \{\lambda \cdot \mathrm{id} \mid \lambda > 0\}$ which consists of homologies with center 0, a contradiction. This shows that $\dim \mathrm{D}_p\Delta \leq 2$.

Case $k = 2$. According to [25], 73.4, the group $\mathrm{D}_p\Delta$ fixes some point ∞ on L_∞. Thus the line $\mathrm{T}_pL := \infty \vee 0$ is fixed by $\mathrm{D}_p\Delta$ as well. We assume that $\dim \mathrm{D}_p\Delta \geq 4$. Then $\dim \Sigma \geq 8$ follows. By the clasification of four-dimensional translation planes, see [25], 73.22, the plane $\mathcal{A}_p$ is either isomorphic to the complex affine plane $\mathcal{A}_2\mathbb{C}$ or to the translation plane $\mathcal{A}_0$ described in [25], 73.10(b). The automorphism group of the affine plane $\mathcal{A}_0$ has dimension 8, whence we get $\mathrm{D}_p\Delta = \Sigma$, because $\mathrm{D}_p\Delta$ is connected. In particular, this shows that $\mathrm{D}_p\Delta$ contains a one-parameter subgroup of homologies with center 0 and we end up again with a contradiction. So let $\mathcal{A}_p \cong \mathcal{A}_2\mathbb{C}$ and denote by $\mathsf{K} \cong \mathbb{T}^r$ a maximal compact subgroup of $\mathrm{D}_p\Delta$. Then $\Sigma \cong \mathrm{GL}_2\mathbb{C}$. By Proposition 1.3 we have $r := \mathrm{rk}\, \mathrm{D}_p\Delta \leq k = 2$. If $r = 2$, then $\mathsf{K} \cong \mathbb{T}^2$. Since the set of fixed lines of K on the spread $\mathcal{S}_p \cong \mathbb{S}_2$ is nonempty, there exists a second fixed line T_pK of K in $\mathcal{S}_p$, see Floyd [6], (4.1) and [5], (5.2). Hence K is a compact subgroup of the group D of all complex diagonal matrices. This implies that $\mathsf{K} = \left\{\mathrm{diag}(e^{is}, e^{it}) \;\middle|\; s, t \in \mathbb{R}\right\}$. Thus the group K contains a one-parameter subgroup of homologies with center O, which is again a contradiction. This implies that $r \leq 1$. The group K acts nontrivially on the spread $\mathcal{S}_p$ or $\mathsf{K} = \mathbb{1}$, for otherwise K would consist of homologies with center 0. Thus, if $\mathsf{K} \neq \mathbb{1}$, then there is some line $\mathrm{T}_pK \in \mathcal{S}_p$ which is moved by K. If $r = 1$, we even get $\mathsf{K}_{\mathrm{T}_pK} = \mathbb{1}$. Of course, this is also true in the case $r = 0$. Hence, if $r \leq 1$, then the torus rank of $\Psi = (\mathrm{D}_p\Delta)^1_{\mathrm{T}_pK}$ is zero (i.e. Ψ has no non-trivial compact subgroup) and we have

$$\dim \Psi = \dim \mathrm{D}_p\Delta - \dim(\mathrm{T}_pK)^{\mathrm{D}_p\Delta} \geq 4 - 2 = 2.$$

Since Ψ fixes two distinct lines of $\mathcal{A}_p$ through 0, this group is again a subgroup of D. Unfortunately, we do not know a priori whether or not Ψ is a closed subgroup of D. If it is closed in D, then Ψ is in fact contained in the two-dimensional subgroup $D_\mathbb{R}$ of D consisting of *real* diagonal matrices. From $\dim \Psi \geq 2$ we infer that $\Psi = D_\mathbb{R}^1$. In particular, the group Ψ contains the one-parameter group Λ which consists of homologies and the proof is completed. It remains to show that Ψ is closed in D. Assume the contrary. Then there exists a closed non-compact one-parameter subgroup $\mathsf{P} = \{\rho(t) \mid t \in \mathbb{R}\}$ of Δ_K which is mapped by D_p onto a non-closed one-parameter subgroup $\mathrm{D}_p\mathsf{P}$ of Ψ (see Hochschild [13], Chap. XVI, Theorem 2.4 and recall that D_p is a covering map since $\dim \ker \mathrm{D}_p = \dim \Delta_{[p,p]} = 0$) which can be represented as the group $\left\{\mathrm{diag}(e^{it}, e^{i\alpha t}) \;\middle|\; t \in \mathbb{R}\right\}$ of diagonal matrices for some irrational real number α. Putting $D_\mathbb{C} = \left\{\mathrm{diag}(e^{it}, e^{it}) \;\middle|\; t \in \mathbb{R}\right\}$, we see that the group $\mathrm{D}_p\mathsf{P} \cap D_\mathbb{C}$ is a dense subgroup of the homology group $D_\mathbb{C}$. In particular, there is a sequence $(t_n)_{n\geq 1}$ of real numbers with $t_n \to \infty$ as $n \to \infty$ such that $\sigma_n := \mathrm{D}_p(\rho(t_n)) \in \mathsf{P} \cap D_\mathbb{C}$ which converges to the identity. Of course, the collineations $\rho(t_n)$ are homologies of $\mathcal{P}$ with center p. Since the derivatives at the point p of these homologies converge to the identity mapping, we may choose compact neighborhoods V and W of p, where V does not intersect any axis of the homologies $\rho(t_n)$, such that $\bigcup_n W^{\sigma_n} \subseteq V$. Now we can apply a lemma of Löwen, [16], (3.8) which states that for any point $q \in W$ and any accumulation point r of q^{σ_n} there is a collineation σ which is an accumulation point of the sequence $(\rho(t_n))$ and which maps q to r. But the one-parameter group P is closed in Δ and $\rho(t_n) \in \mathsf{P}$ does not accumulate at all because of $t_n \to \infty$. This contradiction shows that Ψ is closed and we eventually have finished the case $k = 2$.

Case $k = 3$. We assume that $\dim \mathrm{D}_p\Delta \geq 8$. Then $\dim \mathrm{Aut}(\mathcal{A}_p) \geq 16$ follows. In case of equality we get $\mathrm{D}_p\Delta = \Sigma := \mathrm{Aut}(\mathcal{A}_p)^1$ and thus Δ contains a one-parameter group of homologies with center p, a contradiction. Hence, we may assume that $\dim \Sigma \geq 17$. Thus, the tangent translation plane $\mathcal{A}_p$ is known, see the list in [25], 82.25 or [11]. The following case by case study is based on this list. If $\mathcal{A}_p$ is the classical affine plane over the quaternions, then $\dim \Delta_{[p]} \geq 1$ by 1.6 which is a contradiction. If $\dim \Sigma = 18$, then Σ contains a subgroup isomorphic to $\mathrm{SU}_2\mathbb{C}$. Hence, any maximal solvable subgroup A of Σ has dimension at most 16 and contains the one-parameter subgroup of homologies with center 0 and we get a contradiction as before. If $\dim \Sigma = 17$, then either the kernel of $\mathcal{A}_p$ is isomorphic to $\mathbb{C}$ (cases (iii), (iv) and (vi) of 82.25) and we end up with $\dim \Delta_{[p]} \geq 1$ again, or Σ contains a subgroup isomorphic to $\mathrm{SO}_3\mathbb{R}$. In the latter case we conclude that $\dim \mathsf{A} \leq 15$ for any maximal solvable subgroup A of Σ which is again a contradiction. Thus, $\dim \mathrm{D}_p\Delta \leq 7$ holds.

Case $k = 4$. As in the previous theorem we apply Lemma 2.2 in order to prove the claim. □

The remaining case (III) that may occur for solvable groups can be deduced from the three preceding theorems by means of induction if we recall that the kernel of the action of Δ on a Baer subplane is at most one-dimensional (zero-dimensional in the case $\dim \mathcal{P} = 4$), see M. Lüneburg, [17], I, V.5.

Theorem 2.5. *Let $\mathcal{P}$ be a smooth projective plane of dimension $2^k \geq 4$. Assume that the solvable group Δ leaves some Baer subplane $\mathcal{B}$ invariant. Put $d_{III}(2) = 3$, $d_{III}(3) = 9$, and $d_{III}(4) = 17$. Then the dimension of Δ is at most $d_{III}(k)$.*

By considering the automorphism groups of the four classical planes it can easily be seen that the upper bounds for $\dim \Delta$ are sharp bounds in every case excluding the "otherwise" case which does not occur in the classical planes. Summing up the theorems of this section we get the following table for a *non-classical* smooth projective plane.

$\dim \mathcal{P}$	flag	antiflag	Baer subplane	otherwise
2	3	3	—	2
4	8	6	3	3
8	$\min\{16, 13+t\}$	9	9	7
16	$\min\{29, 22+t\}$	14	17	14

In order to compare our results with the results in the topological case, we also list the bounds of M. Lüneburg [17], p. 36:

$\dim \mathcal{P}$	flag	antiflag	Baer subplane	otherwise
2	5	3	—	4
4	10	6	5	6
8	17	13	11	13
16	30	24	18	24

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Received: April 24, 1997