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On the General Hyperplane Section of a Projective Curve

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Abstract. Here we study cohomological and geometrical properties of the general zero-dimensional section of a nonreduced curve $C \subset \mathbf{P}^n$ and (in positive characteristic) of an integral variety.

0. Introduction

The main actors in this paper will be zero-dimensional subschemes of a projective space $\mathbf{P}^r$. This paper consists of three parts. The link between the three parts is given by the common main actors. The content of the last part (section 6) is used in the second part for the proof of Theorems 5.1 and 5.4.

In the first part of this paper (i.e. in the first three sections) we study the general hyperplane section, X, of a nonreduced curve $Y \subset \mathbf{P}^{r+1}$. This topic was considered in [9]. An interesting question is whether X is in linearly general position (see e.g. [9] and [4] for terminology). The motivation came from Castelnuovo's theory (see [11, Ch. III]) which gives an upper bound on $p_a(Y)$ (see [9, Th. 3.2]). If Y is a double structure on Y_{red}, the linear generality of X was proved in [9, Th. 3.1], in characteristic 0 and studied in positive characteristic in [4]. When Y has high multiplicity at a generic point of Y_{red}, the linearly general position of X may fail just because it fails even for the connected components of X. Hence it seems very useful to have a more general setup. This is done in sections 1 and 2 (which are the main motivation for this paper). As a by-product, under suitable assumptions we get several results on the postulation of X and hence upper bounds for $p_a(Y)$. Indeed there is a machine which takes as input assumptions on Y and gives as output results on the postulation of X. We choose to write explicitely just a few statements, whose proofs show how to use the machine. In these two sections we stress the general setup and the results 1.3, 2.4, 2.5 and 2.6 . In the third section for a given zero dimensional scheme $X \subset \mathbf{P}^r$ we construct a nice curve $Y \subset \mathbf{P}^{r+1}$ with X as hyperplane section.

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Multiple structures supported by curves arose often in the literature ("ribbons", "ropes", construction of vector bundles on 3-folds generalizing Ferrand's construction, and so on). Hence we believe that it is important to understand the cohomological properties of these nonreduced curves. Castelnuovo's theory is the natural first step for this understanding. We believe that the multiplicity sequence, the multiple flag and the corresponding notions of linearly general and strongly linear general position (see Definition 1.1 and the discussion following it) are the natural ones for the Castelnuovo's theory of nonreduced curves.

In the second part (sections 4 and 5) we extend some results from [13] and [14] on syzygies and the index of regularity of varieties. In this part we stress the statements 4.1, 4.2, 4.3.4, 5.1 and 5.2.

In 6.3 we will prove 5.1; the work in 6.3 will complete also the proof of 5.4. In 6.1 and 6.2 we will consider any finite subset S of $\mathbf{P}^r$, $r \geq 2$, in linear semi-uniform position (with the terminology of [2]), but not in linearly general position. In 6.2 we will show with an example how to get in an elementary way results on the postulation of S. In summary in sections 4, 5 and 6 we extend to positive characteristic some results (call them Theorem **A**) using the following recipe.

(a) Look at the proofs in characteristic 0 and see that they work verbatim if the general 0-dimensional or the general 1-dimensional linear section has a good property, say property **D**.
(b) Show that property **D** holds in positive characteristic unless the general linear section has a very strange geometric property **S**;
(c) Show that this geometric property **S** gives numerical or cohomological restrictions so strong that when a curve with property **S** arises as general linear section not only the result is true (and hence Theorem **A** is extended to the positive characteristic case) but here we are far from the borderline cases which show that the statement of Theorem **A** is sharp.

We found that this simple recipe is often very useful for Castelnuovo's type results.

1. Sections of nonreduced curves and linearly general position

Let $Y \subset \mathbf{P}^{r+1}$ be a purely one-dimensional locally Cohen-Macaulay scheme (i.e. a possibly nonreduced curve); assume that $C := Y_{\mathrm{red}}$ is integral. Let H be a general hyperplane of $\mathbf{P}^{r+1}$ and set $X := Y \cap H$, $S := C \cap H$. If Z is a closed subscheme of $\mathbf{P}^{r+1}$ the linear subspace spanned by Z is the intersection of all hyperplanes containing it (i.e. of the hyperplanes whose equation vanishes identically on the scheme Z); if there is no such hyperplane, the linear span of Z is $\mathbf{P}^{r+1}$. By definition the degree $\deg(Z)$ of a zero-dimensional subscheme of $\mathbf{P}^N$ is the integer $h^0(Z, \boldsymbol{O}_Z)$. Recall (see e.g. [9]) that a zero-dimensional subscheme of $\mathbf{P}^N$ is said to be in *linearly general position* if for every linear subspace, M, we have $\deg(M \cap Z) \leq \dim(M) + 1$.

Definition 1.1. Fix $P \in C_{\mathrm{reg}}$; let L be a hyperplane with $P \in L$ and L transversal to C at P. Consider for any integer $t > 0$ the linear subspace $L(t, P)$ of L spanned by the zero-dimensional scheme $L \cap Y \cap C^{(t-1)}$, where $C^{(t-1)}$ is the $(t-1)$-th infinitesimal neighborhood of C in $\mathbf{P}^{r+1}$; set $m(t, P) := \dim(L(t, P))$; this integer does not depend on the choice of L; the increasing stationary sequence $\{m(t,P)\}_{t>0}$ will be called *multiplicity sequence at* P. The *multiplicity sequence* $\{m(t)\}$ (or $\{m(t,Y)\}$) of Y is the multiplicity sequence of a general point of C. Abusing notations, if L is a general hyperplane containing P we will call the flag $\{L(t,P)\}_{t>0}$ of linear subspaces of L the *multiple flag* of Y at P dropping any mention of L. In the same way we define the multiplicity sequence and the multiple flag of a zero-dimensional subscheme of a projective space.

Fix a finite set $Z \subseteq S$ such that for every $P \in T$ and every $t > 0$ we have $m(t,P) = m(t)$. For every $P \in T$ take H as hyperplane for the definition of $m(t,P)$; let $H(t,P)$ be the corresponding linear space. We will say that the set of linear subspaces $\{H(t,Q)\}_{Q\in T,t>0}$ is in *linearly general position* if for every $T'' \subseteq T$ and every choice of integers $t(Q)$ the linear spaces $\{H(t(Q),Q)\}_{Q\in T''}$ span a linear space of dimension

$$\min\Big(r, \operatorname{card}(T'') - 1 + \sum_{Q\in T''} m\big(t(Q)\big)\Big). \tag{1}$$

Let $\{A_j\}$ be a finite collection of subspaces of $\mathbf{P}^r$; we will say that $\{A_j\}$ is in *linearly strongly general position* if for all choices of a subspace B_j of A_j (even the empty one), the family $\{B_j\}$ is in linearly general position. We will say that the multiple flag of X is in *linearly general position* (resp. in *linearly strongly general position*) if the set of linear subspaces $\{H(t,Q)\}_{Q\in S,t>0}$ is in linearly general position (resp. in linearly strongly general position). Note that by definition of multiple flag at a point, if the multiple flag of X is in linearly general position, then X_{red} is in linearly general position.

Remark 1.2. Note that if $P \in C_{reg}$ is a curvilinear point of Y, then for every t we have $m(t,P) \leq m(t+1,P) \leq m(t,P)+1$. Hence the corresponding flag of subspaces of L is a complete flag. Hence if the multiple flag of a general hyperplane section is in linearly general position, then it is in linearly strongly general position.

As in the proof of [4, Th. 0.2], a suitable variation of the proof of [9, Th. 3.1], will prove the following general result.

Theorem 1.3. *Let $C \subset \mathbf{P}^{r+1}$ be an integral nondegenerate curve and Y a locally Cohen-Macaulay curve with $Y_{\mathrm{red}} = C$. Assume that Y is curvilinear outside finitely many points. Assume that C is reflexive. Then the multiple flag of a general hyperplane section of Y is in linearly general position (or equivalently in linearly strongly general position).*

Proof. Let H be a sufficiently general hyperplane; set $S := C \cap H$ and $X := Y \cap H$. Assume by contradiction the existence of $T \subseteq S$ and integers $\{t(Q)\}_{Q\in T}$ such that the linear span, U, of the linear spaces $\{H(t(Q),Q)\}_Q \in T$ has dimension $< \min\big(r, \sum_{Q\in T} m(t,Q) + \operatorname{card}(T) - 1\big)$. Choose $\operatorname{card}(T)$ and the integers $t(Q)$ minimal (after any order of the points of T). Since Y is curvilinear, by Remark 1.2 $\dim(U) = \min\big(r, \sum_{Q\in T} m(t,Q) + \operatorname{card}(T) - 1\big) - 1$. Now to prove Theorem 1.3 it is sufficient to follow step-by-step all the proof of [4, Th. 0.2] (which used heavily the proof of [9, Th. 3.1]). $\square$

2. Sections of nonreduced curves, postulation and genera

By [9, Th. 3.2], the linearly general position of a generic hyperplane section of Y gives Castelnuovo's upper bound for $p_a(Y)$. Theorem 1.3 shows the importance to have similar results when the multiple flag of a generic hyperplane section is in linearly general position and Y has, at a generic point, "controlled" behaviour (see Definitions 2.1 and 2.2). Indeed, in many cases we can check that if the multiple flag of a general hyperplane section is in linearly general position and Y is "good enough" (see Definitions 2.1 and 2.2) at a generic point, but the generic hyperplane section is not in linearly general position, then $p_a(Y)$ is strictly less than Castelnuovo's bound. We do not have a statement which covers all the cases we know. However, all such cases are proved with the same few elementary tricks and we will show these tricks in the cases written down here as Lemmas 2.4 and 2.5 and Theorem 2.6. We will restrict to the curvilinear case. Note that for non-curvilinear schemes, the same method sometimes gives stronger bounds; however, each case seems to require a separate effort.

Definition 2.1. *Let $Z \subset \mathbf{P}^r$ be a 0-dimensional connected curvilinear scheme; set $P := Z_{\mathrm{red}}$; let $\{V_i\}_{0 \le i \le t}$ be the flag of linear subspace associated to Z at P, i.e. let V_i be the linear space spanned by $Z^i := Z \cap \{P\}^{(i-1)}$ (with $t \le r, V_0 = \{P\}$ and V_t the span of Z). Consider the following conditions on Z:*

- (Lk) *k an integer ≥ 1. For every integer $i \ge 0$ the residual scheme R^{i+1} of Z^{i+1} (as subscheme of V_{i+1}) with respect to V_i imposes independent conditions on forms of degree k on V_{i+1}.*
- (LLk) *k an integer ≥ 1. Z satisfies* (Lk) *and R^{i+1} is in linearly general position in V_{i+1}.*
- (At) *t an integer ≥ 0. We have $H^1(\mathbf{P}^r, \boldsymbol{I}_Z(t)) = 0$.*

Note that if Z satisfies (Lk), then $\deg(Z^{i+1}) \le \deg(Z^i) + [(i+1+k)!/((i+1)! \cdot k!)]$; hence

$$\deg(Z) \le (t+k+1)!/(t!(k+1)!) \le (r+k+1)!/(r!(k+1)!). \tag{2}$$

Note that if Z satisfies (Lk) then every subscheme of Z satisfies (Lk).
It seems that (L1) (i.e. (LL1)) is the more important among the definitions (Lk) and that the condition (At) is less interesting.

Definition 2.2. *Let $Y \subset \mathbf{P}^{r+1}$ be a curve. Y is said to satisfy* (Lk) *(resp.* (LLk)*, resp.* (At)*) if for every irreducible component, T, of Y_{red}, for a general $P \in T$ and any hyperplane M with $P \in M$ and M transversal to T at P, the connected component of $Y \cap M$ with support P has property* (Lk) *(resp.* (LLk)*, resp.* (At)*).*

In later proofs any very easy variation of the proof of the following lemma will be called "by Horace" and the details will be left to the reader.

Lemma 2.3. *Let $Z \subset \mathbf{P}^r$ be a 0-dimensional connected curvilinear scheme with property* (Lk)*. Then Z imposes independent conditions on the forms of degree $k+1$ of $\mathbf{P}^r$.*

Proof. The proof is by induction on r. By the inductive assumption we may assume $t = r$ and the lemma true for Z^{r-1} as subscheme of the hyperplane V_{r-1}. Let A be the residual scheme of Z with respect to V_{r-1}. Let $B \subset V_{r-1}$ be the union of $(r+k)!/((r-1)! \cdot (k+1)!) - \deg(Z^{r-1})$ general points of V_{r-1}; set $E := B \cup Z^{r-1}$. Let D be the union of $(r+k)!/(r!k!) - \deg(A)$ general points of $\mathbf{P}^r$; set $F := D \cup A$. It is sufficient to check that $H^0(\mathbf{P}^r, \boldsymbol{I}_{Z \cup B \cup D}(k+1)) = \{0\}$. Take $f \in H^0(\mathbf{P}^r, \boldsymbol{I}_{Z \cup B \cup D}(k+1))$. Note that $f|V_{r-1} \in H^0(V_{r-1}, \boldsymbol{I}_E(k+1)) = \{0\}$ (inductive assumption). Hence f is divided by the equation, z, of V_{r-1}. Since f/z vanishes on F, by condition (Lk) we have $f/z = 0$. □

Lemma 2.4. *Let $X \subset \mathbf{P}^r$ be a zero-dimensional curvilinear subscheme with multiple flag in linearly general position. Let m be the maximal degree of a connected component of X. Then for every integer $t > 0$, X imposes at least* $\min(\deg(X), tr + 1 - (m-1)(r-1))$ *conditions on forms of degree t.*

Proof. Recall that by the assumption on X, the set X_{red} is in linearly general position. Apply the following algorithm. Fix a hyperplane H containing r points of X_{red}. By Remark 1.2 we have $X \cap H = X_{\mathrm{red}} \cap H$ and $X \cap H$ spans H. Let W be the residual scheme of X with respect to H. By Horace we have

$$h^0(\mathbf{P}^r, \boldsymbol{O}(t)) - h^0(\mathbf{P}^r, \boldsymbol{I}_X(t)) \geq h^0(\mathbf{P}^r, \boldsymbol{O}(t-1)) - h^0(\mathbf{P}^r, \boldsymbol{I}_W(t-1)) + r.$$

This means that $h^1(\mathbf{P}^r, \boldsymbol{I}_X(t)) \leq \max\{0, h^1(\mathbf{P}^r, \boldsymbol{I}_X(t-1)) - r\}$. Since W is curvilinear and the multiple flag of W is in linearly general position, we may apply induction on t. The algorithm fails only when we arrive at a subscheme of X with at most $r-1$ connected components. Each of these components gives at least one more condition. □

Lemma 2.5. *Let $X \subset \mathbf{P}^r$ be a zero-dimensional curvilinear subscheme with multiple flag in linearly general position. Set $w := \mathrm{card}(X_{\mathrm{red}})$. Let m, t, a and k be positive integers with $m > 0$, $0 < k < t$, $0 < a \leq \min(w, t-k)$. Assume that every connected component of X has property* (Lk). *Assume that for every $Q \in X_{\mathrm{red}}$, we have* $\deg(X \cap H(r-1, Q)) = m$. *Then X imposes at least*

$$\min(\deg(X), am + (t-a)r + 1 - (m-1)(r-1))$$

conditions on forms of degree t.

Proof. Fix one of the connected components of X_{red}, say Q_j, $1 \leq j \leq a$. Since the multiple flag of X is in linearly strongly general position by Remark 1.2, for every $Q \in X_{\mathrm{red}}$ we have $(X_{\mathrm{red}} \setminus \{Q\}) \cap H(r-1, Q) = \emptyset$. Hence by Lemma 2.3 we may apply a times Horace taking at step j as hyperplanes $H(r-1, Q_j)$. Then we apply $t-a$ times the algorithm used in the proof of Lemma 2.4. □

For a detailed discussion of Castelnuovo's theory, see [11, Ch. III]. By Castelnuovo's method, Theorem 1.3 and the previous lemmas give several non trivial upper bounds on $p_a(Y)$ for curves with Y_{red} reflexive and certain constraints (both numerical and geometric) on their multiple structure. Here we write down just one example, since just elementary arithmetic gives several other examples. We will write $\pi(d, N)$ for Castelnuovo's upper bound (see [11, Ch. III]) of the genus of a nondegenerate degree d integral curve in $\mathbf{P}^N$.

Theorem 2.6. *Let $Y \subset \mathbf{P}^{r+1}$ be a locally Cohen-Macaulay curve with Y_{red} integral, nondegenerate and reflexive. Assume Y curvilinear outside finitely many points and with property* (L1)*. Let m be the multiplicity of Y and set $w := \deg(Y_{\mathrm{red}})$. Assume $m \geq 2r$ and $w \geq 5r$. Then $p_a(Y) < \pi(\deg(Y), r+1)$.*

Proof. Consider the following inequality:

$$m(t - r) > rt - r + 1. \tag{3}$$

The right hand side of (3) is the contribution with respect to degree t hypersurfaces of Castelnuovo's method to obtain $\pi(mw, r+1)$. The left hand side of (3) is less than the corresponding contribution obtained in Lemma 2.5. The inequality on w gives that we may use all integers $t \leq 5w$ before arriving with Castelnuovo's method at a term larger than $\deg(Y)$. Fix the integer i with $0 < i \leq r$; the left hand side of (3) for $t = 5r + i$ (resp. $4r + i$, resp. $3r + i$) dominates the sum of the right hand side for $t = 5r + i$ (resp. $4r + i$, resp. $3r + i$) and $2r + i$ (resp. $r + i$, resp. i). Hence we conclude. □

3. Construction of nonreduced curves with assigned hyperplane section

Now we will show how easy (after [7], of course) it is to construct very nice (e.g. curvilinear and in linearly general position) nonreduced curves having a prescribed curvilinear hyperplane section (Theorem 3.1). The proof of Theorem 3.1 is an extension of the proof of [4, Prop. 6.1]. The fact that this extension is easy seems to be (in our opinion) the proof that we gave a useful setup.

Theorem 3.1. *Fix a zero-dimensional curvilinear subscheme X of $\mathbf{P}^r$ with $S := X_{\mathrm{red}}$ spanning $\mathbf{P}^r$. Fix an integer k such that every connected component of X has degree $\leq (r+k+1)!/(r!(k+1)!)$. See $\mathbf{P}^r$ as a hyperplane, U, of $\mathbf{P}^{r+1}$. Then there is a smooth rational curve C and a locally Cohen-Macaulay curve Y with $X = Y \cap U$ (as schemes), $Y_{\mathrm{red}} = C$, with $p_a(Y)$ very negative, Y curvilinear and with property* (LLk) *and such that the multiple flag of a general hyperplane section of Y is in linearly general position.*

Proof. By [7, Th. 1.6] there is a smooth rational curve, C, with $S = C \cap U$ (as schemes). It is easy to check the existence of a large integer m such that C is contained in the complete intersection, M, of $r - 1$ hypersurfaces of degree m, such that $X \subset M$ and such that M is smooth along C except at the points of S at which X has embedding dimension at least 2. Take as Y a multiple structure of C in the surface M. Fixing any point P of $C \setminus (C \cap U)$ we see that if m is large we may satisfy the condition (LLk) for Y at P (hence at a general point of C). □

Remark 3.2. Using a small extension of [7, Th. 1.6], we may take in the statement of 3.1 a non-curvilinear X. Now C is rational, but singular; $\mathrm{Sing}(C)$ is the union of the points of X_{red} at which X is not curvilinear. The nonreduced curve Y obtained in this way is curvilinear outside $\mathrm{Sing}(C)$.

4. Extensions of results of Maroscia-Nagel-Vogel

The main point of this section is an extension of the range of applications of [13, Lemma 1.5] (The Main Lemma), to positive characteristic in full generality and in particular to general hyperplane sections of very strange integral curves, i.e. by definition [2] to the integral curves $C \subset \mathbf{P}^n$ for which the general hyperplane section is not in linearly general position (see 4.2). Then in the subsection 4.3 we will use this extension of [13, Lemma 1.5], to improve some of the results of [13]. On the use of this lemma, see also the discussion in [14] between the proofs of Proposition 3.2 and Theorem 3.1. For syzygies in positive characteristic, see [15].

Since we are interested both in the "geometric" and the "arithmetic" genera of singular varieties (as in the case of curves) we will use the following definitions and notations. Furthermore, this setup does not depend on the resolution of singularities and allows (at the same price) stronger results than just using the geometric genus of a smooth birational model. Let V be an integral complete variety of dimension n; set $p_a(V) := h^n(V, \boldsymbol{O}_V)$; by [1, Theorem in the introduction], we have $p_a(V) = h^0(V, \omega_V)$ independently of the singularities of V. Let $f : V' \to V$ be a proper birational morphism; since the fibers of f have dimension at most $n-1$, a Leray spectral sequence shows that $p_a(V') \le p_a(V)$; in particular, if V' is smooth, we see that $p_a(V)$ is an upper bound for the geometric genus $p_g(V) := p_a(V')$ of V in the sense of [10]. On Castelnuovo varieties, see [10] and [8].

The following key result follows immediately from the geometric proofs of [6, p. 906], and [16, §1]; the first part of this key result is [5, Lemma 2.1]. We want to stress its power (e.g. it was used in [5] for the discussion of the groups appearing as monodromy group of the generic hyperplane section of a very strange curve).

Proposition 4.1. (Monodromy principle) *Let $C \subset \mathbf{P}^{r+1}$ be an integral nondegenerate curve and let $X \subset \mathbf{P}^r$ be the "generic" hyperplane section of C. Fix an integer m with $1 \le m \le r$. Then the monodromy group G of the generic hyperplane section of C acts transitively on the set of all subsets of m elements of X spanning a linear space of dimension $m-1$. Furthermore, for every linear space, L, spanned by $X \cap L$ and every $\{P, P'\} \subseteq (X \setminus (X \cap L))$ there is $g \in G$ with $g(P) = g(P')$ and $g(X \cap L) = X \cap L$.*

Here is the result corresponding to [13, Lemma 1.5].

Proposition 4.2. *Fix an integral nondegenerate curve $C \subset \mathbf{P}^{r+1}$ and let $X \subset \mathbf{P}^r$ be a general hyperplane section of C. Assume $h_X(t-1) \ge \mathrm{card}(X) - r - 1$. Put $p = \mathrm{card}(X) - h_X(t-1)$. Then there is a subset X' of X with $\mathrm{card}(X') = r+1$, X' spanning $\mathbf{P}^r$, such that for every subset U of X' with $\mathrm{card}(U) \ge p$, we have $h_{X \setminus U}(t-1) = \mathrm{card}(X \setminus U)$.*

Proof. Since by assumption $h^1(\mathbf{P}^r, \boldsymbol{I}_X(t-1)) \le r+1$, there is a subset X'' of X with $\mathrm{card}(X'') = r+1$ and such that $h^1(\mathbf{P}^r, \boldsymbol{I}_{X \setminus X''}(t-1)) = 0$. Assume that there is no such subset with the additional property that it spans $\mathbf{P}^r$. Let $s < r$ be the maximal dimension of a linear space, L, spanned by a subset X'' with the property just considered. Fix $A \subset X''$, A spanning L, with $\mathrm{card}(A) = s+1$. By the maximality of s, for every $P \in (X \backslash (X \cap L))$ and every $Q \in (X'' \backslash A)$, the set $X^* := (X'' \backslash \{Q\}) \cup \{P\}$ has $h_{X \backslash X^*}(t-1) = \mathrm{card}(X) - r - 2$, i.e. P is in the base locus, B, of $H^0(\mathbf{P}^r, \boldsymbol{I}_{(X \backslash X^*)}(t-1))$. Hence B contains $X \setminus X^*$, which is obviously absurd. Hence we may find such a set X'' spanning $\mathbf{P}^r$. If there is a subset U of X with $\mathrm{card}(U) = n - p$ and $h_{X \backslash U}(t-1) = \mathrm{card}(X) - (r-p)$, then every $U' \subset X''$ with $\mathrm{card}(U') = r - p$ has the same property, because by 4.1 the sets U and U' are conjugate under the action of the monodromy group of the generic hyperplane section of C. □

4.3. In this subsection we give a few easy consequences of Proposition 4.2, which improve in an essential way the positive characteristic part of [13]. X will always denote a general hyperplane of any integral nondegenerate curve (no restriction on the characteristic or the curve). The main result is (4.3.4).

4.3.1. By 4.2 the proof of [13, Th. 1.8] works verbatim for X.

4.3.2. By 4.2 [13, Lemma 2.1], and hence [13, Prop. 2.2], are true for X.

4.3.3. By 4.2 and the Main Lemma ([13, Lemma 1.5]), [13, Prop. 2.3] is true for X.

4.3.4. Again we obtain that [14], Prop. 3.2, is true for X. This answers a question raised in [14] in the text between the proofs of [14, Prop. 3.2], and [14, Th. 3.1].

5. Extensions of some results of a paper of Nagel

In this section we give some extensions of [14].

A key result in Castelnuovo's theory is the condition which forces a set of d points in $\mathbf{P}^c$ to be on an elliptic normal curve. The proof given in [11, Prop. 3.20] used the characteristic 0 assumption. In [14, Lemma 4.1], this was proved in positive characteristic if $d \geq 6c - 7$ and $c \geq 3$. We remark that we can prove it in several cases, e.g. if $d \geq 4c - 3$. However, just after the statement of [14, Lemma 4.1], there was announced a new proof (by Green, Harris and Eisenbud) which is expected to hold in positive characteristic. Waiting for weaker numerical assumptions, we just say that $(\$\$)(d, c)$ holds (or more precisely that $(\$\$)(d, c)$ holds over our base field) if over our fixed algebraically closed base field a set $X \subset \mathbf{P}^c$ of d points in uniform position lies on an elliptic normal curve if and only if it imposes $2c + 2$ independent conditions on quadrics of $\mathbf{P}^c$.

Here is the main statement of this section, i.e. an extension of [14, Prop. 4.2 (b)].

Theorem 5.1. *Let $X \subset \mathbf{P}^c$ be a set of d points not lying on a rational normal curve; assume $(\$\$)(d, c)$ (e.g. assume $d \geq 6c - 7$ and $d \geq 25$). Assume $c \geq 3$ and that X is the general hyperplane section of an arithmetically Cohen-Macaulay curve $C \subset \mathbf{P}^{c+1}$. Set $\mu := 1$ if $c+1$ divides d, $\mu := 0$ otherwise. Let e, t be integers defined by $s+1 = e(c+1) - t$ where $1 \leq t \leq c+1$. If $c = 3$ assume that C is not a very strange curve with $d = 2^k$ for some k, $v(1) = 2$ and $v(2) = 4$. Then X satisfies $(N_{e,t-\mu})$. In particular we have $\mathrm{reg}(X) \leq [(d+c)/(c+1)] + 1 + \mu$ and the homogeneous ideal of X is generated by forms of degree $\leq [(d+2+c)/(c+1)]$.*

Theorem 5.1 will be proved in the subsection (6.3) (i.e. we will discuss there the few cases needed to get it from the proof of [14, Prop. 4.2 (b)]). Now, following again [14], we list its several applications.

Remark 5.2. By Theorem 5.1 we have the corresponding extension of [14, Cor. 4.3]. Furthermore (see [14, discussion between the proof of 4.2 and the statement 4.3]) we have the corresponding extension of [17, Lemma 4.4], to a bound for the degrees of higher syzygy module. Furthermore, we have the corresponding extension of [14, Th. 1.2].

Remark 5.3. A general hyperplane section, X, of a very strange (in the sense of [2]) curve $C \subset \mathbf{P}^n$ cannot be contained in a rational normal curve, D, because every finite subset of D is in linearly general position. For very easy numerical reasons the same result is true if the curve D is only assumed to be integral, nondegenerate and with $\deg(D) - n$ small with respect to n (e.g. for every n if $\deg(D) = n + 1$).

Now we extend the last result of [14]; the assumption "U smooth in codimension 2" instead of "U smooth" below comes for free thanks to our set up on $p_a(U)$.

Theorem 5.4. *Take an arbitrary base field. Let $U \subset \mathbf{P}^{a+c}$, $c \geq 3$, be an integral variety of dimension a and degree $d \geq ac+2$; assume that either U is smooth in codimension 2 or that $d \geq 25$. Assume (\$\$)$(\deg(U), c)$. Then U is a Castelnuovo variety if and only if it is arithmetically Cohen-Macaulay and lies on a variety of minimal degree and codimension $c - 1$.*

Proof. If U is integral, arithmetically Cohen-Macaulay and lies on a codimension $c - 1$ minimal degree variety, then all the inequalities in Harris' proof of the upper bound for $p_a(U)$ (not only for the geometric genus) are equalities (since the postulation of any d points of a rational normal curves depends only on d).

Assume that U is a Castelnuovo variety. Let C (resp. X) be a general curve section of U. We claim that under any of the two assumptions on U, the set X is in linearly general position. If U is smooth in codimension 1, i.e. C smooth, the claim follows from [12, Prop. 5]. If $d \geq 25$, the claim follows from the upper bounds on $p_a(C)$ which will be proved in section 6 for very strange curves and then if $\dim(U) > 1$ by Harris' analysis (see the proof of [14, Th. 4.5]). $\square$

6. Points in semi-uniform position

In the second part of this section (subsection 6.3) we will prove 5.1; the work in 6.3 will complete also the proof of 5.4.

In the first part of this section S is just assumed to be any finite subset of $\mathbf{P}^r$, $r \geq 2$, in linear semi-uniform position, but not in linearly general position; we use the notations of [2]. Let $\{v(i,S)\}_{1 \leq i \leq r}$ or just $\{v(i)\}_{1 \leq i \leq r}$ be the invariants of S, i.e. let $v(i)$ be the number of points of S contained in every i-dimensional linear subspace of $\mathbf{P}^r$ spanned by points of S. Fix an integer a with $0 < a \leq r-2$ and any linear space, W, of dimension $a-1$ spanned by points of S; set $A := W \cap S$. The image, S', of $S \setminus A$ under the projection from

W onto a linear subspace, $\mathbf{P}^{r-a}$, disjoint from W is a set in linear semi-uniform position with invariants $\{v(i,S'')\}_{0<i\le r-a}$ depending only from a and the invariants $\{v(i)\}$; set $v^a(i) := v(i,S'')$. These integers satisfy (or are defined by):

$$(v(a+1)-v(a))v^{a+1}(i) = v(a+i+1)-v(a+i). \tag{4}$$

Remark 6.1. We point out here a few congruence relations among the integers $v(i)$ for a set in semi-uniform linear position. Consider any plane, L, spanned by points of S; set $A := L\cap S$; by (4) $v(1)-1$ divides $v(2)-1$. Call "point" an element of A, "line" a line joining two "points" and call "disjoint" two "lines" not meeting in a "point"; since there are $v(2)(v(2)-1)/2v(1)$ "lines" $v(1)$ divides $v(2)(v(2)-1)$. To get other congruence relations project S from a suitable subspace (i.e. use (4)).

6.2. In this subsection we show (with an example) how to get in an elementary way postulation results for a finite set, S, of points of $\mathbf{P}^c$ in semi-uniform linear position but not in linearly general position. These bounds are much stronger than necessary to check, e.g. (answering a very particular case of question raised in [14, Remark 3.3]) that if S is a hyperplane section of an arithmetically Buchsbaum curve $C\subset \mathbf{P}^{c+1}$ with $v(1)>2$, then $\mathrm{reg}(C) < (d/c)$; by the inequality (5) below, indeed this comes for free for very strange curves if $v(i)-i$ is not too small for some i; it is not difficult to lift the corresponding result to higher-dimensional arithmetically Buchsbaum varieties with C as curve section. By [5, Prop. 1.2], we have

$$\mathrm{reg}(S) \le \sum_{1\le a\le r} (v(a+1)-v(a))/(v(a)-v(a-1)). \tag{5}$$

Hence if $v(1)>2$ the only case to consider is the case $c=2$, $v(1)=3$. Set $d=2y+1$. We will use Horace applied to the "lines" several times, at each step using a "line" containing 3 "points" not on the "lines" used previously. The algorithm stops only when we have to control only points in linearly general position. Then, again as Castelnuovo, we use "lines"; however if at some step the remaining "points" are not on a conic, then we can do better using smooth conics through 5 of the remaining "points". There are $(2y+1)y/3$ "lines" and each point is contained in y lines. Hence for any "line" R there are $3y-3$ lines meeting R. Hence if $x\ge 2$ is such that

$$(2y+1)y/3 \le x(3y-3) \tag{6}$$

then we may use x "skew" lines before entering in Castelnuovo's range of inequalities. If $x\ge 4$, i.e. if $d\ge 9$, we gain enough to answer the previously quoted question in [14, Remark 3.3]. If $d<9$, we have $d=7$ by Remark 6.1. Assume $d=7$; then S has the linear structure of $\mathbf{P}^2(F_2)$ and we have $h^1(\mathbf{P}^2,\boldsymbol{I}_S(2))=1$, $h^1(\mathbf{P}^2,\boldsymbol{I}_S(3))=0$, and the homogeneous ideal of S is generated by cubics.

6.3. Here we prove Theorem 5.1. Here S is a general hyperplane section of a nondegenerate degree $d\ge 25$ very strange curve $C\subset\mathbf{P}^{c+1}$. Indeed, we may assume C very strange, because otherwise the monodromy group is $c+1 \geq 4$ transitive, hence (since $d \geq 25$) containing the alternating group and S would be in uniform position. The only missing cases from previous papers are for $c = 4$ and $c = 3$. First note that by [16, Th. 2.5], even if $c > 3$ we have $v(2) > 3$.

6.3.1. First we assume $v(1) = 2$. By [3], every $v(i)$ is a power of 2; furthermore, if $c = 4$, then the characteristic, p, is 2. Using the regularity bound (5), or using its proof to obtain the postulation estimates used in the proof of [14, Prop. 4.2 (b)], we conclude except when $c = 3$ and $v(2) = 4, 8$. Note that we need to check the inequality $\mathrm{reg}(S) \leq 2 + (d/4)$. Assume $c = 3$, $v(2) = 8$. Note that $v^1(1) = 7$. Projecting from a point of S (or using (5)) we get $\mathrm{reg}(S) \leq 7 + (d-2)/6$. Hence we need to check only the case $d = 32$. If $d = 32$ projecting from a point of S we get a plane set S' in semi-uniform position with $v(1) = 7$ and $v(2) = 31$, contradicting the last congruence relation given in Remark 6.1; a direct proof of the postulation results seems to be easy but long. Since the only remaining case is the exception listed in the statement of 5.1, Theorem 5.1 is proven if $v(1) = 2$. However to complete the picture we will discuss in 6.3.3 the exceptional case.

6.3.2. Here we assume $v(1) > 2$. By (5) (or its proof to obtain postulation results) the only bad case is when $c = 3$, $v(1) = 3$ and $v(2) = 7$. Here we find $\mathrm{reg}(S) \leq 3 + ((d-1)/2-1)/6+6$. Hence $\mathrm{reg}(S) \leq d/4$ if $d \geq 35$. Note that $(((d-1)/2-1)/6$ is an integer by Remark 6.1. Hence we may assume $d = 27$. If $d = 27$ projecting from a point of S we would find a set of 13 points in the plane such that any line containing 2 of them contains 6 of them; such a set does not exist (e.g. because always $v(2) - 1 \geq (v(1)-1)^2$, hence we obtain a proof of 5.1.

6.3.3. Here we discuss the exceptional case: $c = 3$, $v(1) = 2$, $v(2) = 4$, $d := v(3) = 2^k$ for some integer k. The curve C is very strange; call P its strange point. Here we assume that C is arithmetically Cohen-Macaulay; hence we need to check only the postulation or the index of regularity of C, or, equivalently, of one (even very particular) hyperplane section. Let m be the multiplicity of C at P. Projecting C from P we find a curve, D; set $x := \deg(D)$ and let q (resp. s) be the inseparable (resp. separable) degree of the projection. We have $d = m + xsq$. Take as hyperplane section of C the general one, Y, containing P. Hence in $\mathbf{P}^3$ we have the following zero-dimensional scheme, B. B has a component of degree m at P. The other connected components have length q and are contained in lines through P. Each line through P containing at least one of them contains exactly s of them. Furthermore, any plane spanned by 3 points of B_{red} but not containing P contains 4 points of B_{red}. Without using the last information, the usual "projection bound" is $\mathrm{reg}(B) \leq m + x/2 + sq + 1$. Since $d = 2^k$, the bound in 5.1 is $\mathrm{reg}(S) \leq 2 + d/4$. Hence we have 5.1 even in this exceptional case if $3m + 2x - 4 \leq (x-4)sq$ (hence for $x \geq 5$ except when m is huge).

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